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FREE VIBRATION ISOGEOMETRIC ANALYSIS OF CAD MAPPED TRUSSES

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Abstract. *With the advent of Isogeometric Analysis (IGA), a new concept of mapping was introduced. In this work the free vibration formulation applied for trusses analysis is presented. The efficiency of IGA is compared with classical Finite Element Method (FEM) and the Generalized Finite Element (GFEM) and the convergence is checked. The proposed method brings advantages related with implementing, by CAD mapping, and also by numerical accuracy.*

Keywords: *Isogeometric Analysis, Free Vibration, Trusses, FEM, GFEM*

1. INTRODUCTION

In the state of art of numerical methods in the last decade isogeometric analysis was introduced as an unification method between FEA (Finite Element Analysis) and CAD (Computer Aided Design) systems due the application of NURBS functions in numerical models Hughes *et al.* (2005). In the first developments performed by Cottrell *et al.* (2006) in structural dynamics AIG demonstrated advantages related with the high continuity property of the NURBS functions, which led to accurate frequency spectra when compared with finite element method (FEM). Extended studies considering the application for reissner-mindlin plates and shells Benson *et al.* (2010a); Thai *et al.* (2012), thin plates Shojaee *et al.* (2012) and framed structures Rauen *et al.* (2017), reinforces IGA advantages related with convergence and also implementation aspects.

Despite these applications have given robustness to IGA, mostly in 2D problems, some applications in 1D problems are still open issues. This work presents a isogeometric formulation of bars in a global mapping context which could be applied for trusses and other curved elements.

2. ISOGEOMETRIC ANALYSIS

Isogeometric analysis (IGA) is discrete numerical method to solve partial differential equations where the domain analysis is exactly described by NURBS functions, from CAD classical literature Farin (1999); Piegel and Tiller (1997). Like FEM, IGA is also originated by an weighted formulation and discretization. The most remarkable difference between IGA and FEM is in the discretization strategy Hughes *et al.* (2005). The usual implementation of FEM is to divide the physical domain in a finite number of locally mapped isoparametric elements. Figure 1 shows the classical FEM mesh mapping strategy, with a local context of mapping and global node coupling. Isogeometric analysis adopts the global mapping concept where the whole parametric space is mapped to the physical domain. Figure 2 shows the basic IGA strategy. Is important to note that FEM performs the discretization process in the physical domain, while IGA performs the discretization in the parametric domain which allows the global mapping by NURBS exact describing geometry.

2.1 B-spline shape functions

The fundamental functions to create the solution and geometric space in isogeometric analysis are the b-splines shape functions. In parametric space, the subdivision of elements is performed by the knots concept. These knots consists in a set of coordinates in the parametric space which holds the number of functions and its continuity. The b-splines shape functions is a set of parametric polynomials defined by a polynomial degree p and a set of knots $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ where knots can appear repeatedly but in an ascending order, and n is the number of b-splines shape functions. The shape functions are recursively defined by De-Boor (1972):

