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OPTIMAL PLACEMENT OF SENSORS FOR THE OUTPUT FEEDBACK CONTROL OF STRUCTURES USING QUADRATIC PERFORMANCE CRITERION

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Abstract. *Optimal control has been studied in many researches of vibration control. The problem with the linear quadratic regulator (LQR) is that, in general, it requires availability of all state variables of the controlled system to be measured, and the observer alternative makes the structure of the controller complex, enlarges the system dimension and is model dependent. A possible way to overcome this problem is to use the restriction of output feedback, which is known as optimal output feedback or partial state feedback. In this paper, this method is studied considering additionally sensors locations as optimization variables. Necessary conditions of optimality are given considering that the set of possible sensors locations is continuous. The dependence on system initial condition is approached using a robust optimization formulation. Control Lyapunov functions are used to study flexible structures stability, which gives the result that the collocated control is a solution of the LQR problem. The method developed in this paper is tested on a cantilever beam modeled using the finite element method. The good results obtained indicates that the proposed method might be an alternative for the LQR or LQG control strategies.*

Keywords: *structural control, sensor placement optimization, optimal output feedback.*

1. INTRODUCTION

Optimal control deals with the problem of determining a control law for a dynamical system in order to minimize a predetermined performance criterion. A formulation considering a cost quadratic in states and control and a linear system leads to the well know linear quadratic regulator. Besides minimizing a cost function that represents a trade-off between control effort and performance, the control law obtained by the solution of the LQR still provides the closed loop system good frequency properties, such as infinite gain margin and at least $60°$ phase margin for single-input systems (Anderson and Moore, 2007). However, its applications in real systems are limited since, in general, it requires that all state variables must be measured. A usual way to deal with this problem is to use an observer to estimate unmeasured states, but this approach still has drawbacks. A state observer requires online computation, doubles the system dimension and is model dependent, which may increase errors. Another alternative to overcome this difficulty, the one that will be studied in this article, is characterized by a restriction of using only measured signals for feedback, which is known as optimal output feedback or partial state feedback.

This problem was first addressed by Levine and Athans (1970), who presented the necessary conditions for an output control to be optimal and an algorithm to solve these conditions. In this paper, and in some subsequent ones (Toivonen, 1985; Rautert and Sachs, 1997), where other algorithms were proposed and the conditions of existence and uniqueness were investigated, it was noted that the optimal output feedback problem posed challenges beyond those encountered in the total state feedback problem. Firstly, to obtain the optimal gain it is necessary to solve a system of three matrix equations, two of which are Lyapunov equations. One of these equations is dependent on the system initial condition, which implies that the optimal output gain is also dependent. The suggestion given in (Levine and Athans, 1970) to solve this problem, which was considered later in many articles, was to adopt the expected value of the cost function considering the initial condition as a random variable uniformly distributed on the surface of a unit sphere of appropriate dimension. However, this approach may not be too accurate, since this set of initial states may not represent well the system states. Additionally, the existence of an optimal output gain is related to the existence of a static output gain that stabilizes the system, problem that, according to the recent review article (Sadabadi and Peaucelle, 2016), is still open. Finally, the problem of optimization of the quadratic cost with output feedback constraint is, in general, non-convex.

Another aspect to consider when it comes to optimal output feedback control is the design of the output matrix, which includes the number, type and position of sensors. In the case of vibration control of structures, several researches have already shown that the location of sensors can significantly affect the performance of the control system (Gupta *et al.*, 2010), and it is clear that this factor can also affect the magnitude of the quadratic cost function. Nonetheless, most of the articles that considered the optimal output feedback to design the control law used another criterion for finding the actuators and sensors locations, such as modal order and the corresponding mode shape function (Lim *et al.*, 1999) and maximization of piezoelectric modal actuating forces (Moon, 2006). An article that considered the quadratic cost function to optimize the sensors locations is (Cai and Lim, 2005). In this paper, the authors proposed a methodology to find the sensors locations and feedback gain divided in two steps. First, the states that have greatest effect on the cost function are selected according to the second order sensitivity, which is the second-order derivative of cost function with respect to control gain. Then, the optimal output gain is obtained by solving the necessary conditions for optimality. This approach can be a good choice when there is discrete set for sensors locations, but it is understandable that it can be improved for continuous systems.

