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CONVECTIVE AND ABSOLUTE INSTABILITY IN A POROUS MEDIUM WITH INCLUDED TEMPERATURE GRADIENT, HORIZONTAL AND VERTICAL THROUGHFLOW

Mateus Sanglard Schuabb Nunes

Leonardo Santos de Brito Alves

Laboratório de Reologia, Universidade Federal Fluminense, Rua Passo da Pátria, 156, Niterói, RJ, 24210-240 Brazil

mateusschuabb@id.uff.br

leonardo.alves@mec.uff.br

Abstract.

The theory of linear absolute and convective instabilities are applied in a model of convection in an extended horizontal layer of a saturated porous medium with inclined temperature gradient, horizontal and vertical throughflow. The critical value of vertical Rayleigh number, R_v , is obtained varying the values of horizontal Rayleigh number, R_h , and Péclet number, Q_v . The computations are performed by using the shooting method for both analyses, guaranteeing a minimum for the convective analysis and a saddle point for the absolute. For some pairs of values of R_h and Q_v in these analyses, there are two modes competing for which is the critical value, then, the marginal curve and the complex wave number plane are plotted to certify that the critical value is the correct. The results of both analyses are compared with the results found in the literature, which in the absolute analysis, there are some different results due this competition of modes. And furthermore, some pairs of values of R_h and Q_v are calculated when two pinching point occur simultaneously.

Keywords: Saturated porous layer, Darcy flow, Inclined temperature gradient, Vertical throughflow, Convective and absolute instabilities

1. INTRODUCTION

The problem of convection in a porous medium with horizontal as well as vertical gradient can be considered of a practical importance owing to a large variety of application in environmental, geophysics, industry and biology. Whether a convective flow is stable or not is also of a practical importance, because the rate of transfer of heat increases when the instability occurs.

The linear stability analysis of convection in a porous medium with inclined temperature gradient was first studied by Weber (1974), considering the ratio of height and length small. However, his analysis is limited to the case of small horizontal temperature gradient. This limitation was removed by Nield (1991) and Nield (1994), who employed a Galerkin approximation of second and eighth order, respectively, to solve the resulting differential equation system. In a later study, Nield (1998) extended this analysis to the case where vertical throughflow is present, and then, being of relevance to the performance of packed bed reactors. He computed the critical value with Galerkin approximation of 12-th order and after that, Brevdo and Ruderman (2009a) solved the same problem using a high-precision pseudo-spectral Chebyshev-collocation method. All these analyses found the imminence of convective instability, but in a second part of their study, Brevdo and Ruderman (2009b), for the first time, investigated the nature of instability and computed the critical values for the imminence of the absolute instability applying the Briggs (1964) collision criterion.

The present paper analyzes the imminence of both convective and absolute instabilities using the shooting method. This computation is used to find the critical values and to plot the marginal curves and the complex wavenumber plane to show the minimum and the pinching point, respectively. All results are compared with the results found by Brevdo and Ruderman (2009a) and Brevdo and Ruderman (2009b).

2. MATHEMATICAL FORMULATION

2.1 Governing equation and basic solution

The present problem consists in a horizontal extended saturated porous layer of height H bounded by two permeable surfaces such that the origin of the Cartesian coordinates are at the mid-height of the layer, the z -axis vertically upwards and the x -axis toward the right (opposite the direction of the applied horizontal temperature gradient). The vertical

throughflow velocity, the vertical temperature difference across the boundaries and the horizontal thermal gradient are denoted by w_v , ΔT and β , respectively.