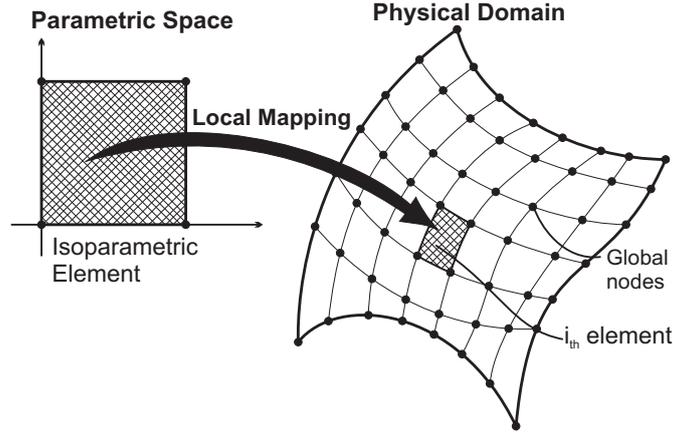


Figure 1. Classical mapping scheme in FEM

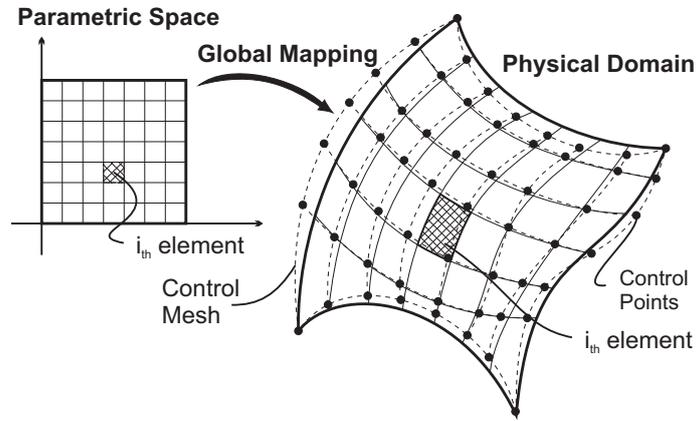


Figure 2. IGA mapping strategy

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise} \end{cases}, \quad (1)$$

starting from an null degree iteration. The iteration for p_{th} degree is given by:

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi). \quad (2)$$

The b-splines shape functions has some important properties. Namely:

- 1) The continuity of the b-splines set in the knot ξ_i are related with its multiplicity in Ξ , i.e the functions have $p - m_i$ continuous derivatives at the knot ξ_i , where m_i is the knot multiplicity;
- 2) The support of a NURBS shape function $N_{i,p}$ is defined in the interval $[\xi_i, \xi_{i+p+1}]$;
- 3) The b-splines shape functions follows the partition of unity in parametric space, i.e $\sum_{i=1}^n N_{i,p}(\xi) = 1$.

The usual knot configuration in isogeometric analysis is to use the open concept Farin (1999), i.e $p + 1$ edge knots. Interior knots also controls the continuity, being that ξ_i repeated p times created an C^0 continuity in that point.

2.2 Curve representation and NURBS functions

A curve defined in a physical domain is mapped by a set o b-splines shape functions by the relation:

$$\mathbf{C}(\xi) = \sum_{i=1}^n N_{i,p} \mathbf{B}_i, \quad (3)$$

where \mathbf{B}_i is the i_{th} control point set which defines the curve. This equation describes the well known *Bézier* curves.

Non uniform rational b-splines (NURBS) functions are originated from b-splines, to increase CAD representation of objects. The control points, used to map a curve in a physical domain appear weighted, being possible to represent a more complex curve than a *beziér* curve Piegl and Tiller (1997). The NURBS functions are defined as a weighted rational set $R_{i,p}$, based in a b-spline set $N_{i,p}$. Given a set of n control points \mathbf{B} and a set of n weights w , NURBS set is the defined as:

$$R_{i,p} = \frac{N_{i,p} w_i}{\sum_{i=1}^n N_{i,p} w_i}. \quad (4)$$

A curve originated by a NURBS set is defined by:

$$\mathbf{C}(\xi) = \sum_{i=1}^n R_{i,p} \mathbf{B}_i. \quad (5)$$

The insertion of weights and the rational concept implied a more flexible possibility to perform curves construction, being possible to control curvature radius, which comes from the projective concept given by weights Farin (1999). From a practical view of implementation in isogeometric analysis (fig. 3), the construction of a NURBS curve can be though as a transformation which maps a point in the parametric space ξ directly to the physical space $\mathbf{C} = (x, y)$.

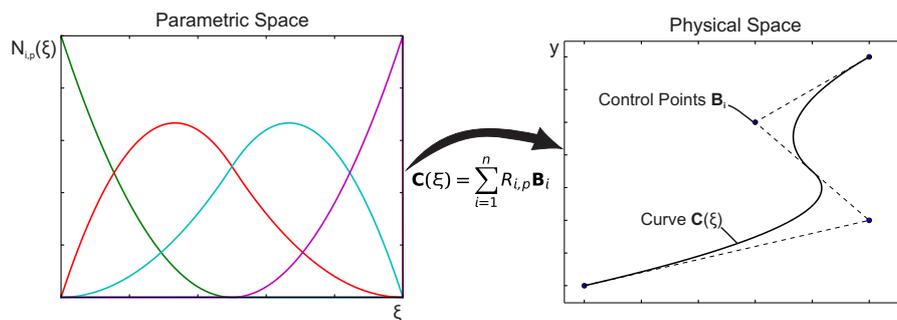


Figure 3. Basic NURBS curve mapping scheme

2.3 Refinements in IGA and CAD

Refinement procedures in IGA are related with the modifications of the parameters Ξ (knot vector) and p (polynomial degree). In CAD procedures, several refinement techniques are discussed and exemplified by Piegl and Tiller (1997) aiming to locate control points and weights properly by impose equality in curve equations. These algorithms implies in automatic mesh refinement.

