24th ABCM International Congress of Mechanical Engineering
December 3-8, 2017, Curitiba, PR, Brazil

24th COBEM - 2017

COBEM-2017-0819

FLEXURAL WAVE PROPAGATION IN SLOWLY VARYING THIN PLATE STRIP USING A FINITE ELEMENT APPROACH

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Abstract. *This work investigates structural wave propagation in a thin plate strip with randomly varying properties along the axis of propagation, specifically when the properties vary slowly enough such that there is negligible backscattering, even if the net change is large. Wave-based methods are typically applied to homogeneous waveguides but the WKB (after Wentzel, Kramers and Brillouin) approximation can be used to find a suitable generalisation of the wave solution in terms of the change of phase and amplitude, but is restricted to analytical solutions of the equations of motion. A wave and finite element (WFE) approach is proposed to extend the applicability of the WKB method to cases where no analytical solution is available. The wavenumber is expressed as a function of the position along the waveguide and a Gauss-Legendre quadrature scheme is used to obtain the phase change while the wave amplitude is calculated using conservation of power. The WFE method is used to evaluate the wavenumbers at each integration point. Random field properties are expressed by a Karhunen-Loève (KL) expansion. Results are compared to a standard FE approach and to an available WKB analytical solution. They show good agreement and require only a few WFE evaluations, providing a suitable framework for spatially correlated randomness in waveguides*

Keywords: *Wave and finite elements, Karhunen-Loève expansion, uncertainty analysis, WKB approximation*

1. INTRODUCTION

Wave-based methods commonly assume that waveguide properties are homogeneous in the direction of the wave propagation, limiting the application of such approaches. This assumption arises mainly because analytical solutions for non-homogeneous waveguides are only possible for very particular cases, for example acoustic horns, ducts, rods and beams, e.g. (Eisenberger, 1991, Guo, and Yang, 2012, Lee, et al., 2007, Li, 2000). Moreover, randomly varying material and geometric properties along the axis of propagation play a significant role in the so-called mid-frequency region.

The wave and finite element (WFE) approach is a wave based method that is used to predict the wavenumbers and wave modes of a waveguide from a FE model, by post processing the mass and stiffness matrices, typically found using a finite element (FE) package. This is particularly useful when no analytical solution is available and conventional FE models of the whole structure became excessively large. This method has been applied to a number of cases in structural dynamics including free and forced vibration (Ichchou, et al., 2007, Mace, et al., 2005, Manconi, and Mace, 2009, Mencik, 2014, Mencik, and Ichchou, 2005, Renno, and Mace, 2010, 2013, Waki, et al., 2009a,b). Even though it can be used to model non-uniform cross-sections, this approach is however limited to homogeneous or piecewise constant waveguides in the direction of the travelling wave.

The classical WKB approximation is a method for finding suitable modifications of plane-wave solutions for non-homogeneous waveguides (Pierce, 1970). Named after Wentzel, Kramers and Brillouin, it was initially developed for solving the Schrödinger equation in quantum mechanics. The formulation assumes that the waveguide properties vary slowly enough such that there are no or negligible reflections due to these local changes, even if the net change is large, and can be extended to include spatially correlated random variability (Fabro, et al., 2015). It maintains the wave-like interpretation of non-uniform waveguides, but it is restricted to available analytical solutions.

In this work, a WFE approach is proposed to extend the applicability of the WKB method to cases where no analytical solution is available. The wave proprieties are calculated using the WFE approach and they are expressed as a function of the position along the waveguide. The phase change is calculated using a Gauss-Legendre quadrature scheme for numerical integration of the local wavenumber. The WFE method is used to evaluate the wavenumbers at each integration point, and these are kept to a minimum to reduce computation cost while being able to capture the non-homogeneity to a given accuracy. The wave amplitude change is calculated using conservation of power.

The numerical example of a plate strip with propagating and evanescent wave modes is considered with non-uniform material and geometrical properties. Two application examples are shown. One considers a deterministic linear variation of the thickness along the plate strip and the other considers the Young's modulus as non-Gaussian random field and it is expressed in terms of a Karhunen-Loève (KL) expansion along with an iterative scheme. Moreover, results show good agreement with full FE approach and require only a few WFE evaluations, providing a suitable framework to account for spatially correlated randomness in waveguides.

2. THE WKB APPROXIMATION

The WKB formulation has been applied in many fields of engineering, including acoustics (Arenas, and Crocker, 2001, Rienstra, 2003) and structural dynamics (Fabro, et al., 2015, Firouz-Abadi, et al., 2007, Pierce, 1970). However, the WKB approximation breaks down if the properties change rapidly or when the travelling wave reaches a local cut-off section where the wave mode ceases to propagate. This transition, also known as a turning point, leads to an internal reflection, breaking down the main assumption in the theory, requiring a different approximation for certain frequency bands (e.g. (Nayfeh, 1973)).

