ICIPE-2024-0097
ADAPTIVE DETERMINATION OF THE DAMPING COEFFICIENT OF THE LEVENBERG-MARQUARDT ALGORITHM USING TYPE-2 FUZZY IN INVERSE HEAT TRANSFER PROBLEMS

Ramin Sajedi
School of Mechanical Engineering, College of Engineering, University of Tehran, Tehran, Iran

Javad Faraji
Faculty of Mechanical Engineering, University of Tabriz, Tabriz, Iran

Forooza Samadi
Department of Mechanical Engineering, The University of Alabama, Tuscaloosa, Alabama, United States
e-mail: fsamadi@ua.edu

Farshad Kowsary
School of Mechanical Engineering, College of Engineering, University of Tehran, Tehran, Iran

Abstract. *In this study, a combination of the Levenberg-Marquardt algorithm and the type-2 fuzzy method is used to solve an inverse heat transfer problem. Considering the essential role of the damping factor in the stability and efficiency of the Levenberg-Marquardt algorithm in solving ill-posed problems, a new method based on fuzzy logic theory is used to update the value of this parameter in the iteration of the solution. A cubic polynomial experimental function is estimated using classical, type-1, and type-2 fuzzy Levenberg-Marquardt methods for the boundary heat flux in a one-dimensional heat transfer problem. The evaluation criteria chosen are convergence speed and robustness. The results show that the use of a new method increases the speed of convergence. The robustness of the estimates is improved by using this new method compared to the conventional Levenberg–Marquardt methods, which are also more accurate.*

Keywords: *Inverse heat transfer; One-dimensional heat transfer; Ill-posed problems, Levenberg-Marquardt algorithm, Type-2 fuzzy logic*

1. INTRODUCTION

Estimation theory is a very common problem in engineering and science, and various methods have been presented over the years (Keighobadi et al. 2020), so that inverse heat transfer problems are no exception. For solving inverse heat transfer problems, generally, a cost function is considered that needs to be minimized. Often, this cost function is formulated as an ordinary least square norm. Given that mathematically, inverse heat transfer problems are categorized as ill-conditioned, certain stabilization methods need to be employed during the solution process. The Levenberg-Marquardt (LM) method is one of the most powerful stabilization methods applied in the estimation process. This method was initially introduced by Levenberg, and later proven by Marquardt (Woodbury et al. 2023). Numerous research have been conducted in the field of inverse heat transfer, encompassing both parameter estimation and function estimation using this method (Lu et al. 2015; Duda 2016). The LM method can be considered as a composite method composed of the steepest-descent and Gauss-Newton methods. Depending on the value of the damping factor, this method will behave similarly to one of the above methods. Thus, it can be stated that the damping factor and its determination play a fundamental role in the accuracy, precision, and speed of the estimation process. Optimizaing the regularization parameter in inverse problems solution techniques is commonly investigated (Samadi et al. 2018; Samadi et al. 2021). However, despite the significant importance of the damping factor, there are few papers in the literature regarding adjusting its value. One of a few papers presented in this regard is the approach proposed by Cui et al. (Cui et al. 2017) for updating the damping factor proportional to the number of solution iterations. In their presented method, the damping factor is directly related to the dimensionless cost function in solving the inverse heat transfer problem. The accuracy, solution efficiency, convergence speed, and stability of the LM method are compared for the damping factor determined using four different methods. Chen et al. (Chen et al. 2003) introduced a new method for determining the damping factor. In this study, a neural network was trained using the LM method, and the damping factor values were determined using the variable decay rate method. According to the presented results, the use of the proposed method significantly increased the convergence speed compared to the classical method for determining the damping factor in the LM algorithm. The solution time using Chen et al.'s method is less than half of the time required in the classical method. A comparison between two different strategies for determining the damping factor, namely, the additive and multiplicative strategies by Lampton, has also been presented (Lampton 1997). The results indicate that the use of the additive damping strategy has a greater impact on improving the speed and stability of ill-posed problems solutions. In addition to these considerations, other papers have been presented on the damping factor and its impact on problem-solving using the LM algorithm in various scientific fields (Kind et al. 2016; Koh and Cheong 2018; Ukrainczyk 2009). Alongside the LM method, which is considered a classical method for solving inverse problems, novel methods have been proposed in recent years for solving inverse and ill-posed problems. Decentralized Fuzzy Inference (DFI), a fuzzy logic-based approach is among these methods. Fuzzy logic theory, introduced by Zadeh, essentially involves utilizing expert knowledge for the control of various systems (Zadeh 1979). Due to the high robustness of the DFI method against noise in input data and its acceptable stability, its application in solving inverse and ill-posed problems, including inverse heat transfer, has yielded satisfactory results. Wang et al. (Wang et al. 2017) utilized the DFI method to estimate unknown boundary conditions in a two-dimensional inverse conduction problem. The comparison of results obtained from the DFI and the LM methods validates the credibility of the new approach. Furthermore, the results demonstrate the independence of this method from initial guesses and its insensitivity to the presence of noise in input data, both of which are considered important advantages for the DFI method. Chen et al. (Chen et al. 2016) used the Decentralized Fuzzy Inference (DFI) method to estimate the shape of the boundary surface where heat transfer between the fluid and solid occurs. Several numerical tests have been conducted to investigate the influence of the initial guess, the number of temperature measurement points, and the presence of noise in the measurement data on the solution results. The accuracy of the DFI method has been validated against the conjugate gradient method (CGM) and genetic algorithm (GA) methods. The results indicate higher solution efficiency of the DFI method compared to the other two methods. Additionally, according to their investigations, the DFI method shows less sensitivity to the initial guess and the presence of noise in the measurement data compared to both other methods. In (Lau et al. 2015), Lau et al. employed the DFI method to estimate the transient heat flux in a participating medium problem. In this study, the radiative heat flux entering the surface is estimated based on the measured temperature of that surface. According to the obtained results, despite noise in temperature measurements, the transient heat flux is accurately determined by the DFI method. Another important result of ref. (Lau et al. 2015) is the high accuracy of the estimation using the fuzzy method compared to other approaches such as GA and Simulated Annealing. Considering the significance of the damping factor in solution efficiency, Sajedi et al. (Sajedi et al. 2021) propose a new approach for determining the damping factor in the LM method using type-1 fuzzy logic for inverse heat transfer problems. Their study acknowledges the advantages of LM method and the positive features of fuzzy logic-based methods like DFI. Their results show that, despite the positive impact of the fuzzy method in significantly reducing the convergence speed and improving solution stability, robustness to noise does not change much. In other words, determining the damping factor using type-1 fuzzy logic does not enhance the solution's robustness to measurement noise compared to classical LM. Therefore, to address this weakness, the use of type-2 fuzzy logic is suggested in the current paper.