Considering NURBS curve refinement in isogeometric analysis, three kind of basic refinement procedures are possible to perform (Cottrell *et al.*, 2007), namely: h refinement, p refinement and k refinement. Like FEM, IGA h refinement is related with point increasing without change polynomial degree. In IGA this means to add new knots in the knot vector and recompute the functions, by the recursive formula (eqs. ??) and the points and weights by the algorithms given by Piegl and Tiller (1997). This results in a higher amount of b-splines shape functions, control points, weights and degrees of freedom in the system. Refinements k and p are related with knot vector changing and also polynomial degree increasing. The basic difference between these refinements techniques is that p refinement performs the polynomial degree increasing aiming to maintain b-spline shape functions continuity at certain knots, a necessary procedure to maintain some kind of geometries like squares and cusps. An detailed guide about geometry and mesh procedures in NURBS construction is given by Farin (2002). Refinement k is related with degree elevation followed by knot insertion, which generates higher continuity functions at the knots. Several works present the advantage of k refinement over p and h refinements in terms of convergence, due mostly to its continuity growing (property 1 of eqs. 1 and 2) with fewer degrees of freedom than p refinement (Hughes *et al.*, 2005; Cottrell *et al.*, 2007). In structural dynamics the works of Cottrell *et al.* (2006) and recently Rauén *et al.* (2017) reinforce this statement.

3. DYNAMIC RESPONSE OF CAD MAPPED BARS

The differential equation which describes the equilibrium of an undamped system in a global domain C comes from dynamic equilibrium imposing in a straight differential (Petyt, 2010), as shown in figure 4 which represents the free body

forces in an infinitesimal element of curve C . Under a time dependent force $p(C, t)$, the structure response in terms of axial forces N is described in terms of displacements \bar{u} considering a linear elastic condition. The inertial mass force $f_i(C, t)$ allows dynamical equilibrium to the system.

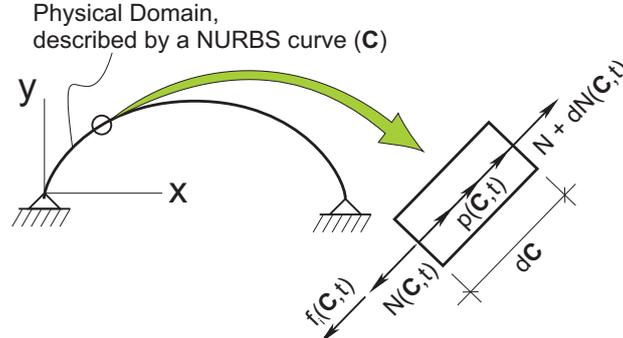


Figure 4. Free body forces for global equilibrium imposing in a curve differential dC

Imposing dynamical equilibrium in the axis direction in dC and considering the linear expression for axial load $N = AE\epsilon$, where ϵ is the axial strain, the dynamic equilibrium differential equation can be written as:

$$\rho A \frac{\partial^2 \bar{u}}{\partial t^2} - \frac{\partial}{\partial C} \left(EA \frac{\partial \bar{u}}{\partial C} \right) = p(C, t), \quad (6)$$

where ρ is the specific mass, A is the cross sectional area, E the Young Modulus and t the time variable.

Like FEM, IGA expressions are obtained from a variational formulation with same procedures as FEM. Details of variational manipulation for FEM in structural dynamics equations are given by Carey and Oden (1984). Applying the projection of a weight function and the minimization of the residual functional, the general expression for the variational form in a Ω domain of eq. 6 is given by:

$$\int_{\Omega} \rho A \frac{\partial^2 \bar{u}}{\partial t^2} w \, d\Omega - \int_{\Omega} \frac{\partial}{\partial C} \left(EA \frac{\partial \bar{u}}{\partial C} \right) w \, d\Omega = \int_{\Omega} p(C, t) w \, d\Omega, \quad (7)$$