Assuming that the Oberbeck-Boussinesq approximation is valid, and the flow in the porous medium is governed by Darcy's law, the governing dimensionless differential equations take the form as shown in Eq. (1)-(4).

$$\nabla \cdot \mathbf{v} = 0 \quad (1)$$

$$\nabla P + \mathbf{v} - T\mathbf{k} = 0 \quad (2)$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla^2 T \quad (3)$$

$$-\infty < x, y < \infty \text{ and } -1/2 < z < 1/2 \quad (4)$$

The boundary conditions are written as Eq. (5).

$$w = Q_v, \quad T = \mp R_v/2 - R_h x \quad \text{at } z = \pm H/2 \quad (5)$$

The non-dimensional parameters that arise in the dimensionless formulation are the vertical and horizontal Rayleigh number and the Péclet number, as can be shown in Eq. (6), respectively.

$$R_v = \frac{\rho_0 g \gamma K H \Delta T}{\mu \alpha_m}, \quad R_h = \frac{\rho_0 g \gamma K H^2 \beta}{\mu \alpha_m}, \quad Q_v = \frac{w_v H}{\alpha_m} \quad (6)$$

Equations (1)-(5) have a steady-state solution of the form

$$T_s = \tilde{T}(z) - R_h x, \quad u_s = U(z), \quad v_s = 0, \quad w_s = Q_v, \quad P_s = P(x, y, z) \quad (7)$$

The stationary solution which satisfies the zero net horizontal flow condition is expressed as

$$u_s(z) = R_h z \quad (8)$$

$$T_s(x, z) = \frac{R_h^2}{2Q_v} \left(z^2 - \frac{1}{4} \right) + \frac{R_h^2}{Q_v^2} z - \frac{Q_v^2 R_v + R_h^2}{2Q_v^2 \sinh(Q_v/2)} [e^{Q_v z} - \cosh(Q_v/2)] - R_h x \quad (9)$$

The base solution for the pressure can be obtained by integrating the Eq. (2).

2.2 Linear instability analysis

The approach of the linear instability analysis consists in adding, into the governing equations, a small perturbation in the steady-state solution, $\mathbf{v} = \mathbf{v}_s + \epsilon \mathbf{v}_p$, $T = T_s + \epsilon T_p$, $P = P_s + \epsilon P_p$, and collect the linear perturbation terms, obtaining the linearized perturbation equations, Eq. (10)-(12).

$$\nabla \cdot \mathbf{v}_p = 0 \quad (10)$$

$$\nabla P_p + \mathbf{v}_p - T_p \mathbf{k} = 0 \quad (11)$$

$$\frac{\partial T_p}{\partial t} + u_s \frac{\partial T_p}{\partial x} + Q_v \frac{\partial T_p}{\partial z} - R_h u_p + \frac{dT_s}{dz} w_p = \nabla^2 T_p \quad (12)$$

Considering the linear perturbation behaves as a wave, the velocities, pressure and temperature perturbations can be modeled as Fourier series, therefore, they can be written as normal modes, Eq (13).

$$[u_p, v_p, w_p, T_p, P_p] = [u_n(z), v_n(z), w_n(z), T_n(z), P_n(z)] \cdot e^{i(kx - \omega t)} \quad (13)$$

In the equation above, k and ω are, respectively, complex wavenumber and complex frequency. Substitution of Eq. (13) into the perturbation equations Eq. (10)-(12) and elimination of u_n , v_n and P_n from the resulting equations lead to the differential dispersion relations, $D(k, \omega, T_n(z), w_n(z), R_v, R_h, Q_v) = 0$,

$$D(k, \omega, T_n(z), w_n(z), R_v, R_h, Q_v) = \begin{cases} k^2 T_n + w_n'' - k^2 w_n; \\ k T_n'' - k Q_v T_n' - k(k^2 + ikR_h z - i\omega)T_n + iR_h w_n' - k \tilde{T}' w_n \end{cases} \quad (14)$$

In the boundaries, the vertical velocity and temperature perturbation vanish, then the boundary condition for the Eq.(14) become the Eq.(15).

$$w_n = T_n = 0 \quad \text{at } z = \pm 1/2 \quad (15)$$

3. NUMERICAL PROCEDURE

The computations to solve the differential dispersion relation are performed using the shooting method, which consists in transforming a boundary value problem in an initial value problem, which is solved through a marching and a root finding procedures.