Assuming a time harmonic solution, $u(x,t) = U(x)\, e^{-i\omega t}$, it is possible to define a local wavenumber $k(x)$. Thus, the *eikonal* function $S(x) = \ln \tilde{U}(x) + i\theta(x)$ is introduced, in order to find wave solutions of the kind (Whitham, 1974)

$$U(x) = e^{S(x)} = \tilde{U}(x) e^{\pm i\theta(x)}. \tag{1}$$

It is possible to define positive $\mathbf{b}^+ = \mathbf{\Lambda}^+(x_a, x_b)\mathbf{a}^+$ and negative going $\mathbf{a}^- = \mathbf{\Lambda}^-(x_b, x_a)\mathbf{b}^-$ propagation matrices for a wave travelling between x_a and x_b. Forced response can be considered as in Fig. 1, where the total wave amplitudes are given at the excitation point by

$$\mathbf{c}^+ = \mathbf{e}^+ + \mathbf{b}^+ \text{ and } \mathbf{b}^- = \mathbf{e}^- + \mathbf{c}^-, \tag{2}$$

where $\mathbf{e}^+$ and $\mathbf{e}^-$ are the amplitude of the waves directly generated from the excitation that can be calculated from equilibrium and continuity conditions. Wave amplitudes at the boundaries are related by the reflection matrices as $\mathbf{a}^+ = \mathbf{\Gamma}_L\mathbf{a}^-$ and $\mathbf{d}^- = \mathbf{\Gamma}_R\mathbf{d}^+$. The traveling waves amplitudes are related by the propagation matrices as $\mathbf{b}^+ = \mathbf{\Lambda}^+(0, L_e)\mathbf{a}^+$, $\mathbf{d}^+ = \mathbf{\Lambda}^+(L_e, L)\mathbf{c}^+$, $\mathbf{a}^- = \mathbf{\Lambda}^-(L_e, 0)\mathbf{b}^-$, $\mathbf{c}^- = \mathbf{\Lambda}^-(L, L_e)\mathbf{d}^-$, $\mathbf{h}^+ = \mathbf{\Lambda}^+(L_e, L_r)\mathbf{c}^+$ and $\mathbf{h}^- = \mathbf{\Lambda}^-(L, L_r)\mathbf{d}^-$.

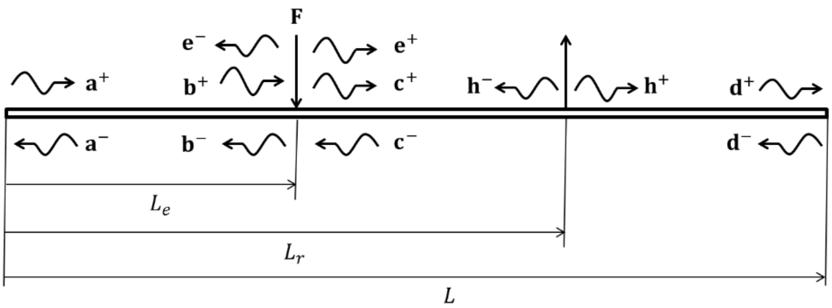

Figure 1. Point excitation and wave amplitudes on a waveguide with slowly varying properties.

These relations can be used to find

$$\mathbf{c}^+ = [\mathbf{I} - \mathbf{\Lambda}^+(0, L_e)\mathbf{\Gamma}_L\mathbf{\Lambda}^-(0, L)\mathbf{\Gamma}_R\mathbf{\Lambda}^+(L_e, L)]^{-1}[\mathbf{e}^+ + \mathbf{\Lambda}^+(0, L_e)\mathbf{\Gamma}_L\mathbf{\Lambda}^-(0, L_e)\mathbf{e}^-], \tag{3}$$

$$\mathbf{c}^- = \mathbf{\Lambda}^-(L, L_e)\mathbf{\Gamma}_R\mathbf{\Lambda}^+(L_e, L)\mathbf{c}^+, \tag{4}$$

from which the input mobility can be calculated. The same rationale can be used to calculate the response at any point in the waveguide from the wave amplitudes $\mathbf{h}^+$ and $\mathbf{h}^-$.

3. THE WAVE AND FINITE ELEMENT METHOD

In this section, a brief review of the WFE method for one-dimensional waveguides is presented. A section of the waveguide of axial length Δ is cut from the structure and, assuming harmonic motion, its dynamic stiffness matrix $\widetilde{\mathbf{D}} = \mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M}$ can be obtained from a conventional FE analysis, such that $\widetilde{\mathbf{D}}\mathbf{q} = \mathbf{f}$, where $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ are, respectively, the stiffness, damping and mass matrices, $\mathbf{q}$ is the vector of nodal degrees of freedom and $\mathbf{f}$ is the vector of nodal forces. The dynamic stiffness matrix $\widetilde{\mathbf{D}}$ can be condensed to eliminate any interior degrees of freedom, leading to the matrix $\mathbf{D}$ that can be partitioned as

$$\begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{f}_L \\ \mathbf{f}_R \end{bmatrix}, \tag{5}$$

relating the degrees of freedom (DOFs) and nodal forces on the left (L) and the right (R) cross-section (Mace, et al., 2005). For a wave freely propagating along the waveguide, a propagation constant relates displacements and forces at the left and right side of the section, i.e. $\mathbf{q}_R^s = \lambda\mathbf{q}_L^s$ and $\mathbf{f}_R^s = -\lambda\mathbf{f}_L^s$. Moreover, from continuity of displacements and equilibrium of forces between sections s and $(s+1)$ it follows that $\mathbf{q}_L^{s+1} = \mathbf{q}_R^s$ and $\mathbf{f}_L^{s+1} = -\mathbf{f}_R^s$. Then, a transfer matrix can be defined such that

$$\begin{bmatrix} \mathbf{q}_L^{s+1} \\ \mathbf{f}_L^{s+1} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{q}_L^s \\ \mathbf{f}_L^s \end{bmatrix}, \tag{6}$$

where

$$\mathbf{T} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & \mathbf{D}_{LR}^{-1} \\ -\mathbf{D}_{RL} + -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1} \end{bmatrix}. \tag{7}$$