2. 1-D TRANSIENT MODEL

For the 1-D transient heat conduction problem, as shown in Fig. 1, a specified heat flux is imposed at $x = 0$ and an adiabatic boundary condition is applied at $x = L$. The governing equations for this problem are as follows:

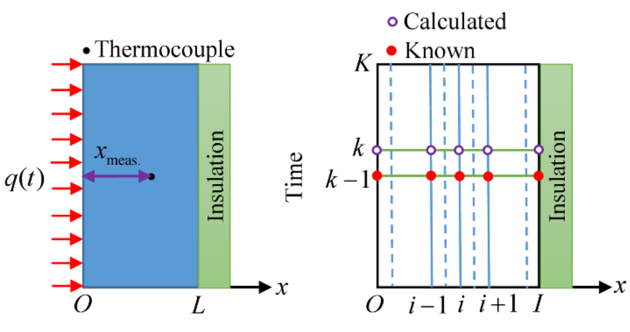

Figure 1. (a) Problem's schematic, (b) Domain discretization (Sajedi et al. 2021)

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial T^2(x,t)}{\partial x^2}, \quad 0 \le x \le L, \quad t \ge 0 \tag{1}$$

$$-\lambda \left. \frac{\partial T(x,t)}{\partial x} \right|_{x=0} = q(t) \tag{2}$$

$$-\lambda \frac{\partial T(x,t)}{\partial x}\bigg|_{x=L} = 0 \tag{3}$$

$$T(x,0) = T_0(x), \quad 0 \le x \le L \tag{4}$$

where $\boldsymbol{\alpha}$ is the thermal diffusivity, T_0 is the initial temperature, $q(t)$ is the heat flux, and λ represents the thermal conductivity. If the heat flux at $x = 0$ is specified, it leads to a direct heat conduction problem. Solving Eqs. (1) to (4) provides the temperature distribution at various points over time. To solve this problem numerically, an implicit finite difference method is applied, resulting in the following discrete equations (WOODBURY ET AL. 2023):

$$\rho c_p \frac{\Delta x}{2} \frac{T_0^k - T_0^{k-1}}{\Delta t} = q(t_k) + \lambda \frac{T_1^k - T_0^k}{\Delta x} \tag{5}$$

$$\rho c_p \Delta x \frac{T_i^k - T_i^{k-1}}{\Delta t} = \lambda \frac{T_{i+1}^k - T_i^k}{\Delta x} + \lambda \frac{T_{i-1}^k - T_i^k}{\Delta x} \tag{6}$$

$$\rho c_p \frac{\Delta x}{2} \frac{T_I^k - T_I^{k-1}}{\Delta t} = q(t_k) + \lambda \frac{T_{I-1}^k - T_I^k}{\Delta x} \tag{7}$$

where $i = 1,2,\ldots,I-1$. To solve the inverse problem presented above, which essentially involves estimating the time-varying heat flux, $q(t)$, the LM method has been employed as an iterative-sequential method. In the heat flux estimation process, additional information related to the measured temperature by an embedded sensor at the location $x = x_{meas}$ and times $t_i = 1,2,\ldots,K$, is utilized using the LM method. In solving the current inverse problem, the unknown heat flux is assumed as follows:

$$q(t) = \sum_{j=1}^{N} P_j C_j(t) \tag{8}$$

where P_j are the unknown parameters, $C_j(t)$ are the known functional forms such as polynomials, spline, etc., and N represents the number of components that are considered unknown. With the assumption made in Eq. (3), the problem of estimating the heat flux on the surface transforms into the estimation of the N unknown parameters of the vector $\mathbf{P} = [P_1,\ \ldots, P_N]^T$. The solution to the inverse heat transfer problem for estimating the unknown parameter vector is based on the minimization of the ordinary least squares norm, defined as follows:

$$S(\mathbf{P}) = \sum_{i=1}^{I} [Y_i - T_i(\mathbf{P})]^2 \tag{9}$$

where S is the objective function to be minimized. $T_i(\mathbf{P}) = T(\mathbf{P}, t_i)$ and $Y_i = Y(t_i)$ are estimated and measured temperatures at time t_i, respectively. Note that the estimated temperature, $T_i(\mathbf{P})$, is the temperature value at the measurement location, x_{meas}, and is obtained by solving the direct heat conduction problem for the estimated values of vector $\mathbf{P}$, at time t_i.

3. OBTAINING THE ITERATIVE SOLUTION PROCESS

By taking the derivative of Eq. (9) with respect to each of the unknown parameters and setting these derivatives equal to zero, the minimization process is implemented.

$$\nabla S(\mathbf{P}) = 2[-\frac{\partial \mathbf{T}^T(\mathbf{P})}{\partial \mathbf{P}}][\mathbf{Y} - \mathbf{T}(\mathbf{P})] = 0 \tag{10}$$

The first term in Eq. (5a) is defined as:

$$\frac{\partial \mathbf{T}^T(\mathbf{P})}{\partial \mathbf{P}} = \begin{bmatrix} \frac{\partial}{\partial P_1} \\ \frac{\partial}{\partial P_2} \\ \vdots \\ \frac{\partial}{\partial P_N} \end{bmatrix} \begin{bmatrix} T_1 & T_2 & \dots & T_N \end{bmatrix} \tag{11}$$

