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Earth-Moon Trajectories based on Variational Equation for Jacobi Integral

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Abstract. *This work proposes a new two-point boundary value problem to determine Earth-Moon bi-impulsive trajectories under the planar bi-circular restricted four-body problem, in which the gravitational attraction of Earth, Moon and Sun are considered. For the trajectory mission, initially, the space vehicle is inserted at the prescribed altitude of a low Earth orbit (LEO). After an application of the first impulsive velocity increment the space vehicle is inserted into a transfer trajectory. The second velocity increment is applied to decelerate and circularize the movement of the space vehicle in the prescribed altitude of a low Moon orbit. To solve this problem a new two-point boundary value problem (TPBVP) is proposed, in which the value of the Jacobi integral at the arrival time is prescribed. Since the Jacobi integral is not a first integral for the four-body problem, its value changes during the space vehicle motion. Therefore, a variational equation for the Jacobi integral in the four-body model is deduced in this work, and, the Jacobi integral is taken as an additional state variable in the description of the dynamics of the space vehicle. Using the initial and the final conditions of the differential equation of motion, analytical expression for the velocity increments are deduced from the Jacobi integral expression. Utilizing the new TPBVP, a numerical procedure is proposed to obtain types of Earth-Moon trajectories with decreasingly fuel consumption.*

Keywords: *Four-body problem, Earth-Moon trajectory, ballistic capture, Jacobi integral.*

1. INTRODUCTION

In the early of 1990s, Belbruno has developed a procedure to determine impulsive trajectories based on the concept of weak stability boundary (WSB) (Belbruno and Miller, 1993), which explores the dynamics of the restricted three-body problem. In this way, low energy external trajectories have been determined for an Earth-Moon mission considering the gravitational influence of the Sun, and, with the space vehicle performing a ballistic capture by the Moon. Despite this procedure be important to determine Earth-Moon trajectories with low fuel consumption, it is limited since others types of trajectories with low consumption can not be determined as, for instance, the trajectories with an intermediary lunar swing-by maneuver and trajectories that perform more revolutions around the Earth. Nowadays, there are modern procedures to determine optimal impulsive trajectories by combining the dynamical approach of the restricted three or four-body problem with optimization techniques (Topputo and Belbruno, 2013). Topputo *et al.* (2005), for instance, utilize Poincaré section and the gradient method to determine optimal transit orbits considering the four-body problem by dividing it in two three-body problems. For this last work, an evolutionary algorithm is utilized to set the initial guess. However, the determination of the initial guess and the setting of the optimization paramaters are hard tasks (McDermott and Fowler, 1977). The work of Topputo (2013), for instance, performs a search in a four dimensional space to generates the initial guess in its algorithm to obtain optimal Earth-Moon trajectories. Gagg Filho and da Silva Fernandes (2015) and Gagg Filho and da Silva Fernandes (2016) have developed patched-conic approximations based on the two-body problem to set the initial guess for more complex planar and tridimensional models considering direct ascent maneuvers in Earth-Moon missions.

The present work explores the dynamic of the restricted four-body problem (Earth, Moon, Sun, and space vehicle) to develop a new procedure to determine bi-impulsive Earth-Moon trajectories with small fuel consumption by formulating a new two-point boundary value problem (TPBVP) with a prescribed value of the Jacobi integral at the final time. Differently from the procedure of Belbruno and Miller (1993) based on the WSB, the new TPBVP allows the determination of any type of trajectory, including external trajectories with swing-by maneuvers and ballistic capture. Due to this wide possibility of trajectories, a good initial guess is necessary. However, this new TPBVP facilitates the intuition of the initial guess since the values of the velocity increments are estimated, before the solving of the TPBVP, by analytical expressions obtained by a mathematical development of the the Jacobi integral.

2. OBJECTIVES

The main purpose of this work is to formulate a new two-point boundary value problem (TPBVP) to obtain bi-impulsive Earth-Moon trajectories based on the planar bi-circular restricted four-body problem with a prescribed value of the Jacobi integral at the final time and with analytical expressions for the velocities increments. A variational equation for the Jacobi integral in the four-body model is deduced in order to take it as an additional state variable in the description of the dynamics of the space vehicle. Also, a procedure is proposed to obtain types of trajectories with decreasingly fuel consumption.