The stiffness term is integrated by parts and the general solution is imposed in the form (Carey and Oden, 1984; Petyt, 2010):

$$\bar{u}(C, t) = e^{i\omega t} u(C). \quad (8)$$

The equation leads 7:

$$\int_{\Omega} \rho A u w \, d\Omega - \int_{\Omega} EA \frac{\partial w}{\partial C} \frac{\partial u}{\partial C} \, d\Omega = \int_{\Omega} p(C, t) w \, d\Omega. \quad (9)$$

The dynamical equilibrium equation described in eq. 9 shows respectively the mass, the stiffness and the force terms. For the application of IGA in the equilibrium sentence (eq. 9) the discretization with Galerkin method is considered by write an approximate \tilde{u} field described by a linear combination of NURBS shape functions $R_{i,p}$. Note that, as discussed in the section above, the discretization process in isogeometric analysis is performed in the parametric domain. This leads two basic advantages, related with the possibility of direct global mapping and to describe any global variable, including material properties, in the parametric domain. The displacement field can also be locally described by

$$\tilde{u}(\xi) = \sum_{i=1}^n R_{i,p}(\xi) \tilde{u}_i. \quad (10)$$

Replacing the discretization process (eq. 10) in eq. 9, the mass, stiffness and force expressions are described in terms of NURBS functions, in a global domain:

$$\mathbf{M} = \int_C \rho A \mathbf{H}^T \mathbf{H} \, dC, \quad (11)$$

$$\mathbf{K} = \int_C EA \mathbf{B}^T \mathbf{B}^T dC, \quad (12)$$

$$\mathbf{f} = \int_C p \mathbf{H}^T dC, \quad (13)$$

where \mathbf{H} and \mathbf{B} are function vectors:

$$\mathbf{H} = [R_{1,p} \quad R_{2,p} \quad \dots \quad R_{n,p}] \quad (14)$$

$$\mathbf{B} = \left[\frac{\partial R_{1,p}}{\partial C} \quad \frac{\partial R_{2,p}}{\partial C} \quad \dots \quad \frac{\partial R_{n,p}}{\partial C} \right]. \quad (15)$$

The local context can be applied by write dC in function of $d\xi$. The slice dC of the curve C is written in terms of global coordinates x and y .

$$\frac{\partial C}{\partial \xi} = J(\xi) = \sqrt{\left(\frac{\partial X}{\partial \xi}\right)^2 + \left(\frac{\partial Y}{\partial \xi}\right)^2} \quad (16)$$

It's important to note that eqs. 11, 12 and 13 can be thought as line integrals, where dC is the mapping term. The local definition is applied by replace eq. 16 in the integrals and apply chain rule in the terms of vector \mathbf{B} .

Like FEM the standard procedure in IGA is to perform numerical integration by quadrature. As each quadrature point is applied in all matrix elements and the mapping term depends only on ξ . The local form can be written as:

$$\mathbf{M} = \int_0^1 \rho A \mathbf{H}^T \mathbf{H} J(\xi) d\xi, \quad (17)$$

$$\mathbf{K} = \int_0^1 EA \hat{\mathbf{B}}^T \hat{\mathbf{B}} \frac{1}{J(\xi)} d\xi, \quad (18)$$

$$\mathbf{f} = \int_0^1 p \mathbf{H}^T J(\xi) d\xi, \quad (19)$$

where the vector $\hat{\mathbf{B}}$ contains the local derivatives of $R_{i,p}$ given by:

$$\hat{\mathbf{B}} = \left[\frac{\partial R_{1,p}}{\partial \xi} \quad \frac{\partial R_{2,p}}{\partial \xi} \quad \dots \quad \frac{\partial R_{n,p}}{\partial \xi} \right]. \quad (20)$$

3.0.1 Decomposing on Degrees of Freedom

Vector decomposing, important for elementar operations such as element coupling and boundary conditions imposing, is performed by trigonometric relation, for a given i th displacement:

$$\tilde{u}_i = \cos(\alpha) \tilde{u}_{x_i} + \sin(\alpha) \tilde{u}_{y_i} = \frac{\partial X}{\partial C} \tilde{u}_{x_i} + \frac{\partial Y}{\partial C} \tilde{u}_{y_i}, \quad (21)$$

where α is the angle related with the global curve $C(\xi)$ first derivative, which varies in function of ξ .