The transpose of the matrix resulting from Eq. (5b) is defined as sensitivity matrix as:

$$\mathbf{J}(\mathbf{P}) = \left[\frac{\partial \mathbf{T}^T(\mathbf{P})}{\partial \mathbf{P}}\right]^T = \begin{bmatrix} \frac{\partial T_1}{\partial P_1} & \frac{\partial T_1}{\partial P_2} & \frac{\partial T_1}{\partial P_3} & \cdots & \frac{\partial T_1}{\partial P_N} \\ \frac{\partial T_2}{\partial P_1} & \frac{\partial T_2}{\partial P_2} & \frac{\partial T_2}{\partial P_3} & \cdots & \frac{\partial T_2}{\partial P_N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial T_I}{\partial P_1} & \frac{\partial T_I}{\partial P_2} & \frac{\partial T_I}{\partial P_3} & \cdots & \frac{\partial T_I}{\partial P_N} \end{bmatrix} \tag{12}$$

By substituting Eqs. (11) and (12) in Eq. (10), it can be written as:

$$-2\mathbf{J}^T(\mathbf{P})\left[\mathbf{Y} - \mathbf{T}(\mathbf{P})\right] = 0 \tag{13}$$

Using Taylor series expansion for $\mathbf{T}(\mathbf{P})$ results in:

$$\mathbf{T}(\mathbf{P}^{k+1}) = \mathbf{T}(\mathbf{P}^k) + \mathbf{J}^k\left(\mathbf{P} - \mathbf{P}^k\right) + O\left(\left(\mathbf{P} - \mathbf{P}^k\right)^2\right) \tag{14}$$

The last term of Eq. (14) can be neglected in inverse heat conduction problems (Ozisik 2000). By substituting Eq. (14) in Eq. (13) and solving for $\mathbf{P}^{k+1}$, the following equation results:

$$\mathbf{P}^{k+1} = \mathbf{P}^k + \left[\left[\left(\mathbf{J}^k\right)^T \mathbf{J}^k\right]^{-1} \left(\mathbf{J}^k\right)^T \left[\mathbf{Y} - \mathbf{T}(\mathbf{P}^k)\right]\right] \tag{15}$$

Given that Eq. (15) is an approximation of the Newton-Raphson method, the main condition for the existence of a solution and obtaining a suitable estimate using this method is the satisfaction of condition $\left|\mathbf{J}^T\mathbf{J}\right| \neq 0$. On the other hand, even in cases where the value of this determinant is non-zero but small, this method will still be ineffective in estimating the desired parameter (Ozisik 2000). Inverse heat transfer problems, which are a subset of ill-posed problems, have a very small value for $\left|\mathbf{J}^T\mathbf{J}\right|$ (Ozisik 2000). Therefore, the direct use of Eq. (15) for estimating the unknown parameters in inverse heat transfer problems is not feasible. The use of the method proposed by LM and adding a corrective term to the equation obtained from the least squares estimation makes it possible to apply it in solving ill-posed inverse heat conduction problems (Ozisik 2000). Thus, the corrective method can be written as follows:

$$\mathbf{P}^{k+1} = \mathbf{P}^k + \left[\left[\left(\mathbf{J}^k\right)^T \mathbf{J}^k + \mu^k \mathbf{\Omega}^k\right]^{-1} \left(\mathbf{J}^k\right)^T \left[\mathbf{Y} - \mathbf{T}(\mathbf{P}^k)\right]\right] \tag{16}$$

where $\mathbf{\Omega}^k = diag\left[\left(\mathbf{J}^k\right)^T \mathbf{J}^T\right]$ is a diagonal matrix. In Eq. (16), μ^k is the damping factor, a positive scalar. The purpose for adding this term to the equation is to dampen oscillations and instabilities arising from the ill-posed nature of the inverse heat transfer problem. In this paper, the stopping criterion for the iterative process of estimating parameter, P_j, is shown below:

$$\left\|\mathbf{P}^{k+1}-\mathbf{P}^{k}\right\| \leq \varepsilon \tag{17}$$

where $\|\mathbf{x}\| = \left(\mathbf{X}^T\mathbf{X}\right)^{1/2}$, and $\varepsilon = 10^{-5}$ is considered as convergence criterion. The LM method is a classical method that combines the steepest descent and Gauss-Newton methods. The larger the damping factor, the closer this method behaves to the steepest descent method. In this case, the length scale of the steepest descent method is inversely proportional to the damping factor of the LM method. Conversely, as the damping factor becomes smaller (approaching zero), the behavior is similar to the Gauss-Newton method. It is known that the steepest descent method is stable but slow, so the LM method also has a speed-limiting factor. Determining the correct value for the damping factor is the most crucial point in inverse heat conduction problem. Generally, in inverse problems if $S\left(\mathbf{P}^{k+1}\right) \geq S\left(\mathbf{P}^{k}\right)$ the damping factor is selected to be 10 times larger than that of the previous step, while if $S\left(\mathbf{P}^{k+1}\right) < S\left(\mathbf{P}^{k}\right)$ it is decided to be 10 times smaller than the previous step (Chen et al. 2016). Considering that the Levenberg-Marquardt method, especially in the vicinity of the initial guess and in the early stages of solving where it is further from its final solution, is in a more ill-posed condition and closer to the steepest descent method, a larger damping factor accelerates the convergence of the solution. In other words, at the beginning of the solution, using a larger damping factor ($\mu = 10$) shortens the time required to reach a suitable estimate for the values of P_j . Moreover, as the solution progresses and the ill-posedness of the inverse heat transfer problem decreases, reducing the damping factor can shorten the convergence time. The algorithm for solving the problem using the classical Levenberg-Marquardt method is presented in Fig. 2.