3. MATHEMATICAL FORMULATION

The following hypotheses are considered to formulate the Earth-Moon transfer problem (da Silva Fernandes and Marinho, 2012) based on the restricted four-body model:

1. Earth and Moon describe circular orbits around the barycenter B of the Earth-Moon system;
2. The barycenter B describes a circular orbit around the barycenter C of the Earth-Moon-Sun system;
3. The flight of the space vehicle lies on the orbital plane of the Moon around the barycenter B;
4. The motion of the space vehicle is influenced by the gravitational fields of the Earth, the Moon and the Sun;
5. The gravitational fields of Earth, Moon and Sun are central and obey the inverse square law;
6. Only bi-impulsive trajectories are considered. The first velocity increment, Δv_{LEO}, applied at the departure time in the LEO, is accelerative and it inserts the space vehicle in a transfer trajectory. The second velocity increment, Δv_{LMO}, applied at the final time, decelerates the space vehicle circularizing its motion in the LMO.

Assuming that the motion of Earth, Moon, Sun and space vehicle are coplanar, and, assuming that the mass of the space vehicle is infinitesimal, the equations of motion of the space vehicle are given by the following expressions in a reference frame B_{XY} centered at the barycenter B

$$\ddot{x}_P = -\frac{\mu_E}{r_{EP}^3}(x_P - x_E) - \frac{\mu_M}{r_{MP}^3}(x_P - x_M) - \frac{\mu_S}{r_{SP}^3}(x_P - x_S) - \frac{\mu_S}{a_S^2}\cos(\omega_S t + \theta_{S0}) \quad (1)$$

$$\ddot{y}_P = -\frac{\mu_E}{r_{EP}^3}(y_P - y_E) - \frac{\mu_M}{r_{MP}^3}(y_P - y_M) - \frac{\mu_S}{r_{SP}^3}(y_P - y_S) - \frac{\mu_S}{a_S^2}\cos(\omega_S t + \theta_{S0}) \quad (2)$$

where the subscripts E, M and S denote quantities related to Earth, Moon and Sun, respectively. In this way, μ_E, μ_M and μ_S are the gravitational parameters of the respective bodies. In the B_{XY} reference frame, the positions of Earth, Moon, Sun and space vehicle are defined by the vectors $\mathbf{r}_E = (x_E, y_E)$, $\mathbf{r}_M = (x_M, y_M)$, $\mathbf{r}_S = (x_S, y_S)$ and $\mathbf{r}_P = (x_P, y_P)$. The dot symbol above the variables indicates time derivative expressions. r_{EP}, r_{MP} and r_{SP} are the radial distances from the space vehicle to Earth, Moon and Sun, respectively; and, θ_{S0} and ω_S are, respectively, the initial phase angle of the Sun and the angular velocity of the Sun as seen by the B_{XY} reference frame.

Considering a counterclockwise LEO, the initial conditions $t = t_0$ for the differential equation system, Eq. (1) and Eq. (2), are:

$$x_P(0) = x_{EP}(0) + x_E(0) = r_{EP}(0)\cos(\theta_{EP}(0)) + x_E(0) \quad (3)$$

$$y_P(0) = y_{EP}(0) + y_E(0) = r_{EP}(0)\sin(\theta_{EP}(0)) + y_E(0) \quad (4)$$

$$\dot{x}_P(0) = \dot{x}_{EP}(0) + \dot{x}_E(0) = -\left[\sqrt{\frac{\mu_E}{r_{EP}(0)}} + \Delta v_{LEO}\right]\sin(\theta_{EP}(0)) + \dot{x}_E(0) \quad (5)$$

$$\dot{y}_P(0) = \dot{y}_{EP}(0) + \dot{y}_E(0) = \left[\sqrt{\frac{\mu_E}{r_{EP}(0)}} + \Delta v_{LEO}\right]\cos(\theta_{EP}(0)) + \dot{y}_E(0) \quad (6)$$

where $\theta_{EP}(0)$ is the initial phase angle of the space vehicle with Earth and with respect to the x-axis of the B_{XY} reference frame. Utilizing the initial conditions, given by Eqs. (3) – (6) in the expression of the Jacobi integral (Szebehely, 1967), the following analytical expression can be derived for the first velocity increment:

$$\Delta v_{LEO} = (\omega r_{EP}(0)) - \sqrt{\frac{\mu_E}{r_{EP}(0)}} \pm \left\{ 2\Big[\frac{1}{2}\omega^2\Big((r_{EP}(0))^2 + \Big(\frac{\mu D}{1+\mu}\Big)^2 - 2\frac{\mu D}{1+\mu} r_{EP}(0)\cos(\theta_{EP}(0))\Big) + \frac{\mu_E}{r_{EP}(0)} + \frac{\mu_M}{r_{MP}(0)}\Big] - C(0) \right\}^{\frac{1}{2}} \quad (7)$$

where the upper and lower signs come from the root square. Only the positive sign is chosen in order to Δv_{LEO} be positive. $C(0)$ is the value of the Jacobi integral at the initial time. By a similar procedure, the final state (at $t = T$) of the space vehicle can be substituted in the Jacobi integral expression in order to obtain the analytical expression for the second velocity increment as it follows

$$\Delta v_{LMO} = -\sqrt{\frac{\mu_M}{r_{MP}(T)}} \mp \left\{ (\omega r_{MP}(T)) \pm \left[\Big[\omega^2\Big((r_{MP}(T))^2 + \Big(\frac{D}{1+\mu}\Big)^2 + 2 r_{MP}(T)\frac{D}{1+\mu}\cos(\theta_{MP}(T) - \omega T)\Big) + \frac{2\mu_E}{r_{EP}(T)} + \frac{2\mu_M}{r_{MP}(T)}\Big] - C(T) \right]^{\frac{1}{2}} \right\} \quad (8)$$

where the first upper and lower signs refer, respectively, to a clockwise arrival at the LMO and to a counterclockwise arrival at the LMO. The second upper and lower signs come from the root square. Only two combinations of signs make sense for the results: the two upper signs for clockwise arrival or the two lower signs for counterclockwise arrival. $C(T)$ is the value of the Jacobi Integral at the final time.

The two-point boundary value problem (TPBVP) concerning to the determination of an Earth-Moon trajectory can be stated as it follows: "Given a value for the Jacobi integral C_f at the final time and given a value for the initial phase angle θ_{S0}, determine the set of variables $(\theta_{EP}(0), \Delta v_{LMO}, T, C(0))$ which satisfies the following constraints:

$$g_1 : \ (x_P(T) - x_M(T))^2 + (y_P(T) - y_M(T))^2 - (r_{MP}(T))^2 = 0 \quad (9)$$

$$g_2 : \ (\dot{x}_P(T) - \dot{x}_M(T))^2 + (\dot{y}_P(T) - \dot{y}_M(T))^2 - \left[\sqrt{\frac{\mu_M}{r_{MP}(T)}} + \Delta v_{LMO}\right]^2 = 0 \quad (10)$$

$$g_3 : \ (x_P(T) - x_M(T))(\dot{y}_P(T) - \dot{y}_M(T)) - (y_P(T) - y_M(T))(\dot{x}_P(T) - \dot{x}_M(T)) - \left(\mp r_{MP}(T)\left[\sqrt{\frac{\mu_M}{r_{MP}(T)}} + \Delta v_{LMO}\right]\right) = 0 \quad (11)$$

$$g_4 : \ C(T) = C_f \quad (12)$$

with the initial conditions defined by Eqs. (3) – (6), in which the first velocity increment Δv_{LEO} is calculated utilizing Eq. (7)." Initially, Δv_{LMO} can be estimated by Eq. (8). The evaluation of the constraint g_4 depends on the calculus of $C(T)$. It is convenient to include an additional differential equation, which describes the temporal evolution of the Jacobi integral, in the system defined by Eq. (1) and Eq. (2). This new differential equation corresponds to the time derivative expression of the Jacobi integral yielding the following expression

$$\dot{C}(t) = 2\left(-\frac{\mu_E}{r_{EP}^2}\dot{r}_{EP} - \frac{\mu_M}{r_{MP}^2}\dot{r}_{MP} \right) - 2(\dot{x}_P\ddot{x}_P + \dot{y}_P\ddot{y}_P) - 2\omega(\ddot{x}_P y_P - x_P\ddot{y}_P) \quad (13)$$

where the variables $r_{EP} = r_{EP}(t)$, $r_{MP} = r_{EP}(t)$, $\dot{x}_P = \dot{x}_P(t)$, $\ddot{x}_P = \ddot{x}_P(t)$, $\dot{y}_P = \dot{y}_P(t)$, and $\ddot{y}_P = \ddot{y}_P(t)$ are evaluated at each time instant.