In this paper, all the problems mentioned above are addressed. The dependence on the initial condition is eliminated by considering a robust counterpart formulation, which consists in specifying a set for the uncertainty, determining the worst case for the cost function in this set and optimizing for this case. This approach has the advantage that it is not necessary to assume a distribution of the state variable, and the determination of the worst case is straightforward. To show that flexible structures are stabilizable using output feedback, it was considered the use of control Lyapunov functions (Sontag, 2013), which additionally gave the unprecedented result, according to our knowledge and researches, that the collocated control is a solution of the linear quadratic regulator problem. The sensors locations are treated as optimization variables of the quadratic cost function, which allows the determination of the necessary conditions for optimal locations. This determination is possible when considering that the output matrix is a differentiable function of sensors positions, which is compatible with series discretization methods (Meirovitch, 1997), such as Ritz-Galerkin, Rayleigh-Ritz and in some cases the finite element method (FEM), which is considered in this paper. This methodology is applied to a cantilever beam, and the results obtained are compared to LQR and collocated control with gain obtained by the optimal output feedback formulation.

2. DESIGN OF SENSORS LOCATIONS AND OUTPUT FEEDBACK GAIN

As stated in the previous section, the optimal output feedback theory deals with some problems, such as the dependence of control gain on the system initial condition, the design of the output matrix, the existence of a static output gain that stabilizes the system and the non-convexity of the optimization. These problems are studied in the following sections.

2.1 Necessary conditions for optimality and optimization algorithm

To obtain the necessary conditions for optimal sensors locations and output feedback gain, it is considered the following linear control system that represents a flexible structure:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\boldsymbol{\alpha})\mathbf{u} \tag{1}$$

$$\mathbf{y} = \mathbf{C}(\boldsymbol{\xi})\mathbf{x} \tag{2}$$

$$\mathbf{u} = -\mathbf{K}\mathbf{y} \tag{3}$$

in which $\mathbf{x} \in \mathbb{R}^{2n}$ is the state vector, $\mathbf{u} \in \mathbb{R}^{a}$ is the control vector, $\mathbf{A} \in \mathbb{R}^{2n\times 2n}$ is the state matrix, $\mathbf{B} \in \mathbb{R}^{2n\times a}$ is the input matrix, $\mathbf{C} \in \mathbb{R}^{s\times 2n}$ is the output matrix, $\mathbf{K} \in \mathbb{R}^{a\times s}$ is the control gain, $\boldsymbol{\alpha} \in \mathbb{R}^{a\times m}$ represents the actuators locations and $\boldsymbol{\xi} \in \mathbb{R}^{s\times m}$ represents sensors locations, in which m is the structure dimension. Equation (3) characterizes the output feedback relation. The control gain and the positions of the control devices are variables to be determined in order to minimize the following cost function:

$$J = \int_0^{\infty} \mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathsf{T}}\mathbf{R}\mathbf{u}\, dt \tag{4}$$

that represents a trade-off between control effort and performance. Considering that the system is stabilizable and using a procedure like the one given in (Lewis *et al.*, 2012), it is possible to rewrite this function in the form:

$$J = \mathrm{tr}\left\{\mathbf{P}\mathbf{X}\right\} + \mathrm{tr}\left\{\mathbf{S}(\mathbf{A}_{\mathbf{c}}^{\mathsf{T}}\mathbf{P} + \mathbf{P}\mathbf{A}_{\mathbf{c}} + \mathbf{Q} + \mathbf{K}^{\mathsf{T}}\mathbf{R}\mathbf{K})\right\} \tag{5}$$

in which $\mathbf{X}$ is a matrix given by the product $\mathbf{x}(0)\mathbf{x}^\mathsf{T}(0)$, $\mathbf{S} \in \mathbb{R}^{2n\times 2n}$ is a symmetric matrix of Lagrange multipliers and $\mathbf{A_c}$ is the closed loop matrix ($\mathbf{A_c} = \mathbf{A} - \mathbf{BKC}$). The first order necessary conditions for optimality are given by the partial derivatives of J with respect to the independent variables:

$$\frac{\partial J}{\partial \mathbf{S}} = \mathbf{A_c}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A_c} + \mathbf{Q} + \mathbf{C}^\mathsf{T}\mathbf{K}^\mathsf{T}\mathbf{RKC} = 0 \tag{6}$$

$$\frac{\partial J}{\partial \mathbf{P}} = \mathbf{A_c}\mathbf{S} + \mathbf{S}\mathbf{A_c}^\mathsf{T} + \mathbf{X} = 0 \tag{7}$$

$$\frac{1}{2}\frac{\partial J}{\partial \mathbf{K}} = \mathbf{RKCSC}^\mathsf{T} - \mathbf{B}^\mathsf{T}\mathbf{PSC}^\mathsf{T} = 0 \tag{8}$$

$$\frac{1}{2}\frac{\partial J}{\partial \xi_i} = \mathrm{tr}\left\{\left(\mathbf{SC}^\mathsf{T}\mathbf{K}^\mathsf{T}\mathbf{RK} - \mathbf{SPBK}\right)\frac{\partial \mathbf{C}}{\partial \xi_i}\right\} = 0 \qquad i = 1,\ldots,ms \tag{9}$$

$$-\frac{1}{2}\frac{\partial J}{\partial \alpha_i} = \mathrm{tr}\left\{\mathbf{KCSP}\frac{\partial \mathbf{B}}{\partial \alpha_i}\right\} = 0 \qquad i = 1,\ldots,ma \tag{10}$$