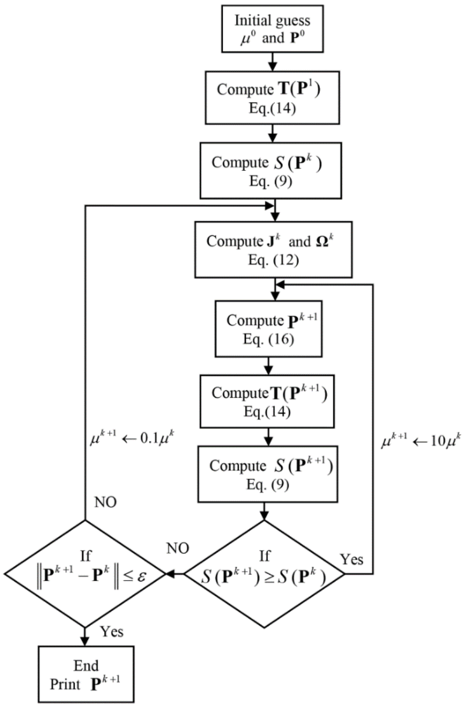

Figure 2. Classical levenberg-marquardt solution algorithm (Sajedi et al. 2021)

Considering the results presented in (Chen et al. 2003; Madsen et al. 2004; Kwak et al. 2011; Torabi and Hosseini 2018), using a fixed length scale in the steepest descent method has an undesirable impact on its convergence speed. Some references (Chen et al. 2003; Madsen et al. 2004; Kwak et al. 2011; Torabi and Hosseini 2018) have introduced the idea of considering a variable damping factor. Chen et al. (Chen et al. 2003) also investigated the influence of using a variable length scale on improving the speed of the classical LM method. The main idea of their approach is to reduce the size of the length scale coefficient in the vicinity of the initial guess and initial steps of the solution. Subsequently, increasing this scale in steps closer to the final solution has led to an acceleration in convergence speed. Therefore, in the present work, using the same approach, the damping factor in the LM method is considered variable with a fuzzy logic-based method, taking into account the percentage error value as a variable step. In this way, a larger damping factor for LM method is considered when the calculated percentage error is higher, and as the solution approaches the final answer, a smaller damping factor is adopted.

4. TYPE-1 FUZZY LOGIC

Fuzzy logic is widely used in dynamic systems. The term "fuzzy" or ambiguous refers to the fact that the logic governing the system can deal with concepts that cannot be expressed as "true" or "false" but rather as "true to some extent". Although other approaches such as genetic algorithms and neural networks can function in many cases similar to fuzzy logic, the unique strength of fuzzy logic lies in presenting and implementing problem solutions in a way comprehensible to human operators. This allows utilizing their experience for design without having a precise mathematical model for the system. Such adaptability makes automating tasks previously successfully performed by humans easier (Buckley and Eslami 2002). The general structure of the type-1 fuzzy system is illustrated in Fig. 3.

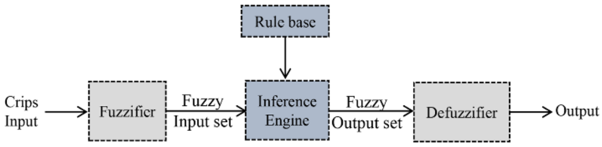

Figure 3. Schematic type-1 fuzzy logic system

As depicted in Fig. 3, in the fuzzification stage, numerical input values are initially transformed into linguistic fuzzy values through the assignment of membership functions to each input and output. The fuzzy rule stage establishes relationships between input and output values through a set of if-then fuzzy rules. Subsequently, the fuzzy inference engine evaluates these fuzzy rules to generate an output based on the input values. In the defuzzification stage, the linguistic fuzzy output is then converted into a numerical output value (Buckley and Eslami 2002). Moreover, the percentage error parameter serves as the input, and the value of the damping factor used in the LM method acts as the output in the fuzzy method. The percentage error parameter is calculated as

$$\text{Percent Error} = \frac{S(\mathbf{P}^{k+1}) - S(\mathbf{P}^{k})}{S(\mathbf{P}^{k})} \times 100 \tag{18}$$

The rules used in the fuzzy method are considered as follows (Sajedi et al. 2021)

1. If Percent Error is NLR then μ is VS
2. If Percent Error is NL then μ is S
3. If Percent Error is NM then μ is SS
4. If Percent Error is NS then μ is LM
5. If Percent Error is Z then μ is M
6. If Percent Error is PS then μ is UM
7. If Percent Error is PM then μ is SL
8. If Percent Error is PL then μ is L
9. If Percent Error is PLR then μ is VL

where NLR stands for negative larger, NL for negative large, NM for negative medium, NS for negative small, Z for zero, PS for positive small, PM for positive medium, PL for positive large, and PLR for positive larger. Additionally, VS denotes very small, S small, SS slightly small, LM lesser medium, M medium, UM more medium, SL slightly large, L large, and VL very large. The problem-solving algorithm using type-1 fuzzy LM method is fully presented in Fig. 4. The membership functions for input and output in the type-1 fuzzy process are illustrated in Figs. 5 and 6, respectively. The fuzzy system components include Mamdani inference engine, singleton fuzzifier, and center average defuzzifier (Buckley and Eslami 2002):

$$\mu^k = \frac{\sum_{l=1}^{M} \bar{y}^l [\prod_{i=1}^{n} \mu_{A_i} l(x_i)]}{\sum_{l=1}^{M} [\prod_{i=1}^{n} \mu_{A_i} l(x_i)]} \tag{19}$$

where $\mu^k \subset \mathbb{R}$ is the damping parameter in step k, $x \subset \mathbb{R}^n$, l is the number of rules. In Eq. (12), n represents the number of inputs, $\bar{y}^l$ the center of the fuzzy system output membership function corresponding to the l^{th} rule, and μ_{A_i} the membership function corresponding to the l^{th} rule correspond to the i^{th} input, respectively.