The eigenvalues/eigenvectors of the transfer matrix are separated into two sets of *n* positive-going (λ_j and $\boldsymbol{\phi}_j^+$) and *n* negative-going ($1/\lambda_j$ and $\boldsymbol{\phi}_j^-$) wave types and the j^{th} eigenvalue is written as $\lambda_j = \exp(-ik_j\Delta)$. The eigenvectors can be rearranged such that $\boldsymbol{\phi}^+ = \begin{bmatrix} \boldsymbol{\phi}_q^+ \\ \boldsymbol{\phi}_f^+ \end{bmatrix}$ and $\boldsymbol{\phi}^- = \begin{bmatrix} \boldsymbol{\phi}_q^- \\ \boldsymbol{\phi}_f^- \end{bmatrix}$ and then are used for a linear transformation of the DOFs and nodal forces

$$\mathbf{q} = \boldsymbol{\phi}_q^+\mathbf{a}^+ + \boldsymbol{\phi}_q^-\mathbf{a}^- \text{ and } \mathbf{f} = \boldsymbol{\phi}_f^+\mathbf{a}^+ + \boldsymbol{\phi}_f^-\mathbf{a}^-, \tag{8}$$

from the wave domain to the physical domain where $\mathbf{a}^+$ and $\mathbf{a}^-$ are respectively the positive-going and negative-going wave amplitudes. Any boundary condition can written as $\mathbf{Af} + \mathbf{Bq} = \mathbf{0}$, then the reflection matrices at the left and right boundaries are given by (Lee, et al., 2007, Renno, and Mace, 2010)

$$\boldsymbol{\Gamma}_L = -\left(\mathbf{A}\boldsymbol{\phi}_{\mathrm{f}}^+ + \mathbf{B}\boldsymbol{\phi}_{\mathrm{q}}^+\right)^{-1}\left(\mathbf{A}\boldsymbol{\phi}_{\mathrm{f}}^- + \mathbf{B}\boldsymbol{\phi}_{\mathrm{q}}^-\right) \text{ and } \boldsymbol{\Gamma}_R = -\left(\mathbf{A}\boldsymbol{\phi}_{\mathrm{f}}^- + \mathbf{B}\boldsymbol{\phi}_{\mathrm{q}}^-\right)^{-1}\left(\mathbf{A}\boldsymbol{\phi}_{\mathrm{f}}^+ + \mathbf{B}\boldsymbol{\phi}_{\mathrm{q}}^+\right). \tag{9}$$

The amplitudes of the positive and negative going wave generated by a point excitation can be calculated by solving

$$\begin{bmatrix} \boldsymbol{\phi}_{\mathrm{q}}^+ & -\boldsymbol{\phi}_{\mathrm{q}}^- \\ \boldsymbol{\phi}_{\mathrm{f}}^+ & -\boldsymbol{\phi}_{\mathrm{f}}^- \end{bmatrix} \begin{Bmatrix} \mathbf{e}^+ \\ \mathbf{e}^- \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_{ext} \end{Bmatrix}, \tag{10}$$

either by direct inversion or by using the orthogonality properties of the left eigenvector of the transfer matrix, for improved numerical conditioning (Waki, et al., 2009a,b). The response to general excitation can be calculated following the procedure given by Renno and Mace (Renno, and Mace, 2010). A number of parameters yielding information about the wave propagation characteristics can be calculated from this approach. In this work, it is particularly interesting to calculate the time average power transmitted through the cross-section, i.e.

$$P = -\frac{1}{2}\mathrm{Re}\{i\omega\mathbf{f}^H\mathbf{q}\} = \frac{\omega}{2}\mathrm{Im}\{\mathbf{f}^H\mathbf{q}\}, \tag{11}$$

where the superscript H stands for the Hermitian.

4. WAVE PROPAGATION WITH SLOWLY VARYING PROPERTIES

For the WKB approximation, it is necessary to calculate the phase change considering the locally defined wavenumber $k_j(x)$ as well as the amplitude change caused by the slowly varying waveguide properties. In this section, the WFE approach is used to estimate $k_j(x)$ at a number of points for calculating the phase change $\theta_j(x_a, x_b)$ from x_a to x_b. A numerical integration using a Gauss-Legendre quadrature scheme is applied, i.e.

$$\theta_j(x_a, x_b) = \int_{x_a}^{x_b} k_j(x)\, dx \approx \sum_{i=1}^{N_{gl}} G_i k_j(x_i), \tag{12}$$

where G_i are the weights and $k_j(x_i)$ is the j^{th} wavenumber calculated at the sampling point x_i defined from the Gauss-Legendre quadrature. The proprieties are evaluated at x_i from a given function describing the spatial variability and then assumed constant within the WFE cross-section. This is equivalent to a mid-point discretization for the spatial variability given by a random field, (Der Kiureghian, and Ke, 1988, Stefanou, 2009, Sudret, and Der Kiuereghian, 2000). The integration scheme gives the exact integral for a polynomial of a given order depending on the number of points N_{gl}. Therefore, this is equivalent to a polynomial fitting of the wavenumber over the waveguide between x_a and x_b. The number of points used by the quadrature must be kept to a minimum number of evaluations, to avoid excessive computational cost. No re-meshing of the FE model is necessary for each WFE evaluation.