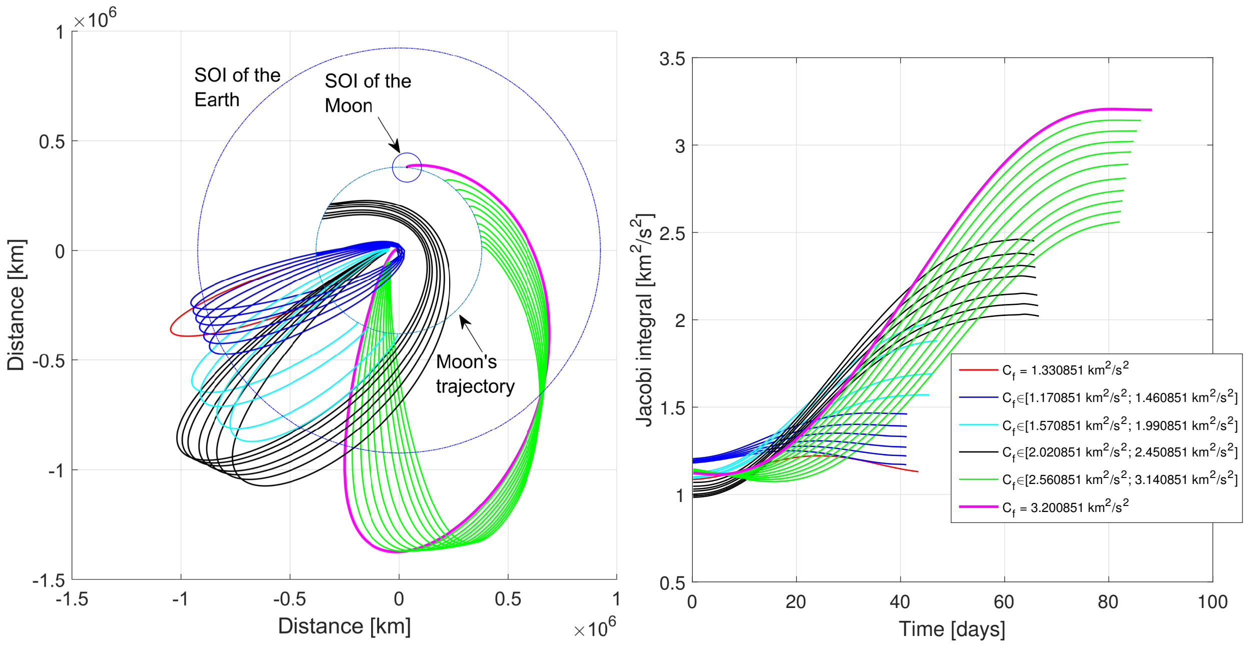
4. RESULTS

The analysis of the expressions given by Eq. (3) and Eq. (4) provides that if all the parameters in these expressions are constants but the Jacobi integral at the terminal times, then, the velocity increments decrease with the increase of the Jacobi integral. For instance, in Eq. (3), if larger values of $C(0)$ is set, smaller values of Δv_{LEO} is achieved independently from the values of the Jacobi integral of other time instant. The same occurs in Eq. (4), in which for larger values of $C(T)$, smaller values of Δv_{LMO} is achieved. Therefore, higher the Jacobi integral at the terminal times, smaller is the fuel consumption, which is represented by $\Delta v_{Total} = \Delta v_{LEO} + \Delta v_{LMO}$ (Marec, 1979). This facts motivates the procedure described below, in which the trajectories are obtained with close values of $C(0)$ and with increasingly values of $C(T)$. According to this discussion, trajectories with smaller fuel consumption are determined.