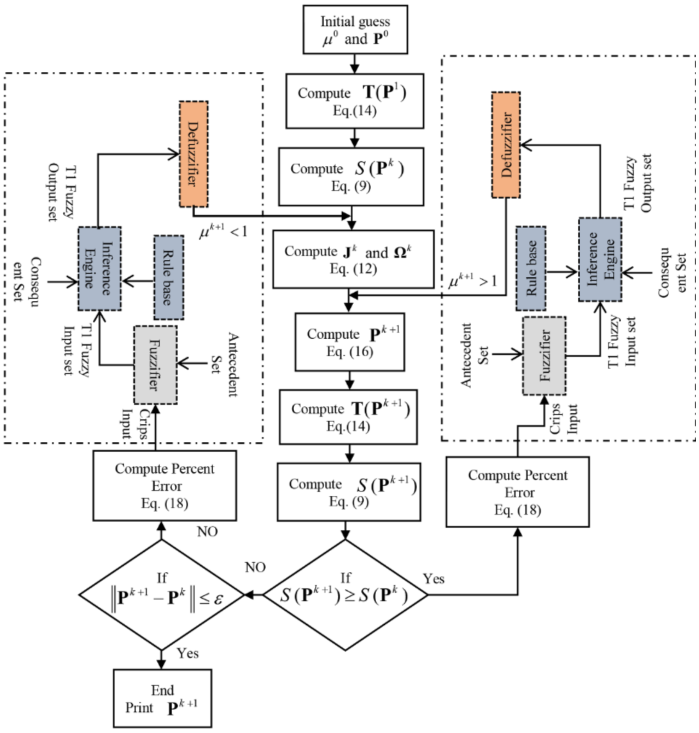

Figure 4. Flowchart of the type-1 fuzzy approach in lm algorithm

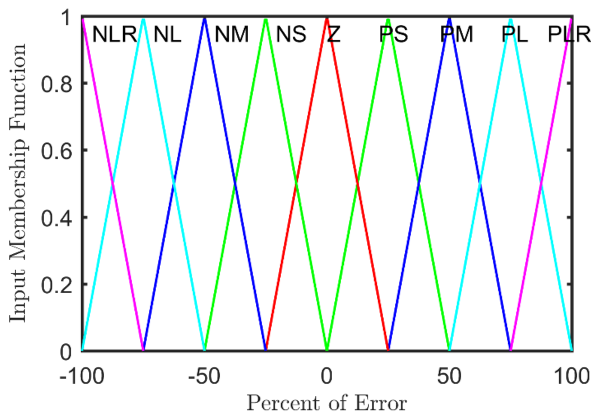

Figure 5. Input parameter’s membership function (percent error) for type-1 fuzzy method

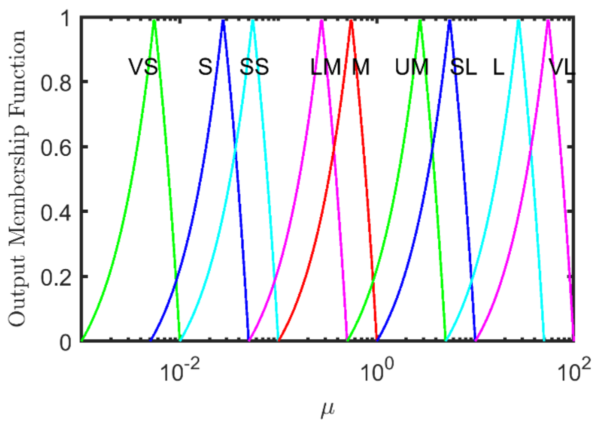

Figure 6. Output parameter’s membership function (damping factor) for type-1 fuzzy method

5. TYPE-2 FUZZY LOGIC

Fuzzy logic Type-2 was earnestly introduced and applied after the publication of a comprehensive book on the subject of intersection, union, and complementation of Type-2 fuzzy systems, as well as the presentation of a comprehensive algorithm for output calculation and defuzzification (Buckley and Eslami 2002). Subsequently, comprehensive information for the computation of Type-2 fuzzy systems, including reduction order relationships, defuzzification, and Type-2 fuzzy sets, was provided (Buckley and Eslami 2002; Mendel et al. 2006; Mendel and Liu 2013). Various models of Type-2 fuzzy systems have been proposed, all sharing the commonality of Type-1 fuzzy membership degree. Essentially, Type-2 fuzzy systems serve as a method to enhance the ability to deal with imprecise information in a sound and logical manner. Therefore, in this paper, as mentioned before, Type-2 fuzzy systems have also been utilized considering having noisy data. In Fig. 7, the structure of a Type-2 fuzzy system, along with the various steps of its execution and the sequence of performing them, is illustrated. By comparing Figs. 3 and 7, it is evident that there is a significant resemblance between Type-1 and Type-2 fuzzy systems in terms of their execution steps, and to some extent, their governing logic is similar. However, in the Type-2 fuzzy system, an extra step involving order reduction needs to be carried out, in addition to the Type-1 fuzzy system. For a more in-depth exploration of Type-2 fuzzy systems, please see Refs. (Mendel et al. 2006; Mendel and Liu 2013). Figure 8 depicts the flowchart illustrating the LM method with Type-2 fuzzy. The membership functions for input and output in the type-2 fuzzy process are illustrated in Figs. 9 and 10, respectively.