The amplitude change can be calculated from the energy conserving property as a consequence of the WKB approximation (Nielsen, and Sorokin, 2014, Pierce, 1970). Therefore, for a positive-going wave travelling from x_a, with amplitude a^+, to x_b, with amplitude b^+, as shown in Fig. 2, assuming no damping, the time average power transmitted through the cross-section, Eq. (11), at both positions must be equal, leading to

$$\left|a_j^+\right|^2 Re\{i\omega \boldsymbol{\Phi}_{f,j}^{+H}(x_a)\boldsymbol{\Phi}_{q,j}^{+}(x_a)\} = \left|b_j^+\right|^2 Re\{i\omega \boldsymbol{\Phi}_{f,j}^{+H}(x_b)\boldsymbol{\Phi}_{q,j}^{+}(x_b)\}. \tag{13}$$

This relation is written in order to define the amplitude change, giving

$$\gamma_j(x_a, x_b) = \log\left(\frac{|b^+|}{|a^+|}\right) = \frac{1}{2}\log\left(\frac{Re\{i\omega \boldsymbol{\Phi}_{f,j}^{H}(x_a)\boldsymbol{\Phi}_{q,j}(x_a)\}}{Re\{i\omega \boldsymbol{\Phi}_{f,j}^{H}(x_b)\boldsymbol{\Phi}_{q,j}(x_b)\}}\right). \tag{14}$$

Both $\theta_j(x_a, x_b)$ and $\gamma_j(x_a, x_b)$ are then used to calculate to positive and negative going propagation matrices by

$$\begin{aligned} \boldsymbol{\Lambda}^+(x_a, x_b) &= diag\left[\exp\left(-i\theta_j(x_a, x_b) + \gamma_j(x_a, x_b)\right)\right], \\ \boldsymbol{\Lambda}^-(x_b, x_a) &= diag\left[\exp\left(-i\theta_j(x_b, x_a) - \gamma_j(x_b, x_a)\right)\right], \end{aligned} \tag{15}$$

where $diag[\cdot]$ stands for diagonal matrix.

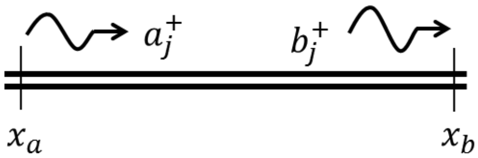

Figure 2. Positive-going wave travelling from x_a, with amplitude a_j^+, to x_b, with amplitude b_j^+, in an infinite waveguide.

5. RANDOM VARIABILITY

Random field theory can be used to model spatially distributed randomness using a probability measure. There are a number of methods available in the literature for generating random fields (Ghanem, and Spanos, 2012, Stefanou, 2009, Sudret, and Der Kiuereghian, 2000, Vanmarcke, 2010), including formulations using series expansions that are able to represent the field using deterministic spatial functions and random uncorrelated variables. The KL expansion is a special case where these deterministic spatial functions are orthogonal and derived from the covariance function.

A Gaussian homogeneous random field $H(x, p)$ with a finite, symmetric and positive definite covariance function $C_H(x_1, x_2)$, defined over a domain D, has a spectral decomposition in a generalized series as (Ghanem, and Spanos, 2012)

$$H(x) = H_0(x) + \sum_{j=1}^{\infty} \sqrt{l_j}\, \xi_j f_j(x), \tag{16}$$

where ξ_j are Gaussian uncorrelated random variables, l_j and $f_j(x)$ are eigenvalues and eigenfunctions. The eigenvalues and eigenfunctions can be ordered in descending order of eigenvalues and the KL expansion is then calculated with a finite number of terms N_{KL}, chosen by the accuracy of the series in representing the covariance function (Huang, et al., 2001). As a rule of thumb, N_{KL} can be chosen such that $l_{N_{KL}}/l_1 < 0.1$, and N_{KL} will depend on the correlation length of the random field.

In general, this problem can only be solved numerically by discretizing the covariance function. However, for some families of correlation functions and specific geometries, there exist analytical solutions. One such case is the one dimensional exponentially decaying autocorrelation function, $C(x_1, x_2) = e^{-|x_1 - x_2|/l_c}$, where l_c is the correlation length, in the interval $-L/2 \leq x \leq L/2$, where L is the length of the domain and where x_1 and x_2 are any two points within the interval. In this case, the KL expansion, for a zero-mean random field, can be written as

$$H(x) = \sum_{j=1}^{N_{kl}} \left[\alpha_j \xi_{1j} \sin(w_{1j} x) + \beta_j \xi_{2j} \cos(w_{2j} x)\right] \tag{17}$$

where ξ_{1j} and ξ_{2j} are Gaussian zero-mean, unity standard-deviation, independent random variables with the properties $\langle \xi_{1j} \rangle = \langle \xi_{1i} \rangle = 0, \langle \xi_{1i} \xi_{2j} \rangle = 0\,, \langle \xi_{1i} \xi_{1j} \rangle = \delta_{ij}$ where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$, and $\alpha_j = \sqrt{l_{1j} / \left(\frac{L}{2} - \frac{\sin(w_{1j} L)}{2 w_{1j}}\right)}$, $\beta_j = \sqrt{l_{2j} / \left(\frac{L}{2} + \frac{\sin(w_{2j} L)}{2 w_{2j}}\right)}$, $l_{1j} = 2c/(w_{1i}^2 + c^2)$, $l_{2j} = 2c/(w_{2i}^2 + c^2)$, where $c = 1/b$ and w_{1i} and w_{2i} are the i^{th} roots of the transcendental equations $c \tan w_1 + w_1 = 0$ and $w_2 \tan w_2 - c = 0$, respectively. This expansion is truncated to N_{KL} terms according the weight of the higher order eigenvalues in the series. A complete derivation can be found in the book by Ghanem and Spanos (Ghanem, and Spanos, 2012).