Equations (6-8) are the necessary conditions given by Levine and Athans (1970) and the others give the optimal locations for sensors and actuators. In this paper, only Equations (6-9) are considered in order to investigate exclusively sensors locations. Differently from the case of full state feedback, it is not possible to manipulate these equations in order to obtain one equation in function of a single variable, which means that they must be solved simultaneously. With this purpose, there are proposed two algorithms to obtain a solution. In the first one, it is used a least squares method with the Levenberg-Marquardt algorithm. Like the algorithms proposed for the problem of optimal output feedback (Moerder and Calise, 1985), only Equations (8,9) are used in the least squares method, so that in the i-th iteration, given the values of ξ_i and K_i, Equations (6,7) are solved for P_i and S_i using the algorithm proposed by Bartels and Stewart (1972) to solve Sylvester equations.

Because it only uses information about the gradient of the cost function, this method has the disadvantage of not distinguishing points of maximum, inflection and minimum. As an alternative, it is used a sequential quadratic programming method in order to optimize the cost function. In this method, Equations (8,9) are used as a cost function gradient and the Hessian is estimated as in the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm. A comparison between these two proposed algorithms may be useful to identify whether there are maximum points. For instance, if both algorithms converge to the same point for every initial guess, it may indicates that the problem does not have maximum points.

2.2 Flexible structures stabilization and algorithm initial guess

The existence of an optimal output feedback control is related to the existence of a static output feedback that stabilizes the system. In this section, this investigation about stability is done by specifying the system given in Eq. (1) and using control Lyapunov functions, which allows to determine not only which control law stabilizes flexible structures, but also an optimal control law. With this purpose, it is considered the following system in the modal form with proportional damping:

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\eta} \\ \dot{\boldsymbol{\eta}} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{\Lambda} & -\mathbf{D} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\phi}(\boldsymbol{\alpha}) \end{bmatrix} \tag{11}$$

in which $\boldsymbol{\eta} \in \mathbb{R}^n$ is the modal coordinate vector, $\mathbf{I} \in \mathbb{R}^{n\times n}$ is the identity matrix, $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times n}$ is a diagonal matrix of system eigenvalues or natural frequencies squared, $\mathbf{D} \in \mathbb{R}^{n\times n}$ is a diagonal matrix of damping and $\boldsymbol{\phi} \in \mathbb{R}^n$ is the vector of eigenfunctions or approximated eigenfunctions when it is used a discretization method. The form considered of the input matrix means that the actuator acts as a concentrated force in location $\boldsymbol{\alpha}$. To show that this system is stabilizable, it is used the following Lyapunov function:

$$V = \mathbf{x}^\mathsf{T}\begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\mathbf{x} \tag{12}$$

which is a function associated with system energy in modal coordinates. Using Einstein notation, the time derivative of this function can be expressed as:

$$\dot{V} = -2\left\{d_r\dot{\eta}_r^2 - [\phi_r(\boldsymbol{\alpha})\dot{\eta}_r]\,u\right\} \tag{13}$$

in which d_r, η_r e ϕ_r represents the diagonal elements of $\mathbf{D}$, the elements of vector $\boldsymbol{\eta}$ and the elements of vector $\boldsymbol{\phi}$, respectively. Using collocated control with negative velocity feedback this equation becomes:

$$\dot{V} = -2\left\{d_r\dot{\eta}_r^2 + k\,[\phi_r(\boldsymbol{\alpha})\dot{\eta}_r]^2\right\} \tag{14}$$

in which k is the control gain. Equation (14) shows that the collocated control stabilizes the system given in Eq. (11). Furthermore, it shows that the function considered to investigate this system stability is a control Lyapunov function, which means that it is a solution of the Hamilton-Jacobi-Bellman (HJB) equation (Freeman and Kokotovic, 2008). Using this fact, it is possible to determinate an optimal control law u^* for this system:

$$u^* = -\frac{1}{2}\nabla V\mathbf{B} = -\boldsymbol{\phi}^{\mathsf{T}}(\boldsymbol{\alpha})\dot{\boldsymbol{\eta}} \tag{15}$$

which is exactly the collocated control with negative velocity feedback. The substitution of the control Lyapunov function on the HJB equation:

$$q(\mathbf{x}) + \nabla V(\mathbf{A}\mathbf{x}) - \frac{1}{4}[\nabla V\mathbf{B}]^2 = 0 \tag{16}$$

gives also the cost function minimized by the optimal control law u^*, which is of the type given in Eq. (4) with weighting matrices:

$$\mathbf{Q} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & 2\mathbf{D} + \boldsymbol{\phi}(\boldsymbol{\alpha})\boldsymbol{\phi}^{\mathsf{T}}(\boldsymbol{\alpha})\end{bmatrix}, \qquad R = 1 \tag{17}$$

This result shows that, for a structure with low damping, a cost function given by the velocity squared of one of its points plus the control squared is minimized by the collocated control with negative velocity feedback of the same point. Another interesting aspect of this result is that it is a solution of the LQR problem with only one sensor. The results obtained here can be easily extended for the case of more than one collocated pair using the HJB equation for multi-input systems.