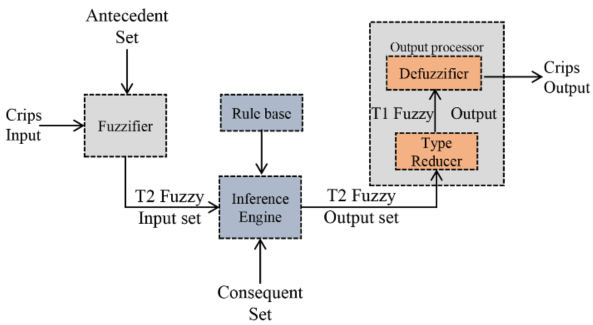

Figure 7. Schematic type-2 fuzzy logic system

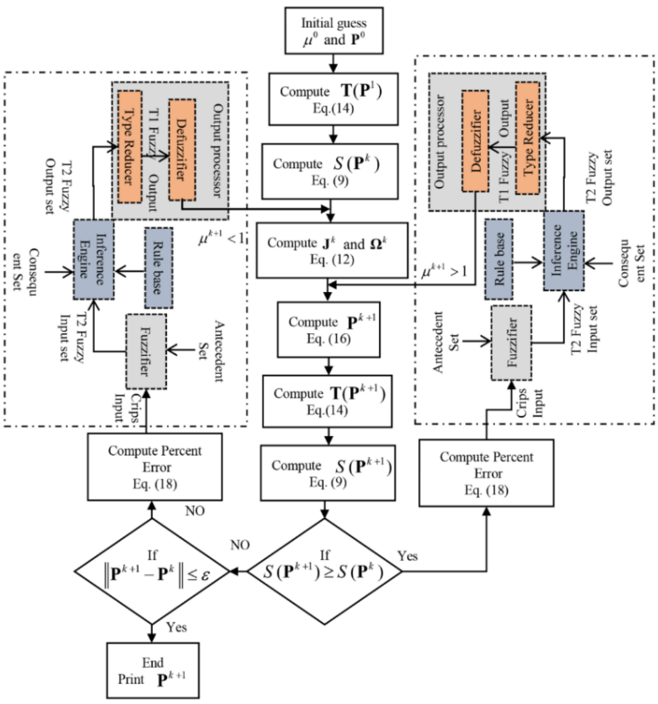

Figure 8. Flowchart of the type-2 fuzzy approach in lm algorithm

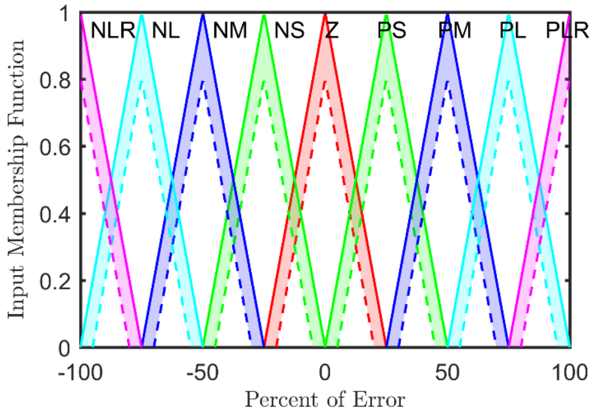

Figure 9. Input parameter's membership function (percent error) for type-2 fuzzy method

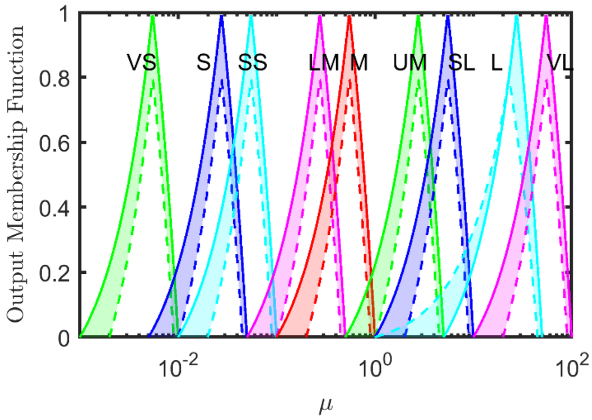

Figure 10. Output parameter's membership function (damping factor) for type-2 fuzzy method

6. RESULTS AND DISCUSSIONS

This paper addresses the solution to the 1-D transient inverse heat conduction problem using the LM method. Three different approaches have been employed to determine the damping factor. In the course of solving this problem, the values for damping factor have been initially determined using the classical LM method. Subsequently, the coefficient has been reevaluated twice, leveraging both the expert knowledge and its incorporation into the Type-1 and Type-2 fuzzy logic methods. The one-dimensional transient inverse heat conduction problem is discussed through numerical experiments outlined in Section 2. Each simulation involves $\lambda = 40W/(mK)$, $\rho c_p = 4\times 10^6$, $L = 0.025m$, T_0=20°C, $\Delta t = 0.03s$, $\Delta x = 1.3\times 10^{-3}$, and $K = 500$ time steps. Additionally, the sensor location is assumed to be $x_{masu} = 11.7\times 10^{-3}m$. A third-order polynomial in the form of $q(t) = 10^3 t^3$ has been employed as a trial function for the heat flux imposed on the left surface. The parameters considered for evaluating the performance of the three algorithms are the estimation speed and the robustness of the solution. Note that no experimental measurement or data collection has taken place in this article. The temperatures, $Y(t_i)$, have been obtained through the numerical solution of the transient one-dimensional heat conduction problem. Figures 11 to 14 showcase the historical plots of the noisy temperature and the estimated surface heat flux obtained from fuzzy methods and the classical method. The mean value is set to zero, and the standard deviation is unity within the [−2.576 2.576] interval, aiming for a 99% confidence coefficient. In all cases, the noise value adheres to the following equation, where η is a random number between –1 and 1.

$$Y_{noisy} = Y_{exact} \times (1 + \zeta\eta / 2.576) \tag{20}$$

where ζ is a constant value coefficient, which determines the intensity of the noise. Comparing the convergence rates for estimating the parameters P_j indicates that the time needed to estimate the four required parameters for damping factor using type-2 fuzzy method takes 38 times less than the classical LM method. However, there is no significant difference between the use of type-2 and type-1 fuzzy algorithms.