If $H(x)$ is a Gaussian random field, $\xi_{1j,2j}$ are always independent zero mean unit standard deviation Gaussian random variables. If, on the other hand, the random field is not Gaussian, then $\xi_{1j,2j}$ are not independent and have unknown joint PDF. Therefore, it is not possible to use the KL expansion to directly generate a non-Gaussian random field. However, an iterative scheme can be used to overcome this issue (Phoon, et al., 2002, 2005). It has the advantage of directly using the KL expansion and can simulate both stationary and non-stationary random fields as well as strongly non-Gaussian targeted CDFs (Li, et al., 2007). Moreover, if the target CDF is approximately Gaussian, only one iteration might be enough to achieve convergence.

In the numerical examples, the Young's modulus is chosen to be random as $E(x) = E_0(1 + H(x))$, where E_0 is the nominal value and $H(x)$ is Gamma distributed, with PDF given by $f_H(x) = \exp(y/b_0)/(b_0^{a_0}\, \Gamma(a_0))$, for $x > 0$, where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt\,, x > 0$ is the Gamma function and the parameters of the distribution are given by $a_0 = 1/\delta_E^2$ and $b_0 = E_0 \delta_E^2$, where δ_E is the dispersion parameter. This choice of distribution follows the Maximum Entropy argument, and ensures the second-order statistic of the response are finite (Soize, 2017). Equation (17) is used to simulate the random field and generated using a Latin Hypercube sampling (LHS) scheme (A. M. J. Olsson and G. E. Sandberg 2002, Rubinstein, and Kroese, 2007).

6. APPLICATION EXAMPLE: PLATE STRIP

In this example, a thin plate strip with simple supported edges and free at both ends is considered. The proposed finite element approach for wave propagation is applied for slowly varying material and geometrical properties, considering deterministic properties and random variability.

6.1. Analytical solution

For the example shown in Fig. 3, it is possible to derive an analytical governing equation, from which the eikonal equation for the phase change is

$$\left[\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial}{\partial y}\phi(x,y)\right)^2\right]^2 = \frac{\rho h(x,y)}{B(x,y)}, \tag{18}$$

and $\rho h(x,y)$ is the product of the mass density and thickness, and $B(x,y)$ is the plate bending stiffness. Also assuming the material varies only in the direction of propagation, i.e. the x axis, $B(x,y)=B(x)$, and $\rho h(x,y)=\rho h(x)$, then it is possible to derive the wavenumbers and the wave amplitude ratio (Fabro, 2014)

$$k_{x1m,x2m}(x)=\pm\sqrt{\pm k_p^2(x)-k_{ym}^2},\quad A_{x0}(x)=\frac{A_{x0}(x_0)\sqrt{k_{xm}(x_0)}}{\sqrt{k_{xm}(x)}}. \tag{19}$$

Moreover, expressions for force $\boldsymbol{\phi}_f^{\pm}$ and displacement $\boldsymbol{\phi}_q^{\pm}$ wave modes and propagation matrices $\boldsymbol{\Lambda}^{\pm}(x_a,x_b)$ are given in the Appendix, from which the directly excited waves $\mathbf{e}^{\pm}$ and reflection matrices $\boldsymbol{\Gamma}_{R,L}$ can be calculated.

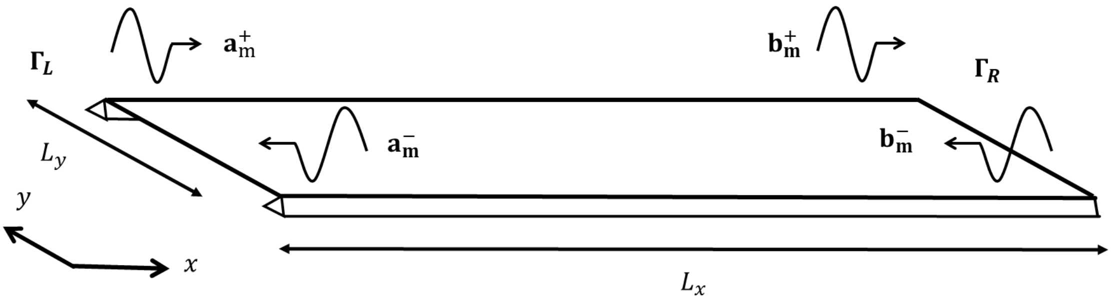

Figure 3. Finite length plate strip undergoing flexural wave behaviour, with slowly varying material properties and considering no internal reflections.