For an easy implementation the decomposing process occurs during the integration process. For each quadrature point a linear transformation is applied in eqs. 17, 18, 19 aiming to create stiffness, mass and force terms considering decomposed degrees of freedom, given by:

$$\hat{\mathbf{K}} = \mathbf{T}^T \mathbf{K} \mathbf{T}, \quad (22)$$

$$\hat{\mathbf{M}} = \mathbf{T}^T \mathbf{M} \mathbf{T}, \quad (23)$$

$$\hat{\mathbf{f}} = \mathbf{T}^T \mathbf{f}, \quad (24)$$

where $\hat{\mathbf{K}}$, $\hat{\mathbf{M}}$ and $\hat{\mathbf{f}}$ is the decompsed stiffness, mass and force terms. A selective decomposing is considered in the construction of \mathbf{T} . Let \mathbf{r} be a set of integers which represents the index i of all selected displacements for decomposing and \mathbf{s} a set of integers in the form $\{1, 2, 3, \dots, n\}$. The transformation \mathbf{T} consists in a $(n + \bar{r}) \times (n)$ dimension matrix, with null dominance, where $\bar{r} = \dim(\mathbf{r})$. The non-zero values of \mathbf{T} , are given by:

$$T_{i,i} = \frac{\partial X}{\partial C}(\xi) \text{ and } T_{i+1,i} = \frac{\partial Y}{\partial C}(\xi) \text{ for } i \text{ in } \mathbf{r}, \quad (25)$$

and

$$T_{i,i} = 1 \text{ for } i \text{ in } \mathbf{s} - \mathbf{r}, \quad (26)$$

where the operation $\mathbf{s} - \mathbf{r}$ denotes all elements on \mathbf{s} which are not in \mathbf{r} .

4. NUMERICAL EXAMPLE

The example developed consists in a seven bar truss, firstly developed by Zeng (1998a). The material properties considered is $A = 0.001m^2$, $\rho = 8000kg/m^3$ and $E = 2.10^{11}N/m^2$. The geometric properties are shown in Fig. 5.

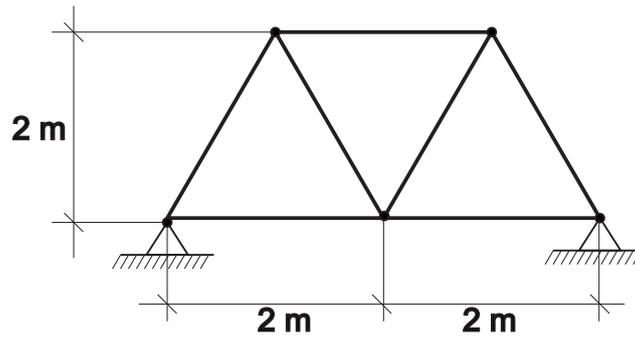


Figure 5. Truss geometry properties

The truss in isogeometric concept is mounted by two distinct elements. The common nodes are decomposed and its nodal contributions are coupled. Figure 6 shows the minimal geometry to describe the structure and the coupling scheme. As the system is formed by straight bars coupling, the minimal geometry can be simple described by NURBS with $p = 1$, which means a set of hat functions of FEM. However the main nodes must be defined with C^0 continuity aiming to set a geometric cusp, as illustrated in Fig. 7.

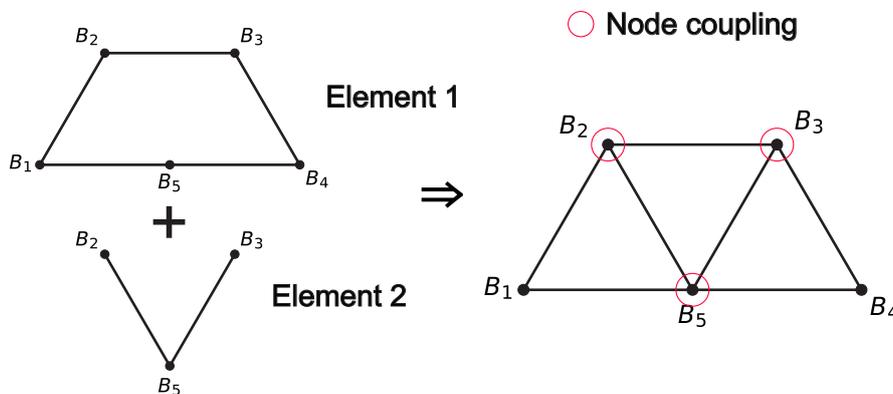


Figure 6. Truss mounting scheme

The truss is modeled with IGA with degrees $p = 3$ and $p = 4$ and the first 20 free vibration frequencies are compared with the results obtained by Arndt *et al.* (2010) for Generalized Finite Element Method (GFEM) and the Composite Element Method (CEM). Figure 8 shows the relative error in functions of the mode number.