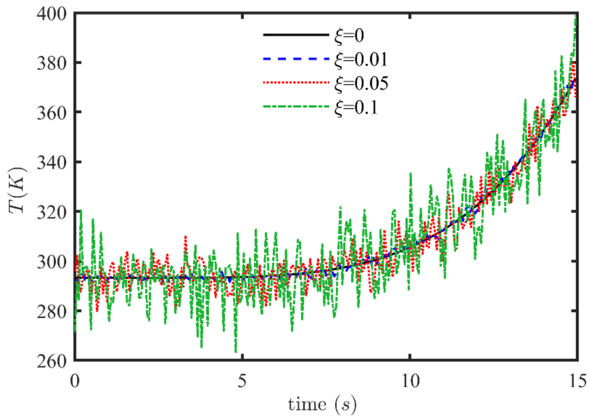

Figure 11. Temperature obtained through numerical solution for a 1-D transient heat conduction problem assuming a cubic polynomial for heat flux

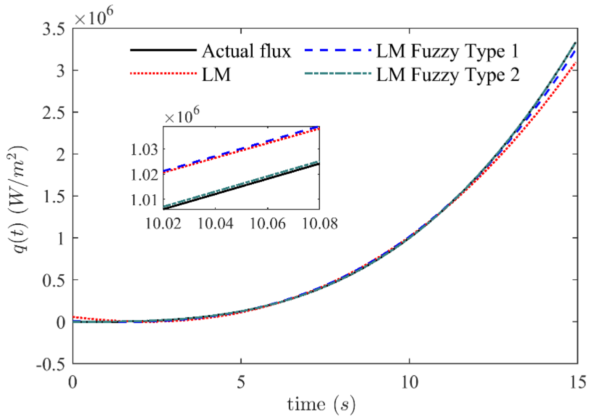

Figure 12. Surface heat flux estimation using the classical and fuzzy logic lm methods, $\zeta = 0.01$

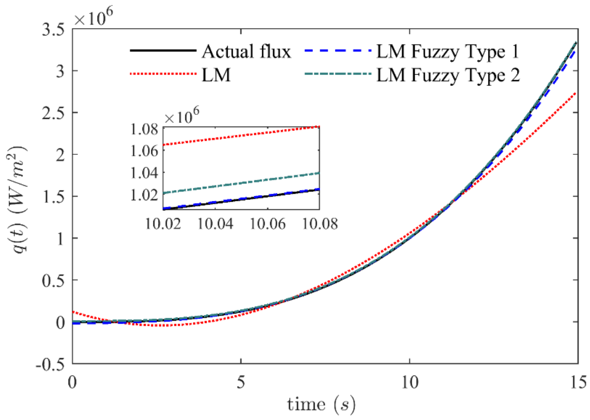

Figure 13. Surface heat flux estimation using the classical and fuzzy logic lm methods, $\zeta = 0.05$

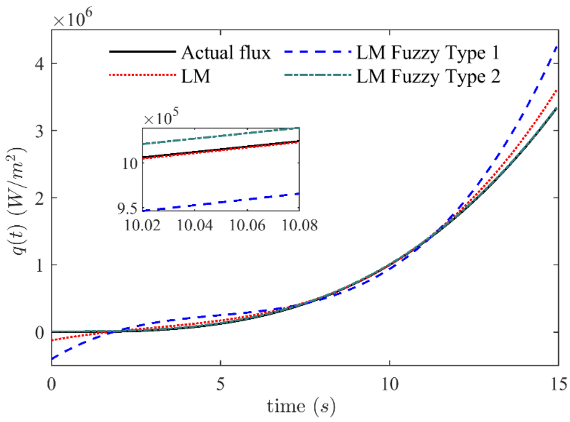

Figure 14. Surface heat flux estimation using the classical and fuzzy logic lm methods, $\zeta = 0.1$

The robustness of the utilized method against measurement noise is investigated in this section. Accordingly, based on Eq. (20), noisy data have been generated by solving the transient one-dimensional heat conduction equation and adding three different levels of noise to it. As illustrated in Fig. 11, to generate this noisy data, values of $\zeta = 0.01, 0.05, 0.1$ were applied in Eq. (20). Also, the initial guess of 0.001 has been assumed for data generation. The increase in noise is clearly discernible with the increase in ζ values. For instance, the temperature sensor data are noise-free in the case of $\zeta = 0$, while they are heavily affected by noise in the case of $\zeta = 0.1$. Figure 12 depicts the estimated surface heat flux using type-1 and type-2 fuzzy methods and the classical method for a cubic polynomial function under the influence of noise with $\zeta = 0.01$. Figures 13 and 14 also present the estimated surface heat flux for $\zeta = 0.05$ and $\zeta = 0.1$. These plots demonstrate the robustness of the solution to the presence of noise in temperature sensor data. According to the presented results, even though the use of fuzzy methods leads to relative improvement and better alignment of estimates with accurate values, the significant impact arising from the use of type-1 or type-2 fuzzy methods is not apparent, as the classical LM method provides accurate estimates. The type-2 fuzzy method outperforms the type-1 fuzzy method in estimating heat flux with superior precision. Particularly, under the highest noise intensity, as shown in Fig. 14, the type-2 fuzzy method excels in providing accurate estimates, while the other methods struggle to estimate surface heat flux accurately. In other words, the fuzzy approach enhances the LM method's robustness to sensor noise, resulting in more accurate estimates. To quantify the presented observations and conduct a more detailed examination and comparison of the results in the plots, Table 1 provides the root mean square error (ERMS) for various noise intensities. According to this table, the most significant positive impact of using the type-2 fuzzy method occurs at $\zeta = 0.1$. This results in a reduction of errors by more than 50%. While the impact of using type-1 and type-2 fuzzy methods varies across the three scenarios, overall, employing the LM method with damping factor values determined using type-2 fuzzy rules demonstrates greater resistance to noise and, simultaneously, faster performance compared to the classical and type-1 fuzzy methods.

Table 1. Root mean square error in heat flux estimation

Method	Error in estimation for different noise levels			
	$\zeta = 0$	$\zeta = 0.01$	$\zeta = 0.05$	$\zeta = 0.1$
Classical LM	7.2×10^{-7}	1773	4864	20029
Type-1 FL LM	6.6×10^{-7}	1792	3459	16466
Type-2 FL LM	2.6×10^{-8}	1636	2096	9658

7. CONCLUSION

In this study, three methods are employed to determine the damping factor in the Levenberg-Marquardt method used in solving inverse heat transfer problems. The classical method proposed by Marquardt, and a new method based on type-1 and type-2 fuzzy logic theories, whose principles are explained in the paper. With the aim of reducing estimation time in the LM method, the newly introduced method updates the damping factor at each iteration. To assess the effectiveness of the new method, surface heat flux in a one-dimensional heat transfer problem is estimated using a third-degree polynomial test function with three methods: classical LM, type-1 fuzzy, and type-2 fuzzy. In addition to the convergence speed, the robustness of heat flux estimation against noisy data is investigated. According to the results, the robustness of the solution to noisy data using the type-2 fuzzy method has increased compared to the other two methods, and the estimation speed of this method is higher than classical LM and on par with type-1 fuzzy. Therefore, it can be concluded that the use of the new method based on type-2 fuzzy logic for determining the damping factor in the LM method improves its performance, with very accurate and reliable results.