In Eq. (14), k as well as ω have real and imaginary part, and will be represent by subscript R and i, respectively. If $\text{Im}[\omega] < 0$ or $\text{Im}[\omega] > 0$, the flow is said to be, respectively, stable or unstable regardless the value of the complex wavenumber, thus the value of interest of $\text{Im}[\omega]$ to imminence of both instabilities analysed in this paper is equal to zero, but the value of imaginary part of k is different in each analysis. The perturbation grows spatially when $-\text{Im}[k] > 0$, and it decreases when $-\text{Im}[k] < 0$, hence the transition from stable to convectively unstable occurs when $-\text{Im}[k] = 0$ and in the transition from convectively to absolutely unstable, $-\text{Im}[k] > 0$.

3.1 Convective analysis

The convective critical value can be obtained plotting marginal curves ($R_v \times k_R$) using the shooting method in the Eq.(14) and deriveting it to find a minimum. Instead of derivating the marginal curve, the procedure used in this paper is to derivate the dispersion relation with respect k_R (given that $k_i = 0$), considering the derivatives of the dimensionless parameters equal to zero and $\partial\omega/\partial k_R = \partial\omega/\partial k$ the group velocity and then solve Eq. (16) and (17) simultaneuos with Eq. (14) and (15).

$$\frac{dD}{dk} = D_k \left(k, \omega, \frac{\partial\omega}{\partial k}, T_n(z), \frac{\partial T_n(z)}{\partial k}, w_n(z), \frac{\partial w_n(z)}{\partial k}, R_v, R_h, Q_v \right) = 0, \quad k_i = 0 \quad \text{and} \quad \omega_i = 0 \quad (16)$$

$$\frac{\partial w_n(z)}{\partial k} = \frac{\partial T_n(z)}{\partial k} = 0 \quad \text{at} \quad z = \pm 1/2 \quad (17)$$

3.2 Absolute analysis

The physical difference between the convective and the absolute instability is that the perturbation propagates just in downstream direction in the convective instability, however, in the absolute instability, it propagates in both direction, downstream and upstream, and this imminence occurs when the group velocity is zero. To impose this condition, it's necessary derivate Eq. (14) again, however, the imaginary part of the complex wavenumber is not equal to zero and $\partial\omega/\partial k = 0$. Then, to find the absolute critical value, the Eq. (14) and (18) must be solve simultaneously, together with their boundary conditions, Eq. (15) and (17) . This method guarantees a saddle point, however, to be a pinching point, the collision criterion (Briggs, 1964) must be satisfied.

$$\frac{dD}{dk} = D_k \left(k, \omega, T_n(z), \frac{\partial T_n(z)}{\partial k}, w_n(z), \frac{\partial w_n(z)}{\partial k}, R_v, R_h, Q_v \right) = 0 \quad \text{and} \quad \omega_i = 0 \quad (18)$$

4. RESULTS

4.1 Convective

The Fig. 1 shows some neutral curves and points which are the critical values computed using the shooting method for $R_h = 20$ and different values of Q_v . It is easy to observe that the minimum of the neutral curves coincide with the black points.

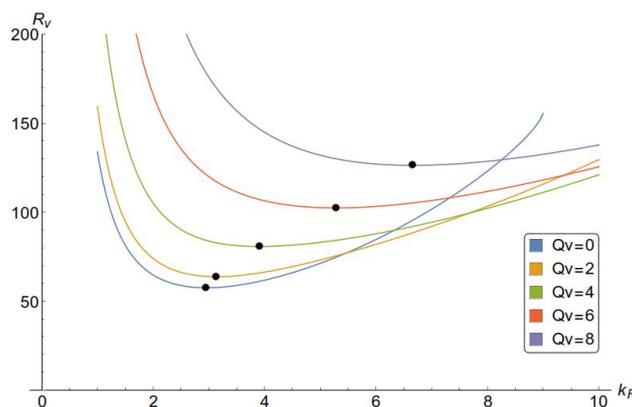


Figure 1. Netral curves and the critical value points for $R_h = 20$

The critical result found through the derivative of the neutral curves was used just to give a good estimative for the shooting method to find the critical value, i.e., just for $Q_v = 0$ this approach was used, and then a interpolation and extrapolation was used to estimate a good shoot for different values of Q_v and the same value for R_h .