The problem to use only collocated control with negative velocity feedback as an initial guess for the algorithm is that the optimization problem may be non-convex. An alternative to this problem is to generate random positions and gains until the closed loop system is stable, and then optimize using the algorithm mentioned in the previous section. However, using the fact that it is possible to obtain an optimal control with lower number of sensors than states when using the formulation in modal coordinates, we propose an optimization to find the initial guess. Considering that only velocity feedback will be used and based on the fact that position feedback does not affect energy dissipation, it is proposed an optimization in order to approximate the product between output gain $(\mathbf{K})$ and output matrix $(\mathbf{C})$ from the total state feedback gain $(\mathbf{G})$ obtained by the LQR:

$$\min_{\boldsymbol{\xi}} \|\mathbf{G} - \mathbf{K}\mathbf{C}(\boldsymbol{\xi})\| \tag{18}$$

In this case, the variable $\mathbf{K}$ was not considered as an optimization variable since given a position $\boldsymbol{\xi}$, $\mathbf{K}$ can be determined using the least squares method. This optimization is non-convex and can generate different initial guesses to be used in the algorithms proposed in the previous section. Although there is no guarantee that this optimization will converge to a point that makes the closed loop system stable, during the application of this procedure to the problem studied, for every random position used in the optimization given in Eq. (18) it was possible to find a local optimum that stabilizes the system.

2.3 System initial condition

The last problem about the methodology proposed in this article is the dependence of the cost function and the optimal solution on system initial condition. The method proposed here to overcome this difficulty is based on a robust formulation, which consists in optimizing the cost function for the worst case of the uncertainty (Ben-Tal *et al.*, 2009). To present the methodology, the cost function is first rewritten using the fact that for every pair $(\boldsymbol{\xi}, \mathbf{K})$ that makes the closed loop system stable, the second term on Eq. (5) vanishes:

$$J = \mathrm{tr}\left\{\mathbf{PX}\right\} \tag{19}$$

then, by defining the following set to parametrize the uncertainty:

$$\chi = \left\{\boldsymbol{\zeta} \in \mathbb{R}^{2n} \mid \|\boldsymbol{\zeta}\|_\infty \leq 1\right\} \tag{20}$$

it is possible to specify a set that may represent well the possible system initial conditions, which has the following form:

$$\Omega = \left\{\mathbf{z} \in \mathbb{R}^{2n} \mid \mathbf{z} = \mathbf{x}_0 + \sum_{i=1}^{2n} \zeta_i \boldsymbol{\delta}_i\right\} \tag{21}$$

in which $\mathbf{x}_0$ is a central value, ζ_i is the i-th component of the vector $\boldsymbol{\zeta}$ and $\boldsymbol{\delta}_i$ is a vector in which the i-th component is a positive constant δ_i, and the others are null. In other words, the initial condition belongs to a box of dimension $2n$ centered at $\mathbf{x}_0$. Defining the variable:

$$\boldsymbol{\Delta} = \sum_{i=1}^{2n} \boldsymbol{\delta}_i \tag{22}$$

and replacing $\mathbf{z}$ in Eq (19), it is possible to determine the cost function for the worst case $(\bar{J})$:

$$\bar{J} = \mathrm{tr}\left\{\mathbf{P}\left(\mathbf{x_0}\mathbf{x_0}^\mathsf{T} + \mathbf{x_0}\boldsymbol{\Delta}^\mathsf{T} + \boldsymbol{\Delta}\mathbf{x_0}^\mathsf{T} + \boldsymbol{\Delta}\boldsymbol{\Delta}^\mathsf{T}\right)\right\} \tag{23}$$

3. NUMERICAL EXAMPLE

The methods outlined in this paper were tested on a cantilever beam. It was considered one actuator that acts as a concentrated force at the beam free end and sensors whose signals are processed in order to give the velocity of measured points. The inertia and possible rigidity that sensors can add to the structure were neglected. It was considered a steel beam with properties given in Tab. 1.

Table 1. Beam properties.