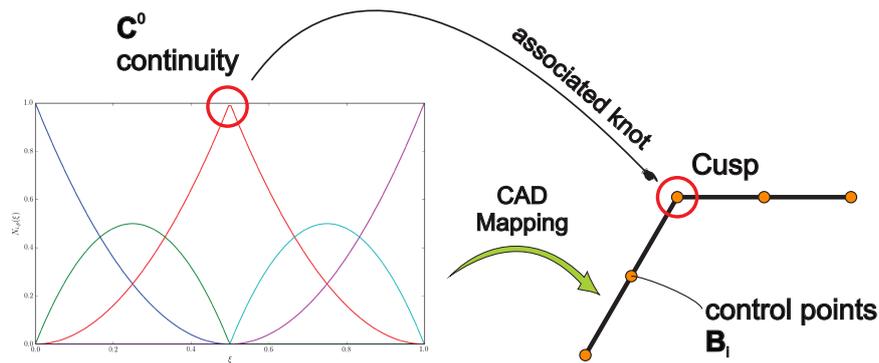


Figure 7. Geometric cusp set on NURBS

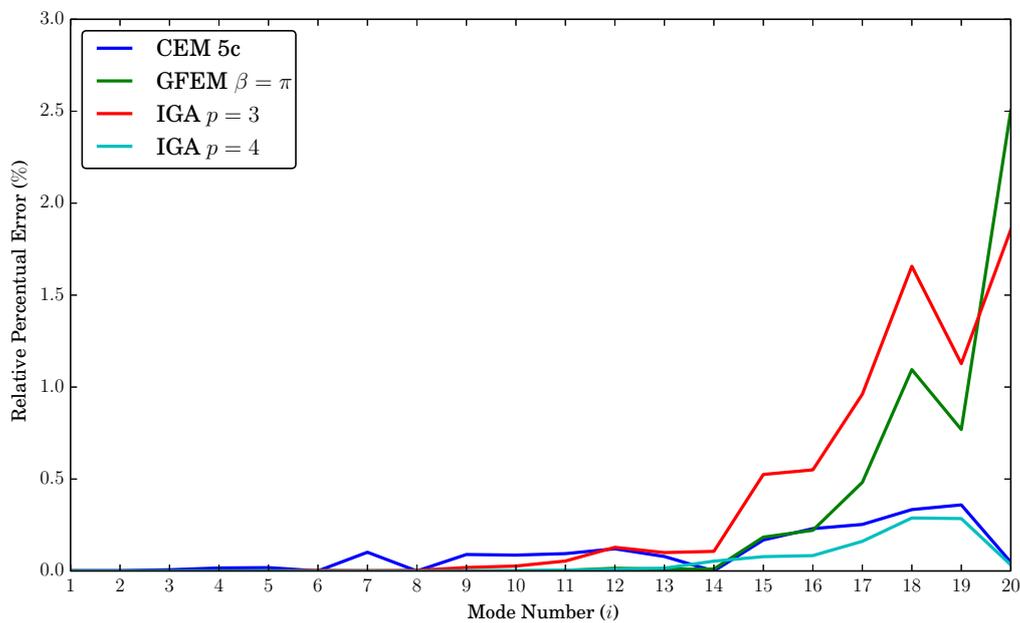


Figure 8. Relative error on free vibration frequencies

5. CONCLUSIONS

This paper presented a formulation of mapped bars applied to isogeometric analysis of trusses. By the numerical results, the isogeometric model with polynomial degree $p = 4$ presented the most accurate frequency spectrum and a good convergence capacity on k refinement. The method shown a capability of exactness on mapping, giving ease in unidimensional mesh representations.

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