Utilizing the TPBVP stated in the previous section, the following procedure is proposed: a first trajectory with a given value of C_f is obtained through the solution of the TPBVP. This first solution is, then, taken as an initial guess to the same TPBVP, however with a value of C_f a little larger. The new solution is then utilized to obtain another solution with an increased value of C_f. By repeating this process, trajectories with increasingly values of C_f is determined. The TPBVP is solved by means of the Newton-Raphson algorithm. The procedure described above to obtain trajectories with increasingly values of C_f becomes more interesting if a heuristic over $C(0)$ during the iterations of the Newton-Raphson is set. Note that $C(0)$ is unknown and, therefore, it is iterated in the Newton-Raphson algoritm together with others three unknowns – $\theta_{EP}(0)$, Δv_{LMO} and T. This heuristic establishes a limit value for $C(0)$ not allowing a huge variation of it during the convergence of the results. Therefore, if $C(0)$ does not have a large variation and C_f has increasingly values, then, trajectories with smaller fuel consumption is obtained during this procedure. The increment between the original value of C_f and the new value C_f must be small (0.01 km^2/s^2, for instance) to allow the old solution be used as initial guess for the new solution. Figures 1 and 2 illustrate two sets of trajectory solutions determined by this procedure. For the first ans the second set of solutions, the initial phase angle θ_{S0} of the Sun is prescribed at $85.729°$ and $157.039°$, respectively.

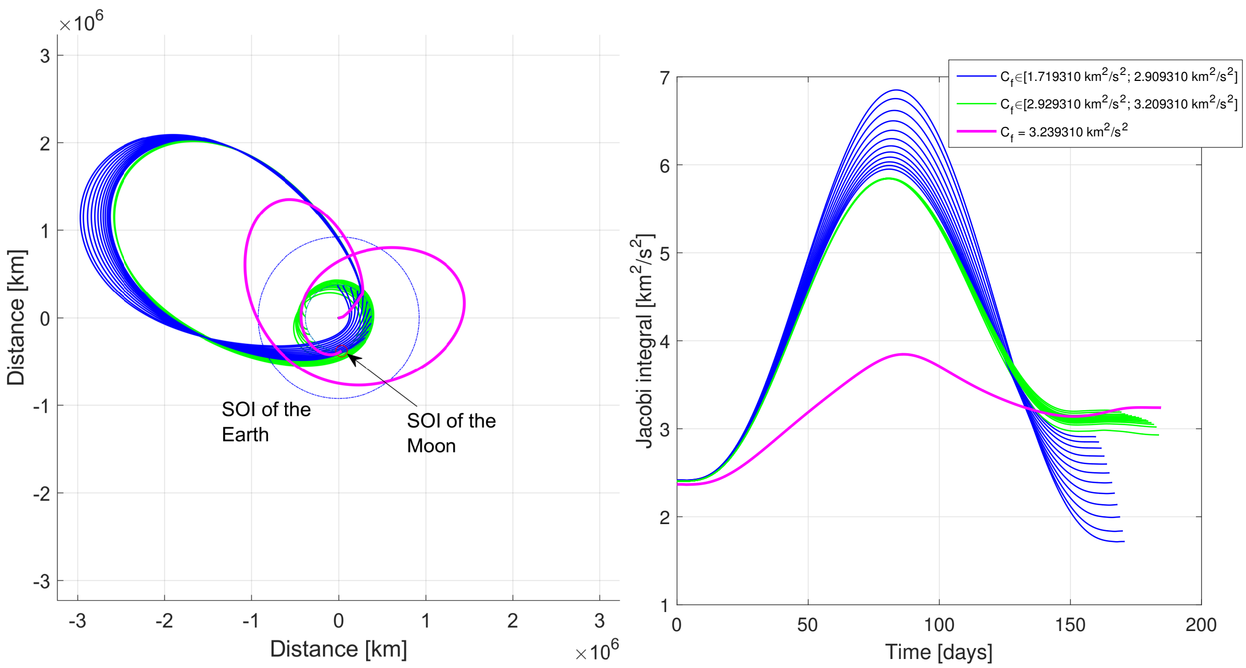
Note that this procedure allows the convergence of trajectories with smaller fuel consumption by obtaining different types of trajectories. The types of trajectories are depicted by color in Figs. 1 and 2. The trajectories with the smallest fuel consumption is depicted in magenta color. In this way, the smallest fuel consumption of the first set of solutions (Fig. 1) corresponds to a trajectory with $\Delta v_{Total} = 3.859124\ km/s$ with a time of flight equal to 88.264 $days$; and, the smallest fuel consumption of the second set of solution (Fig. 2) corresponds to a trajectory with $\Delta v_{Total} = 3.784136\ km/s$ with a time of flight equal to $184.545\ days$. Particularly, in the second set of solutions, Fig. 2, the trajectories perform an intermediary lunar swing-by maneuver. Figures 1b and 2b show the behavior of the time evolution of the Jacobi integral of such trajectories. Note that, for both Figs. 1b and 2b, the initial value C_0 of the Jacobi integral for each set of solution is almost the same. As the trajectory evolves in time, the value of the Jacobi integral is changed. For the final time, the value of the Jacobi integral is C_f and it is different for each trajectory in one set of solution: the trajectory with the smallest fuel consumption has the largest value of C_f.