6.2. Numerical results

In this section, two numerical results are presented. The first considers a linear variation of the plate thickness along its length while the second involves random variability of the Young's modulus along the plate strip. The Young's modulus is considered as a random field with Gamma distribution and exponentially decaying autocorrelation, $C(x_1,x_2)=e^{-|x_1-x_2|/l_c}$. The statistics of the response for the stochastic analysis are calculated from a MC scheme with 500 samples. Both examples use a steel plate undergoing flexural vibration with uniform mass density $\rho=7800$ kg/m^3 and nominal Young's modulus $E_0=210$ GPa, with $L_T=5$ m total length, $L_x=1.2$ m wide and point excitation at the left boundary. A total of six wave modes are used for all of the numerical examples.

In the first case, Fig. 4 presents the thickness variation, from 10 mm to 21 mm, along with the GL points, and the magnitude of the frequency response obtained from the analytical model and the WFE approach. There is a very good agreement with $N_{gl}=3$ points of validation plus one at each boundary.

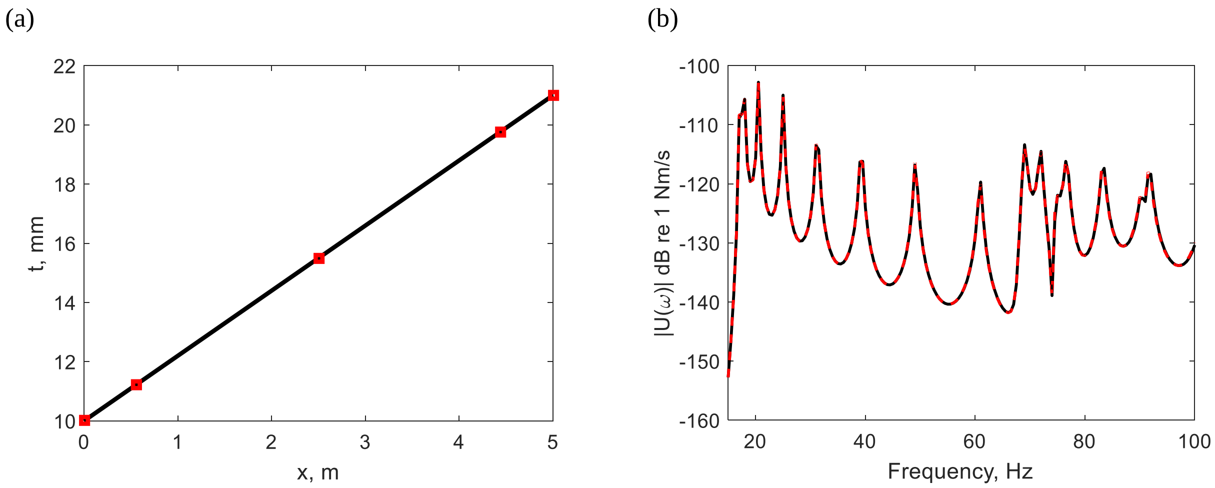

Figure 4. (a) Thickness variation along the plate strip and GL points (red squares) and (b) magnitude of the frequency response function obtained from the analytical model (black) and from the WFE approach (dashed red).

In the second case, Fig. 5 presents a sample of the random field with the $N_{gl}=8$ points plus one at each boundary. Fig. 6 shows the phase change and attenuation constant and Fig. 7 (a) presents the amplitude of the frequency response function obtained from this single sample. Moreover, Fig. 7 (b) shows the 5th and 95th percentiles obtained from the LHS. Good agreement between the WFE and analytical model can be seen, both in the single sample and the percentile bounds, except in the narrow band around the cut on frequencies. Responses in this frequency bands have to treated separately and will be subject of future work.

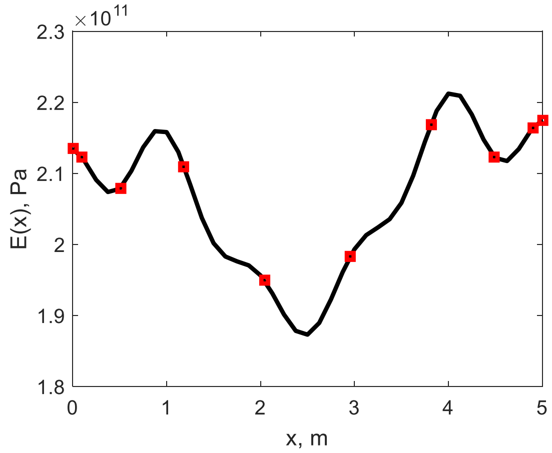

Figure 5. Random field sample using $l_c = L_x$ and GL points for integration scheme and at the boundaries.

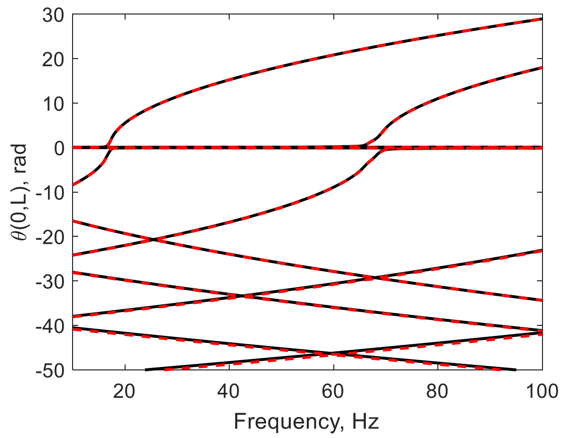

Figure 6. Phase change (positive values) and attenuation constant (negative values) over the plate length for $k_{1m}(\omega, x)$ and $k_{2m}(\omega, x)$ from the analytical model (black) and from the WFE approach (dashed red). Positive values are the real part (propagating waves) and negative values are the imaginary part (non-propagating waves).