8. REFERENCES

Buckley, J.J., and Eslami, E. 2002. *An introduction to fuzzy logic and fuzzy sets*. (Springer Science & Business Media).

Chen, H., Cao, D., Wang, G., Wan, S., and Li, Y., 2016. "Fuzzy estimation for unknown boundary shape of fluid-solid conjugate heat transfer problem". *International Journal of Thermal Sciences*,Vol. 106, pp. 112-121.

Chen, T.-c., Han, D.-j., Au, F.T., and Tham, L. 2003. "Acceleration of Levenberg-Marquardt training of neural networks with variable decay rate." In *Proceedings of the International Joint Conference on Neural Networks*, 1873-1878. IEEE.

Cui, M., Zhao, Y., Xu, B., and Gao, X.-w., 2017. "A new approach for determining damping factors in Levenberg-Marquardt algorithm for solving an inverse heat conduction problem". *International Journal of Heat and Mass Transfer*,Vol. 107, pp. 747-754.

Duda, P., 2016. "A general method for solving transient multidimensional inverse heat transfer problems". *International Journal of Heat and Mass Transfer*,Vol. 93, pp. 665-673.

Keighobadi, J., Hosseini-Pishrobat, M., Faraji, J., and Langehbiz, M.N., 2020. "Design and Experimental Evaluation of Immersion and Invariance Observer for Low-Cost Attitude-Heading Reference System". *IEEE Transactions on Industrial Electronics*,Vol. 67, No. 9, pp. 7871-7878.

Kind, N., Berthel, B., Fouvry, S., Poupon, C., and Jaubert, O., 2016. "Plasma-sprayed coatings: Identification of plastic properties using macro-indentation and an inverse Levenberg–Marquardt method". *Mechanics of Materials*,Vol. 98, pp. 22-35.

Koh, J.M., and Cheong, K.H., 2018. "Automated electron-optical system optimization through switching Levenberg–Marquardt algorithms". *Journal of Electron Spectroscopy and Related Phenomena*,Vol. 227, pp. 31-39.

Kwak, Y.-t., Hwang, J.-w., and Yoo, C.-j., 2011. "A new damping strategy of Levenberg-Marquardt algorithm for multilayer perceptrons". *Neural Network World*,Vol. 21, No. 4, pp. 327.

Lampton, M., 1997. "Damping–undamping strategies for the Levenberg–Marquardt nonlinear least-squares method". *Computers in Physics*,Vol. 11, No. 1, pp. 110-115.

Lau, H.C., Nakandala, D., and Zhao, L., 2015. "Development of a hybrid fuzzy genetic algorithm model for solving transportation scheduling problem". *JISTEM-Journal of Information Systems and Technology Management*,Vol. 12, pp. 505-524.

Lu, T., Han, W., Jiang, P., Zhu, Y., Wu, J., and Liu, C., 2015. "A two-dimensional inverse heat conduction problem for simultaneous estimation of heat convection coefficient, fluid temperature and wall temperature on the inner wall of a pipeline". *Progress in Nuclear Energy*,Vol. 81, pp. 161-168.

Madsen, K., Nielsen, H.B., and Tingleff, O., 2004. "Methods for non-linear least squares problems".

Mendel, J.M., John, R.I., and Liu, F., 2006. "Interval type-2 fuzzy logic systems made simple". *IEEE transactions on fuzzy systems*,Vol. 14, No. 6, pp. 808-821.

Mendel, J.M., and Liu, X., 2013. "Simplified interval type-2 fuzzy logic systems". *IEEE transactions on fuzzy systems*,Vol. 21, No. 6, pp. 1056-1069.

Ozisik, M.N. 2000. *Inverse heat transfer: fundamentals and applications*. (Crc Press).

Sajedi, R., Faraji, J., and Kowsary, F., 2021. "A new damping strategy of Levenberg-Marquardt algorithm with a fuzzy method for inverse heat transfer problem parameter estimation". *International Communications in Heat and Mass Transfer*,Vol. 126, pp. 105433.

Samadi, F., Woodbury, K. A., and Beck, J. V., "Evaluation of generalized polynomial function specification methods". International Journal of Heat and Mass Transfer, 2018, 122: pp. 1116-1127.

Samadi, F., Woodbury, K., and Kowsary, F. "Optimal combinations of Tikhonov regularization orders for IHCPs". International Journal of Thermal Sciences, 2021, 161, 106697.

Torabi, M., and Hosseini, M.-M., 2018. "A new descent algorithm using the three-step discretization method for solving unconstrained optimization problems". *Mathematics*,Vol. 6, No. 4, pp. 63.

Ukrainczyk, N., 2009. "Thermal diffusivity estimation using numerical inverse solution for 1D heat conduction". *International Journal of Heat and Mass Transfer*,Vol. 52, No. 25-26, pp. 5675-5681.

Wang, G., Wan, S., Chen, H., Lv, C., and Zhang, D., 2017. "A double decentralized fuzzy inference method for estimating the time and space-dependent thermal boundary condition". *International Journal of Heat and Mass Transfer*,Vol. 109, pp. 302-311.

Woodbury, K.A., Najafi, H., De Monte, F., and Beck, J.V. 2023. *Inverse Heat Conduction: Ill-Posed Problems*.

Zadeh, L.A., 1979. "Fuzzy sets and information granularity". *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers*, pp. 433-448.

9. RESPONSIBILITY NOTICE

The author(s) is (are) the only responsible for the material included in this paper.