Table 1 and Tab. 2, represents, respectively, the results of trajectories of the first and third set of solutions. Also, these tables highlights in yellow the trajectories that have negative Kepler's energy at the arrival of the LMO, characterizing a ballistic capture. From the last column of these tables, it is possible to observe if the ballistic capture occurs by the Lagrangean point L_1 or L_2. In this way, for both set of trajectories, the ballistic captures are performed by the L_2. This fact enhances the importance of the gravitational influence of the Sun for the occuring of such trajectories.



(a) Trajectories at the inertial reference frame B_{XY}. (b) Time evolution of Jacobi integral.

Figure 1: First set of solution with prescribed C_f.



(a) Trajectories at the inertial reference frame B_{XY}. (b) Time evolution of Jacobi integral.

Figure 2: Third set of solution with prescribed C_f.

Table 1: Results of the trajectories depicted by Fig. 1. Counterclockwise arrival.

Δv_{LEO} $[km/s]$	Δv_{LMO} $[km/s]$	Δv_{Total} $[km/s]$	Time of flight $[days]$	$\theta_{EP}(0)$ $[degrees]$	C_0 $[km^2/s^2]$	C_f $[km^2/s^2]$	Arrival[1]
3.201777	1.073940	4.275717	43.388	18.204	1.085680	1.130851	L_2
3.197439	1.066517	4.263956	41.070	13.654	1.181008	1.170851	L_1
3.197301	1.057224	4.254525	41.092	15.657	1.184025	1.220851	L_1
3.197141	1.047899	4.245040	41.114	17.654	1.187539	1.270851	L_1
3.196920	1.036665	4.233585	41.145	20.072	1.192393	1.330851	L_1
3.196668	1.025385	4.222052	41.184	22.545	1.197942	1.390851	L_1
3.196332	1.012163	4.208495	41.242	25.550	1.205321	1.460851	L_1
3.201365	0.991262	4.192627	45.442	30.381	1.094741	1.570851	L_2
3.201072	0.968257	4.169328	46.022	33.758	1.101183	1.690851	L_2
3.200403	0.931409	4.131812	47.010	39.684	1.115882	1.880851	L_2
3.199845	0.909831	4.109677	47.658	43.834	1.128128	1.990851	L_2
3.206419	0.903908	4.110327	66.562	37.325	0.983640	2.020851	L_1
3.206023	0.892033	4.098057	66.405	38.333	0.992341	2.080851	L_1
3.205597	0.880103	4.085700	66.249	39.449	1.001715	2.140851	L_1
3.204802	0.860091	4.064893	66.000	41.633	1.019196	2.240851	L_1
3.204259	0.848006	4.052265	65.863	43.214	1.031127	2.300851	L_1
3.203538	0.833833	4.037371	65.731	45.451	1.046968	2.370851	L_1
3.202533	0.817534	4.020067	65.662	48.897	1.069067	2.450851	L_1
3.199059	0.794946	3.994005	82.070	98.874	1.145404	2.560851	L_2
3.199051	0.782534	3.981586	82.324	97.294	1.145563	2.620851	L_2
3.199057	0.770059	3.969116	82.598	95.666	1.145438	2.680851	L_2
3.199077	0.757518	3.956595	82.893	93.982	1.144993	2.740851	L_2
3.199122	0.742802	3.941924	83.272	91.936	1.144014	2.810851	L_2
3.199206	0.725872	3.925078	83.765	89.467	1.142179	2.890851	L_2
3.199313	0.710958	3.910271	84.267	87.166	1.139807	2.960851	L_2
3.199438	0.698098	3.897536	84.772	85.061	1.137062	3.020851	L_2
3.199601	0.685166	3.884768	85.382	82.798	1.133483	3.080851	L_2
3.199814	0.672162	3.871976	86.162	80.328	1.128809	3.140851	L_2
3.200049	0.659074	3.859124	88.264	78.250	1.123637	3.200851	L_2

[1] It defines if the entrance of the space vehicle into the SOI of the Moon occurs by L_1 Lagrangean point or by L_2 Lagrangean point.