In Table 1 - 3, the computation results for the critical values of vertical Rayleigh number, wavenumber and the real frequency for various values of the Péclet and horizontal Rayleigh number are shown. All these results were compared with Brevdo and Ruderman (2009a), and the greater relative error was about 0.036%

Table 1. Convective critical values of R_v for different values of R_h and Q_v .

	Rh=0	10	20	30	40	50	60
Qv=0	39.478	43.932	57.528	80.883	114.65	156.23	174.71
1	40.875	45.371	59.094	82.638	116.58	143.29	162.66
2	45.078	49.690	63.739	87.628	116.06	137.92	156.99
3	52.068	56.830	71.198	94.192	116.71	136.99	155.39
4	61.666	66.518	80.728	100.63	120.24	139.02	156.61
5	73.415	78.164	91.282	108.27	125.83	143.20	159.89
6	86.619	90.986	102.43	117.13	132.95	148.99	164.75
7	100.58	104.37	114.14	127.00	141.27	156.06	170.88
8	114.83	118.03	126.36	137.67	150.55	164.17	178.06

Table 2. Convective critical values of k_R for different values of R_h and Q_v .

	Rh=0	10	20	30	40	50	60
Qv=0	3.142	3.098	2.956	2.710	2.396	4.593	5.162
1	3.179	3.136	2.999	2.767	2.488	4.830	5.564
2	3.292	3.254	3.139	3.000	4.448	5.235	6.017
3	3.490	3.463	3.416	3.839	4.946	5.713	6.496
4	3.785	3.787	3.907	4.672	5.472	6.226	6.990
5	4.196	4.249	4.585	5.309	6.029	6.760	7.498
6	4.733	4.846	5.285	5.930	6.608	7.309	8.017
7	5.379	5.529	5.974	6.561	7.202	7.870	8.548
8	6.092	6.249	6.663	7.205	7.808	8.443	9.092

Table 3. Convective critical values of ω_R for different values of R_h and Q_v .

	Rh=0	10	20	30	40	50	60
Qv=0	0	0	0	0	0	43.21	75.42
1	0	0.3226	0.7638	1.414	2.379	50.91	86.53
2	0	0.7768	1.886	4.016	30.34	60.70	98.48
3	0	1.519	3.912	13.06	39.93	71.48	111.0
4	0	2.753	7.755	23.80	49.63	82.89	124.0
5	0	4.697	13.61	32.61	59.78	94.80	137.4
6	0	7.418	20.10	41.29	70.35	107.2	151.4
7	0	10.69	26.66	50.17	81.30	119.9	165.8
8	0	14.21	33.30	59.29	92.59	133.2	180.8

From the results presented in Tab.1, it can be observed the stabilization effect with increasing vertical throughflow, Q_v , for $R_h = 0, 10, 20$ and 30 , nevertheless, for $R_h = 40, 50$ and 60 this effect just occurs when $Q_v \geq 2, 3$ and 3 , respectively. Furthermore, this effect occurs with the increase of horizontal thermal gradient for all values of Q_v .

In Tab. 2, the critical real wavenumber is an increasing function of Q_v regardless the value of R_h . The results of Tab. 3 show that if $R_h = 0$ for all values of Q_v and $0 \leq R_h \leq 40$ for $Q_v = 0$, the transition to convective instability is non-oscillatory and for other pair of values, the frequency is an increasing of both R_h and Q_v .

4.2 Absolute

The results of the computation of imminence to absolute instability are present in Tab. 4-6. To obtain these results from Eq. (14) and (18), it is necessary a good initial shoot, therefore, the convective critical values were used. Comparing

the convective and absolute results, it can be observed that the nature of instability at the onset of convection is absolute for $R_h = 0$ for all values of Q_v and for $R_h \leq 40$ for $Q_v = 0$, and it is non-oscillatory whether the product $R_h Q_v$ is nil and, oscillatory if it is not. The stabilization effects occur increasing both R_h and Q_v for all values of vertical throughflow and $R_h \leq 40$, and for other values of R_h , the critical value R_v decreases for just few values of Q_v .