Density (kg/m^3)	7860
Elastic Modulus (GPa)	200
Length (mm)	300
Width (mm)	30
Height (mm)	3

For the beam model, it was used the FEM considering Euler-Bernoulli hypothesis. Although other series discretization methods are more common when there is a need to determine system's eigenfunctions, the FEM was chosen in order to show that the methodology proposed here can also be applied to this method. Nonetheless, except by the convergence and embedding properties of mass and stiffness matrices, when functions that satisfies the requirements of Rayleigh's quotient are used, the FEM can be treated as a Rayleigh-Ritz method (Meirovitch, 1997). Hermite cubics were used as interpolation functions and beam was divided in fifty finite elements. The approximated eigenfunctions were calculated as in the Rayleigh-Ritz method, since Hermite cubics satisfies the requirements of Rayleigh's energy form quotient. The model was truncated in the tenth mode, which gives a frequency bandwidth of approximately $7\,kHz$. The frequency range studied is larger than it is usually considered in practical problems, however, the objective of working with a greater number of modes is to analyze the methodology proposed considering a small ratio between number of sensors by the number of states. Additionally, it was adopted a damping factor of 0.5% for every mode.

The weighting matrix Q was adopted as suggested in (Meirovitch *et al.*, 1983):

$$\mathbf{Q} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{24}$$

which has the same form of the Lyapunov function used in previous section, and consequently has the same interpretation. For the matrix R, after some simulations, the value 0.1 was chosen. In order to determine a possible set of uncertainties for the initial condition, it was considered the application of a force at the beam tip until this point reached a displacement equal to the beam's height. The modal displacements were evaluated and adopted for each corresponding δ_i, while the constants δ_i associated with modal velocities were considered null. This case represents the largest modal displacement for every mode when a concentrated force of a given magnitude is applied at any point of the beam. For $\mathbf{x}_0$ it was adopted the null vector in order to include the case of symmetrical excitation.

Firstly, the dependence of the control gain on system initial condition was analyzed. With this purpose, it was considered the collocated control, whose optimization of control gain is convex, so there is an assurance that the optimal solution has been determined. To evaluate the optimal output gain, the robust approach using the uncertainties set chosen above and the expected value of the cost function considering the initial condition as a random variable uniformly distributed on the surface of a unit sphere, which represents the proposal given in (Levine and Athans, 1970), were considered. For the robust case, the control gain obtained was 9.34×10^{-1}, while for the other case, which henceforth will be referred to as spherical distribution, it was 5.72×10^{-2}. These cases were compared according to the cost function magnitude, using the following relation:

$$\mathbf{P}_2 - \mathbf{P}_1 \succ 0 \iff \mathbf{x}^{\mathsf{T}}\mathbf{P}_2\mathbf{x} > \mathbf{x}^{\mathsf{T}}\mathbf{P}_1\mathbf{x} \iff \operatorname{tr}\{\mathbf{P}_2\mathbf{X}\} > \operatorname{tr}\{\mathbf{P}_1\mathbf{X}\} \quad \forall \mathbf{x} \in \mathbb{R}^{2n} \tag{25}$$

So, for two different pairs $(\boldsymbol{\xi}_1, \mathbf{K}_1)$ and $(\boldsymbol{\xi}_2, \mathbf{K}_2)$ that stabilizes the system, if the corresponding matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ satisfies Eq. (25), then for every initial condition the cost J_1 is lower than J_2. Moreover, if this relation is not satisfied, the eigenvectors whose corresponding eigenvalues of the matrix $(\mathbf{P}_2 - \mathbf{P}_1)$ are positive span a subspace in which J_2 is greater than J_1. Using these considerations, it was possible to show that robust case was better for initial conditions that are approximately given by the first and second modes, while the spherical distribution case was better for the other modes.

Additionally, the cost function for these cases was compared to the cost for the LQR. First, it was determined for which initial condition the cost function for both cases would have the worst value relative to the LQR cost. The relative cost was evaluated for each eigenvector of the matrix difference given in Eq. (25), and the largest value obtained was taken as the worst condition. Although this calculation does not give the exact worst condition, it was found numerically using many different initial conditions that it can be a good parameter to indicate the largest relative difference. Using this consideration, the largest difference for robust case obtained was 75.15%, which corresponds to an initial condition that is approximately given by the tenth mode. For the spherical distribution case, this difference reached 280.53% for an initial condition which is almost equal to the first mode. Moreover, in order to compare the cost function value for these cases with the cost for the LQR for more initial conditions, ten thousand initial conditions uniformly distributed in the uncertainties set were generated. For each element of this set, the cost function value was compared to the case of full state feedback, which is optimal for every initial condition. For the robust case, the mean (μ) obtained for the relative difference was 0.91% with standard deviation (σ) of 2.02%, while for the spherical distribution case, the mean of the relative difference was 255.56% with standard deviation of 51.11%. Figure 1 illustrates the time response and control effort for the initial condition used to generate the uncertainties set. The cost function difference relative to the LQR for this initial condition was 0.17% for the robust case and 276.33% for the spherical distribution case.