(a) (b)

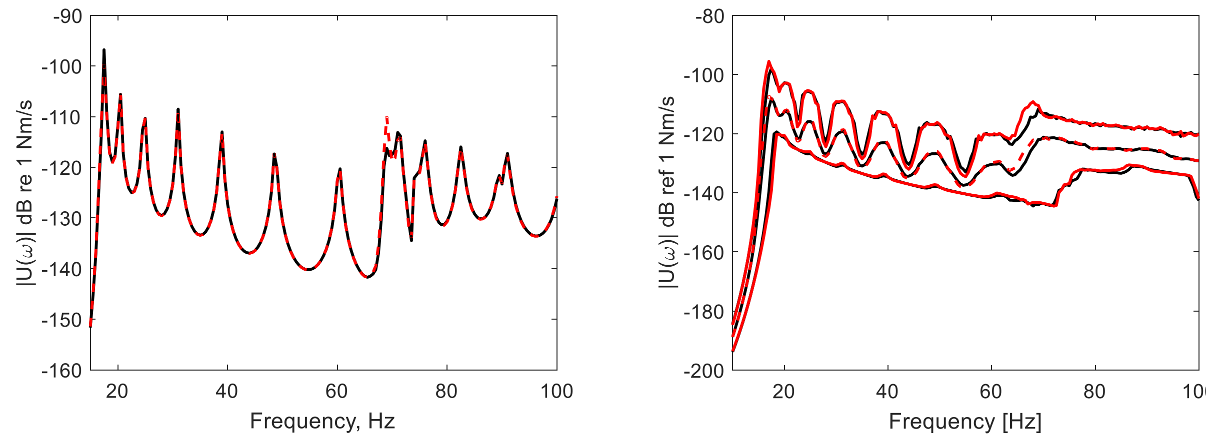

Figure 7. (a) Magnitude of the frequency response function obtained from the analytical model (black) and from the WFE approach (dashed red) and (b) 5^{th} and 95^{th} percentile of frequency response function amplitude obtained from the analytical model (black) and from the WFE approach (dashed red).

7. CONCLUDING REMARKS

A method is proposed to extend the applicability of the WKB approach to cases where no analytical solution exists by using a FE approximation. The phase change and the attenuation constant require the numerical evaluation of the locally defined wavenumber at various points, which are kept to a minimum, and are evaluated at locations defined by a Gauss-Legendre quadrature scheme. Also, the WKB solution implies conservation of power which is used to calculate the amplitude change.

An example of a thin plate strip with simple supported edges undergoing flexural vibration is presented for two cases. In the first, a deterministic, linearly varying thickness is assumed while in the second, the spatially correlated non-Gaussian random variability of the Young's modulus is expressed by the KL expansion for random fields. Even though an analytical KL expansion is used, the method can be extended straightforwardly to a numerical solution of the KL expansion and therefore to different correlation functions or probability density functions. Results are compared to available analytical WKB solution and a good agreement is seen for both examples.

Further steps include extending the proposed approach to more complex waveguides, with different wave modes, exploring other random field types and the sensitivity to the random field discretization.

8. ACKNOWLEDGEMENTS

The authors gratefully acknowledge the financial support of the Brazilian National Council of Research (CNPq) Process number 445773/2014-6, the Federal District Research Foundation (FAPDF) Process number 0193001040/2015 and the Royal Society for the Newton International Exchanges Fund.

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APPENDIX

This section presents the analytical solution of the WKB approach for a plate strip with slowly varying properties, summarized from (Fabro, 2014). Considering a thin plate undergoing out-of-plane flexural vibration with varying material properties and assuming harmonic motion, it is possible to derive the governing equation (Graff, 1991)

$$\frac{\partial^2}{\partial x^2}\left[B(x,y)\left(\frac{\partial^2}{\partial x^2}w(x,y)+\nu\frac{\partial^2}{\partial y^2}w(x,y)\right)\right]+\frac{2\partial^2}{\partial x\partial y}\left[B(x,y)(1-\nu)\frac{\partial^2}{\partial x\partial y}w(x,y)\right] + \frac{\partial^2}{\partial y^2}\left[B(x,y)\left(\frac{\partial^2}{\partial y^2}w(x,y)+\nu\frac{\partial^2}{\partial x^2}w(x,y)\right)\right]=\rho h(x,y)\omega^2 w(x,y), \tag{20}$$

where $w(x,y)$ is the out-of-plane harmonic displacement, $\rho h(x,y)$ is the product of the mass density and thickness, and $B(x,y)$ is the plate bending stiffness. Setting a small parameter ϵ such that $\epsilon^{-4}=\omega^2$ and assuming solutions of the kind $w(x,y)=A(x,y)e^{-i\frac{\phi(x,y)}{\epsilon}}$ where the amplitude is expanded in terms of the parameter ϵ as $A(x,y)=A_0(x,y)+A_1(x,y)\epsilon$ it is possible to match the terms with equal power of ϵ on both left and right hand side of Eq. (20). This procedure leads to a number of differential equations to solve for each term on the expansion. From the ϵ^0 order, one can find the eikonal equation for the phase change, Eq. (18) and the higher orders terms of ϵ give expressions for the amplitude terms. However, if a plate strip with simply supported boundary conditions at $y=0$ and $y=L_y$ is considered, also assuming the material variability only in the direction of propagation, i.e. the x axis, $B(x,y)=B(x)$, and $\rho h(x,y)=$