Table 4. Absolute critical values of R_v for different values of R_h and Q_v

	Rh=0	10	20	30	40	50	60
Qv=0	39.478	43.932	57.528	80.883	114.65	159.12	214.26
1	40.875	45.425	59.373	83.529	118.79	165.60	224.13
2	45.078	49.936	65.069	92.268	134.52	189.13	229.14
3	52.068	57.499	75.102	110.80	150.46	185.87	222.11
4	61.666	67.961	89.884	122.16	153.57	186.48	220.20
5	73.415	80.680	102.81	129.82	159.10	189.92	221.71
6	86.619	94.343	114.15	138.88	166.28	195.37	225.59
7	100.58	108.01	125.90	148.85	174.65	202.28	231.16
8	114.83	121.65	138.04	159.50	183.92	210.27	237.98

In Brevdo and Ruderman (2009b), this same analysis is done, however, using other methodology and a comparison with the critical values is done. Although the majority of the values match with a relative error around 0.008%, five of them have a meaningful difference and these pair of value (R_h , Q_v) are (40, 2), (50, 0), (50, 1), (60, 0) and (60, 1). In convective analysis, the smaller value is the critical, however, in the absolute analysis, not necessarily the smaller is the critical, it also has to show that the branches involve in the pinching point have to cross the imaginary complex wavenumber plane. Because of that, a careful analysis must be done to reveal what value is the critical, and plotting the complex wavenumber plane, it was observed that there modes competing to be the critical value.

All values found by Brevdo and Ruderman (2009b) and this analysis, the branches cross the complex wavenumber plane, therefore, the critical has the smaller value of R_v . Figure 2 shows just one of these cases, confirming that the value found using the shooting method procedure is correct.

Table 5. Absolute critical values of (k_R , $-k_i$) for different values of R_h and Q_v

	Rh=0	10	20	30	40	50	60
Qv=0	(3.142, 0)	(3.098, 0)	(2.956, 0)	(2.71, 0)	(2.39, 0)	(2.07, 0)	(1.801, 0)
1	(3.179, 0)	(3.134, 0.1129)	(2.987, 0.2338)	(2.73, 0.344)	(2.39, 0.404)	(2.069, 0.4147)	(1.799, 0.4056)
2	(3.292, 0)	(3.246, 0.2431)	(3.084, 0.5181)	(2.77, 0.790)	(2.39, 0.978)	(5.600, 4.497)	(6.311, 5.753)
3	(3.490, 0)	(3.446, 0.4120)	(3.267, 0.9424)	(2.825, 1.584)	(5.688, 3.934)	(6.345, 4.962)	(7.107, 5.997)
4	(3.785, 0)	(3.764, 0.6468)	(3.635, 1.813)	(5.474, 3.271)	(6.300, 4.212)	(7.080, 5.252)	(7.845, 6.270)
5	(4.196, 0)	(4.265, 0.9576)	(5.088, 2.384)	(6.111, 3.410)	(6.978, 4.445)	(7.798, 5.500)	(8.571, 6.533)
6	(4.733, 0)	(4.982, 1.232)	(5.870, 2.474)	(6.799, 3.573)	(7.674, 4.654)	(8.507, 5.727)	(9.290, 6.782)
7	(5.379, 0)	(5.753, 1.372)	(6.596, 2.539)	(7.496, 3.733)	(8.372, 4.845)	(9.213, 5.939)	(10.01, 7.014)
8	(6.092, 0)	(6.502, 1.451)	(7.311, 2.707)	(8.195, 3.881)	(9.070, 5.021)	(9.915, 6.136)	(10.72, 7.231)