This particular example demonstrates the dependence of the control gain on initial condition and suggest that the robust approach can solve this problem efficiently. Now, another case is discussed in which sensors locations are also considered as design variables. Although different solutions were obtained for different initial guesses, it was noticed that both the least squares method and the direct optimization of the cost function converged to the same point for every initial guess used. This result may indicates that the optimization problem is non-convex because the set of parameters $(\boldsymbol{\xi}, \mathbf{K})$ that stabilizes the system is not connected, and that there are no maximum points. Another important factor of the methodology proposed is that, by using a continuous set to represent sensors locations, it is possible to use line search or trust-region based algorithms, which usually converge faster than heuristic methods.

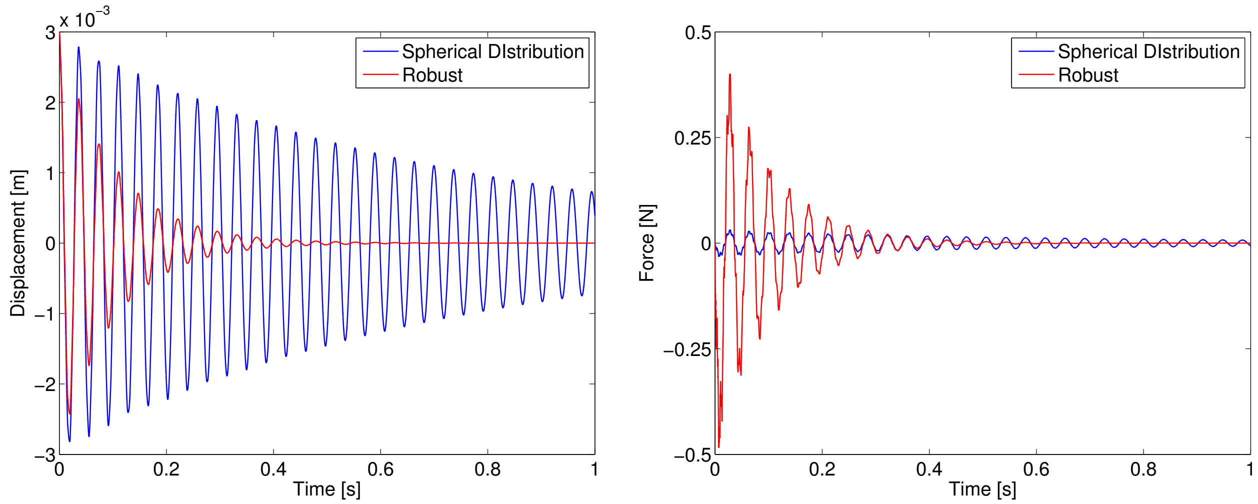

Figure 1. Time response of the beam tip displacement (left) and control effort (right) for the initial condition used to generate the uncertainties set.

As an example, a case considering two sensors whose locations and control gains are to be optimized is studied. After some iterations using different initial conditions, a relevant local optimum was found, which is shown in Tab. 2 together with collocated control. These cases were compared as in the previous analyses of the dependence of the gain on initial condition. By using Eq. (25), it was possible to show that the collocated control was better for initial conditions that are approximately given by the first mode, while the case with two sensors was better for the other modes. Using the eigenvectors of the matrix difference given in Eq. (25), the highest difference between cost functions for the case with two sensors and LQR was determined, which corresponds to an initial condition equivalent to the fifth mode. Comparing the cost with LQR, using again ten thousand initial conditions uniformly distributed in the uncertainties set, the mean and standard deviation of the relative difference to the cost obtained using LQR were evaluated. All this results are shown in Tab. 2. It interesting to notice that by using the feedback of the output of only two sensors, it was possible to achieve almost the same performance of full state feedback (LQR), which, in this case, would require 20 sensors. Figure 2 exemplifies the time response and control effort for one initial condition contained in the uncertainties set. The cost function difference relative to the LQR for this initial condition was 0.12% for the case with two sensors and 40.76% for the collocated control. Additionally, a comparison between open loop, LQR and the case with two sensors was performed. For that, an impulse at free end of the beam was considered and the results are plotted in Fig. 3. Analyzing this figure it is possible to see that the curves referring to the control with two sensors and LQR are practically indistinguishable.

Table 2. Comparison between optimal collocated control and optimal output control with two sensors.