Table 2: Results of the trajectories depicted by Fig. 2. Clockwise arrival.

Δv_{LEO} $[km/s]$	Δv_{LMO} $[km/s]$	Δv_{Total} $[km/s]$	Time of flight $[days]$	$\theta_{EP}(0)$ $[degrees]$	C_0 $[km^2/s^2]$	C_f $[km^2/s^2]$	Arrival[(1)]
3.140981	0.952971	4.093952	170.696	233.982	2.418059	1.719310	L_1
3.141028	0.929711	4.070739	169.978	233.911	2.417025	1.839310	L_1
3.141095	0.898365	4.039460	168.934	233.812	2.415557	1.999310	L_1
3.141157	0.870615	4.011772	167.932	233.723	2.414216	2.139310	L_1
3.141214	0.844570	3.985785	166.922	233.641	2.412955	2.269310	L_1
3.141267	0.820284	3.961551	165.916	233.567	2.411810	2.389310	L_1
3.141313	0.797808	3.939121	164.923	233.503	2.410801	2.499310	L_1
3.141353	0.777194	3.918547	163.945	233.447	2.409933	2.599310	L_1
3.141386	0.758489	3.899876	162.981	233.401	2.409199	2.689310	L_1
3.141418	0.739637	3.881055	161.898	233.357	2.408507	2.779310	L_1
3.141442	0.724871	3.866313	160.914	233.323	2.407976	2.849310	L_1
3.141467	0.712141	3.853608	159.764	233.289	2.407429	2.909310	L_1
3.141541	0.707890	3.849430	183.777	233.196	2.405820	2.929310	L_2
3.141534	0.688631	3.830164	182.867	233.201	2.405975	3.019310	L_2
3.141532	0.682175	3.823707	181.900	233.202	2.406006	3.049310	L_2
3.141532	0.677861	3.819393	180.923	233.202	2.406015	3.069310	L_2
3.141532	0.673539	3.815071	179.667	233.202	2.406017	3.089310	L_2
3.141532	0.671375	3.812907	178.936	233.201	2.406015	3.099310	L_2
3.141532	0.667042	3.808574	177.218	233.200	2.406009	3.119310	L_2
3.141532	0.664871	3.806403	176.168	233.200	2.406005	3.129310	L_2
3.141532	0.662699	3.804231	174.917	233.199	2.406003	3.139310	L_2
3.141532	0.660524	3.802057	173.448	233.199	2.406005	3.149310	L_2
3.141532	0.658348	3.799880	172.007	233.199	2.406017	3.159310	L_2
3.141531	0.656170	3.797700	170.885	233.200	2.406039	3.169310	L_2
3.141528	0.651808	3.793335	169.487	233.202	2.406103	3.189310	L_2
3.141523	0.647438	3.788961	168.786	233.207	2.406209	3.209310	L_2
3.143266	0.640870	3.784136	184.545	230.812	2.368113	3.239310	L_2

(1) It defines if the entrance of the space vehicle into the SOI of the Moon occurs by L_1 Lagrangean point or by L_2 Lagrangean point.

5. CONCLUSION

This work proposes a new two-point boundary value problem with a prescribed value of the Jacobi integral at the final time to determine Earth-Moon trajectories considering the planar circular restricted four-body model. The expression of the Jacobi integral is utilized with the initial conditions of the differential equation of motion of the space vehicle to deduce analytical expressions for the velocity increments. In this way, an estimate of the fuel consumption is possible before the determination of the trajectories. Also, a variational equation for the Jacobi integral in the four-body model is deduced in order to take it as an additional state variable in the description of the dynamics of the space vehicle. Despite the determination of trajectories is done through a two-point boundary value problem, a procedure is proposed in which types of trajectories with decreasingly fuel consumption are obtained. In this way, external trajectories of Belbruno type, with ballistic capture by the L_2 Lagrangean point, are obtained. Also, a trajectory with a fuel consumption of $\Delta v_{Total} = 3.784136\ km/s$ and with a time of flight equal to $184.545\ days$ is determined by performing an intermediary lunar swing-by maneuver and a ballistic capture by the L_2 Lagrangean point. Moreover, this study allows an analysis of the Kepler's energy at the arrival of the LMO with the Jacobi integral and the Hill's regions which will be done in future work. This kind of study permits to establish some limits values of the Jacobi integral about possibilities of ballistic capture by the Moon.

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