$\rho h(x)$, then it is possible to assume the phase change, of the form $\phi(x,y)/\epsilon = \phi_{xm}(x) + k_{ym}y$, where $k_{ym} = m\pi/L_y$ is the m^{th} wave mode in the y direction and $\phi_{xm}(x)$ is the phase change in the x direction. This expression can be used for direct integration of the eikonal equation leading to

$$\phi_{xm}(x_0, x) = \pm \int_{x_0}^{x} k_{xm}(\xi)\, d\xi, \tag{21}$$

with four solutions for the phase change in the x direction, where $k_{xm}(x) = \sqrt{\pm k_p^2(x) - k_{ym}^2}$, $k_p(x) = \sqrt{\omega}\left(\frac{\rho h(x)}{D(x)}\right)^{1/4}$ is the local free bending wavenumber in a thin plate and x_0 is an arbitrary point in the waveguide, from which it is possible to distinguish two wave types $k_{x1m}(x)$ and $k_{x2m}(x)$.

For a plate strip with simply supported edges, the displacement can be written as

$$w(x,y) = \sum_{m=1}^{\infty} w_m(x)\sin(k_{my}y), \tag{22}$$

and therefore for the m^{th} wave mode, the amplitude change can be written using only the first order as $A(x,y) = [A_{x0}(x) + A_{x1}(x)\epsilon]\sin(k_{my}y)$. The amplitude term $A_{x0}(x,y)$ is found from matching the ϵ^1 order of the expansion and using the solution for the phase term $\phi(x,y)$ and for $k_p^2(x) \gg 0$, it is possible to obtain the differential equation,

$$\frac{2}{A_{x0}}\frac{\partial A_{x0}}{\partial x} + \frac{1}{\phi_{xm}(x)}\frac{\partial \phi_{xm}(x)}{\partial x} = 0, \tag{23}$$

whose solution gives the typical WKB wave amplitude change expression, Eq. (19).

The displacement and force vectors from the wave domain to the physical domain are given by

$$\mathbf{q} = \begin{bmatrix} w \\ dw/dx \end{bmatrix} = \sum_{\boldsymbol{m}} \boldsymbol{\Phi}_{qm}^{+}\mathbf{a}_{\boldsymbol{m}}^{+} + \boldsymbol{\Phi}_{qm}^{-}\mathbf{a}_{\boldsymbol{m}}^{-} \text{ and } \mathbf{f} = \begin{bmatrix} Q \\ M \end{bmatrix} = \sum_{\boldsymbol{m}} \boldsymbol{\Phi}_{\boldsymbol{fm}}^{+}\mathbf{a}^{+} + \boldsymbol{\Phi}_{fm}^{-}\mathbf{a}^{-}, \tag{24}$$

where w is the out-of-plane displacement Q is the net force and M is the bending moment on the boundaries (Graff, 1991). Propagation matrices for the m_{th} mode are given by

$$\begin{aligned} \boldsymbol{\Lambda}_{\boldsymbol{m}}^{+}(x_a, x_b) &= diag\left[\exp\left(-i\phi_{xm}(x_a, x_b) + \gamma_m(x_a, x_b)\right)\right], \\ \boldsymbol{\Lambda}_{\boldsymbol{m}}^{-}(x_b, x_a) &= diag\left[\exp\left(-i\phi_{xm}(x_a, x_b) - \gamma_m(x_a, x_b)\right)\right], \end{aligned} \tag{25}$$

where $\gamma_m(x_a, x_b)$ is obtained from Eq. (19). The displacement and force wave mode matrices

$$\begin{aligned} &\boldsymbol{\Phi}_{\mathrm{qm}}^{+} = \sin(k_{my}y)\begin{bmatrix} 1 & 1 \\ -ik_{x1m} & -ik_{x2m} \end{bmatrix}, \boldsymbol{\Phi}_{\mathrm{qm}}^{-} = \sin(k_{my}y)\begin{bmatrix} 1 & 1 \\ ik_{x1m} & ik_{x2m} \end{bmatrix}, \\ &\boldsymbol{\Phi}_{\mathrm{fm}}^{+} = B(x)\begin{bmatrix} k_{x1m}^2 + \nu k_{my}^2 & k_{x2m}^2 + \nu k_{my}^2 \\ -ik_{x1m}\left(k_{x1m}^2 + (2-\nu)k_{my}^2\right) & -ik_{x2m}\left(k_{x2m}^2 + (2-\nu)k_{my}^2\right) \end{bmatrix}, \\ &\boldsymbol{\Phi}_{\mathrm{fm}}^{-} = B(x)\begin{bmatrix} k_{x1m}^2 + \nu k_{my}^2 & k_{x2m}^2 + \nu k_{my}^2 \\ ik_{x1m}\left(k_{x1m}^2 + (2-\nu)k_{my}^2\right) & ik_{x2m}\left(k_{x2m}^2 + (2-\nu)k_{my}^2\right) \end{bmatrix}. \end{aligned} \tag{26}$$

Reflection matrices at both boundaries and the amplitudes of the positive and negative going wave generated by a point excitation can be calculated by Eqs. (9) and (10), respectively.