	Location (mm)	Gain	Relative Cost Difference	Worst Condition Difference
Collocated	$\xi = 300$	$k = 9.34 \times 10^{-1}$	$\mu = 0.911\%$, $\sigma = 2.024\%$	75.15%
Two Sensors	$\xi_1 = 241.0$ $\xi_2 = 285.6$	$k_1 = 3.39 \times 10^{-1}$ $k_2 = 7.42 \times 10^{-1}$	$\mu = 0.038\%$, $\sigma = 0.004\%$	0.41%

Lastly, other important aspects of the LQR are its frequency properties, such as infinite gain margin and at least $60°$ phase margin. Although these properties are not encompassed by the optimal output controller, it is expected that the properties of both systems are closer when the values of the cost functions are also (Anderson and Moore, 2007). This property of stability when the feedback gains are varied, and in the present case of sensors positions also, was analyzed for the case of two sensors. First, the optimal locations given in Tab. 2 were fixed, the gains of both sensors were varied and the correspondent cost function was calculated for one initial condition contained in the uncertainties set. The cost function surface obtained is shown in Fig. 4. Then, to analyze system stability and performance when sensors locations are varied, the output gain vector was fixed and the corresponding cost function was calculated for each sensors pair location. The results obtained for this case are shown in Fig. 5.

Figure 4 shows that even when the control gain is increased one order of magnitude, the system remains stable. For the sensor which is furthest from the actuator, the gain margin is smaller and it has the greatest impact on the cost function. In Fig. 5 it is possible to see that there is also a margin to move both sensors so that the closed loop system remains stable, and the sensor which is closer to the actuator and has the largest gain has the greatest effect on system performance. An important conclusion that can be drawn from these results is that if the position and optimal gains can not be realized exactly in the real system, there is a certain assurance that the closed loop system will remain stable.

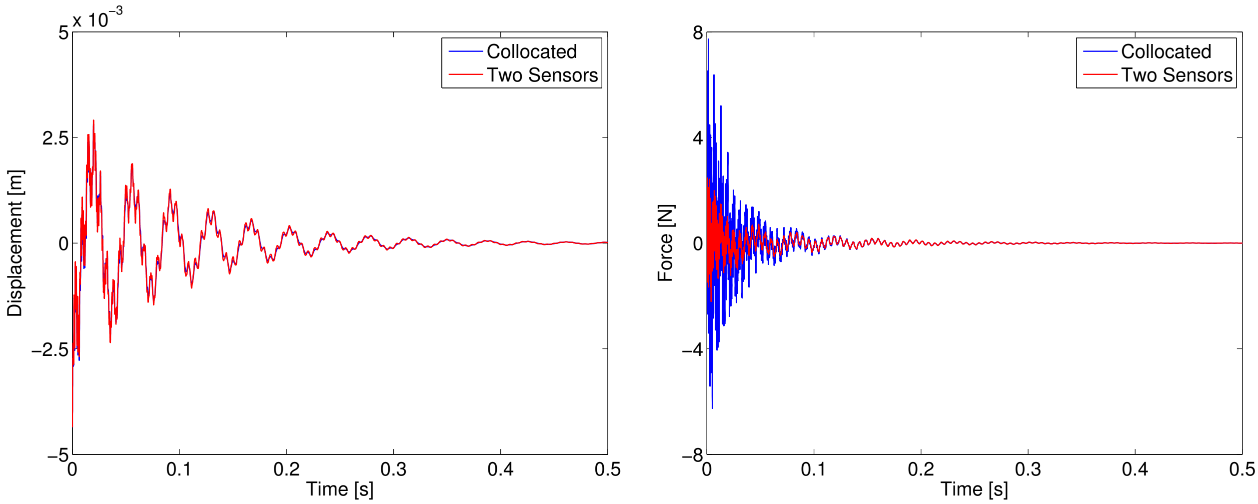

Figure 2. Time response of the beam tip displacement (left) and control effort (right) for one initial condition contained in the uncertainties set.