Table 6. Absolute critical values of ω_R for different values of R_h and Q_v

	Rh=0	10	20	30	40	50	60
Qv=0	0	0	0	0	0	0	0
1	0	0.3235	0.7697	1.416	2.225	3.110	4.060
2	0	0.7847	1.946	3.937	7.607	75.81	120.1
3	0	1.555	4.238	11.61	53.76	92.41	140.4
4	0	2.876	9.609	33.90	66.60	108.8	160.1
5	0	5.056	18.27	43.82	79.54	125.0	179.6
6	0	8.150	25.39	53.73	92.48	141.1	199.0
7	0	11.66	32.28	63.62	105.4	157.1	218.3
8	0	15.27	39.15	73.54	118.4	173.2	237.6

Due the existence of two modes competing, there are values for R_h and Q_v which occur two pinching point at the same time. To find these values, a monitoring of saddle point of each mode varying Q_v was done, for then do an intersection of it. The results of this computations is shown in Tab. 7.

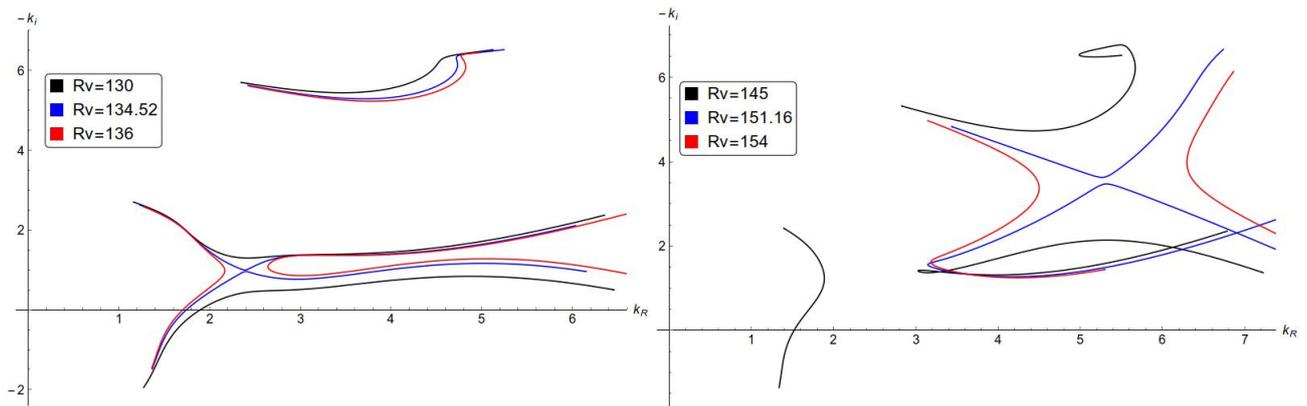


Figure 2. Netral curves and the critical value points for $R_h = 40$ and $Q_v = 2$

Table 7. Set of values when occur double pinching points

R_h	20	30	40	50	60
Q_v	4.200	3.286	2.467	1.839	1.382
R_v	93.170	118.25	150.22	190.08	237.38

5. CONCLUSION

In this paper, the two-dimension instabilities at the onset of convection in a saturated porous medium with inclined thermal gradient, horizontal and vertical throughflow were studied. The first analysis revealed the stabilization effect of the horizontal applied temperature gradient, R_h , at all values of the throughflow velocity, Q_v , and the same effect was observed to increasing Q_v for $R_h < 40$. In the second analysis, the critical value, R_v is an increasing function of both horizontal thermal gradient and throughflow velocity for values of horizontal Rayleigh number below 50, and the rate of spatial growth increase with increasing Q_v as well R_h . In both analyses, the frequency increases with the increasing of R_h and Q_v for both different to zero, and also, the instability is non-oscillatory when $R_h = 0$. Comparing the two analyses, it can conclude that the nature of the destabilization is absolute in the absence of horizontal thermal gradient and also in the absence vertical throughflow for values of $R_h < 50$.

The computations used in both analyses proved to be a good method to found critical values for the imminence of convective and absolute instability. A methodology using the convective critical values for a good initial shooting for the absolute analysis has proved to be capable of capturing critical value very well, until the values were not found in literature. And also, a monitoring due the existence of a competing of modes was done to capture values of Q_v and R_h for when two pinching point occur simultaneously.

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