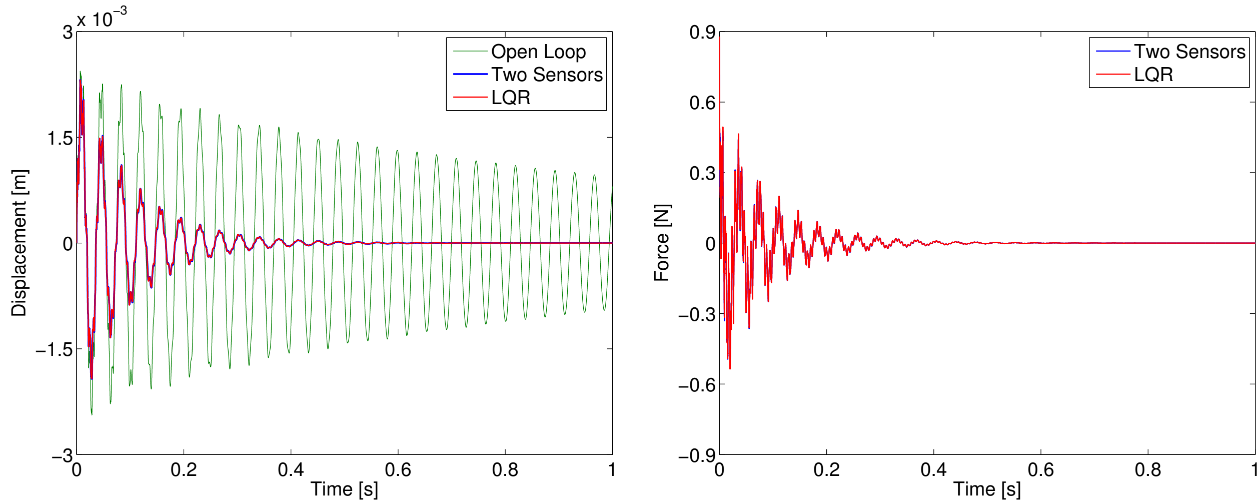

Figure 3. Time response of the beam tip displacement (left) and control effort (right) for an impulse input at the beam tip.

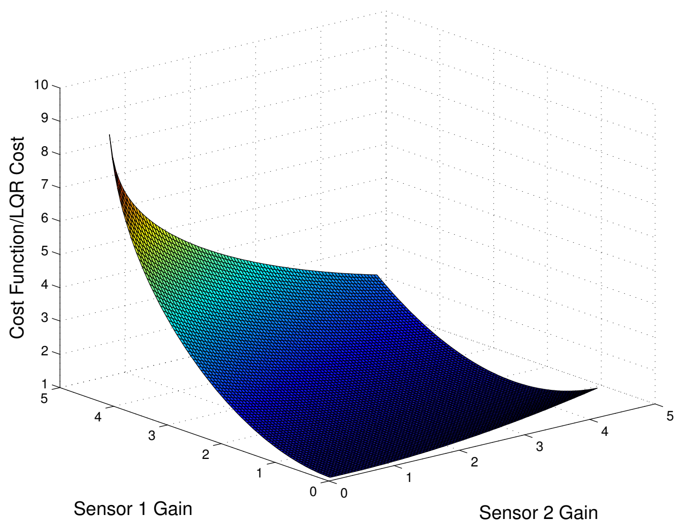

Figure 4. Cost function relatively to LQR cost when sensors gains are varied.

4. CONCLUSIONS

This paper studied the optimal output feedback problem considering sensors locations as optimization variables. Necessary conditions for optimality were determined considering that the set of possible sensors locations is continuous. The

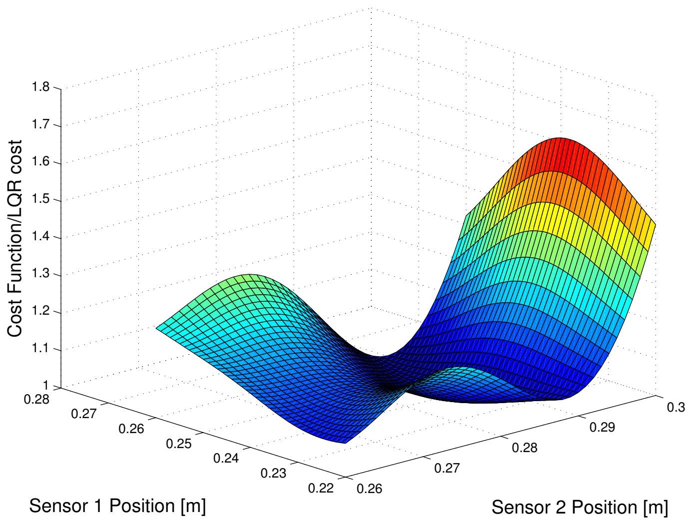

Figure 5. Cost function relatively to LQR cost when sensors position are varied.

existence of a static output gain was treated specifying the system to a modal form and using control Lyapunov functions, which additionally gave the result that the collocated control with negative velocity feedback is a solution of the LQR problem. Moreover, for every random position used in the optimization to find an initial guess (Eq. (18)), it was possible to find a static output gain and sensors locations that makes the closed loop system stable. The dependence of the optimal solution on system initial condition was eliminated by considering a robust formulation, which does not require assumptions on uncertainties distribution and has a simple determination of the worst case. The application of the proposed methodology to the control of a cantilever beam indicated that it is a powerful technique to find control gains and sensors locations. The example also illustrates that for a small ratio between number of sensors by number of states it was possible to achieve almost the same performance as the full state feedback controller. These results indicate that the proposed methodology may be a good alternative to LQR and LQG control strategies. Future works will be directed to the extension of this methodology to more complex structures, sensors and actuators.

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