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Behavior of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms as objective functions in the positioning of active elements in vibration control

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Abstract. *In this paper, the problem of finding the optimal solution for the optimization problem of placement of non-collocated sensors and actuators in a mechanical system is studied. To solve this problem, an algorithm based on Genetic Algorithms, named here as GAP Method, is used. In this method, the variables are adopted as binary and the crossover of one point and punctual mutation is applied. The difference between GAP Method and the traditional Genetic Algorithm is the verification of the generation of valid chromosomes. The goal is to evaluate the behavior of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms as objective functions in the positioning problem of actuators and sensors, i.e., the goal is to obtain the positions of the active elements from the analysis of these two norms. Furthermore, the aim is to verify the computational processing time.*

Keywords: *active elements, Genetic Algorithms, $\mathcal{H}_\infty$ norm and $\mathcal{H}_2$ norm*

1. INTRODUCTION

One of the challenges to solve the positioning problem of sensors and actuators in vibration control is to define an optimization method that achieves the optimal solution but is not expensive in computational terms. For discrete positioning, the purely combinatorial method ensures the optimal solution of the problem, but the processing time can be high in cases the problem has many possible positions to be analyzed in order to allocate the active elements.

Genetic Algorithm (GA) can be an alternative to solve this kind of optimization problem (Bruant et al. (2010); Chakraborty et al. (2012)). GA is an optimization technique developed based on biology. In this technique, the optimization variables are named chromosomes, which can also be called individuals. GA is implemented using techniques inspired by evolutionary biology such as crossover (exchange of genetic information between two individuals) and mutation (changes in the genetic material of the chromosome).

The positioning problem of active elements was studied by many authors as shown in Gupta et al. (2010). In the referred paper, the authors present many optimization criterion used for researchers to solve the allocation problem. According to Guo et al. (2004), the problem of optimal allocation of the active elements involves two important decisions that are the definitions of the objective functions and the restrictions that need to be adopted, and the choice of the optimization method to be used to obtain the solution.

Some objective functions already used to solve the positioning problem are based on gramiams of observability and controllability. Another possibility is to use as objective function the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of the systems (Gawronski, 2004).

The problem of finding the placement of active elements on a continuous structure is presented in Arabyan and Chemishkian (1998). In their paper, the goal was to minimized the $\mathcal{H}_\infty$ norm to reduce the vibrations of the flexible structure. The authors Ambrosio et al. (2012) used the $\mathcal{H}_2$ norm to solve the allocation problem. And Liu et al. (2006) used the spatial $\mathcal{H}_2$ norm as objective function and GA to solve the allocation problem.

The problem of finding the optimal solution of the discrete optimization problem for the positioning of sensors and actuators is studied in this paper. The interest is to obtain the positions of the active elements from the analysis of the closed-loop system, i.e., the simultaneous determination of the positioning of active elements, from an optimization problem, and the controller design.

In this paper, the positioning problem is solved using an algorithm based on GA (Soubhia and Serpa, 2017), and the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms are used as objective functions. The goal is to obtain the allocation of the active elements and to compare the results using these two norms.

2. PROBLEM DEFINITION

The sensors and actuators positioning optimization problem consists of defining the best place to allocate the active elements, with the goal of achieve the best system behavior. However, depending on the optimization problem, the possible places to allocate the actuators and sensors can be large and a combinatorial technique can be prohibitive due to the number of possible combinations. The number of possibilities is calculated using the equation of the combination without repetition as follow

$$C^{n_{a,s}}_{d_{p_{a,s}}} = \frac{n_a\,!}{d_{p_a}\,!(n_a - d_{p_a})\,!} \times \frac{n_s\,!}{d_{p_s}\,!(n_s - d_{p_s})\,!}; \tag{1}$$

where n_a and n_s are the total numbers of possible positions of actuators and sensors, respectively, d_{p_a} is the desired number of actuators and d_{p_s} is the desired number of sensors. If the number of combinations (Eq. (1)) that needs to be evaluated is huge, the computational time can be very high.

The goal of this paper is to obtain the placement of the active elements in discrete positions using an optimization technique. In this paper, a technique based on Genetic Algorithms (GA) is presented in order to reduce the number of combinations. The $\mathcal{H}_\infty$ norm and $\mathcal{H}_2$ are used as objective functions. The goal is to compare the results and the computational processing time using these norms.

3. CONTROL SYSTEMS

In this section, the control systems studied in this paper are presented. After, the $\mathcal{H}_\infty$ norm and $\mathcal{H}_\infty$ control and $\mathcal{H}_2$ norm and $\mathcal{H}_2$ control will be treated.

Figure 1 shows the usual block diagram to represent the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control. In this figure, $w(t)$ represents the external disturbance vector, $z(t)$ represents the vector of the system performances, $u(t)$ represents the control input vector (actuators inputs) and $y(t)$ represents the output vector (sensors outputs) that are used for dynamic output feedback. It is possible to write the system state-space model as

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t); \tag{2}$$
$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t); \tag{3}$$
$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t); \tag{4}$$

where all the matrices have compatible dimensions. A is the dynamic matrix of the system and matrices D_{11} and D_{22} are null in this work, reflecting that disturbances are not directly related to the performance and control signal does not affect directly the measured signal.

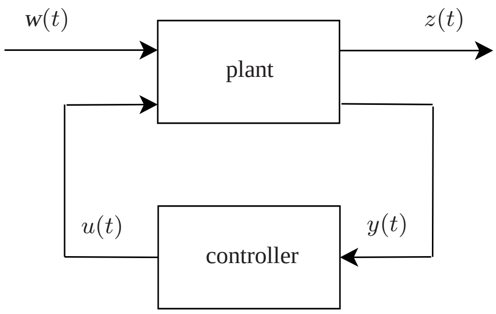

Figure 1. Output feedback control scheme

The system described by equations (2), (3) and (4) can be written using transfer functions as

$$Z(s) = P_{zw}(s)W(s) + P_{zu}(s)U(s);$$
$$Y(s) = P_{yw}(s)W(s) + P_{yu}(s)U(s);$$
$$U(s) = K(s)Y(s).$$

In the case of multiple-inputs and multiple-outputs (MIMO) system, $P_{zw}(s)$ is the transfer matrix between the output $Z(s)$ and the input $W(s)$; $P_{zu}(s)$ is the transfer matrix between the output $Z(s)$ and the input $U(s)$; $P_{yw}(s)$ is the transfer matrix between the output $Y(s)$ and the input $W(s)$; and $P_{yu}(s)$ is the transfer matrix between the output $Y(s)$ and the input $U(s)$.

The relation between $W(s)$ and $Z(s)$ of the closed-loop system can be written as

$$Z(s) = T_{zw}(s)W(s);$$

where

$$T_{zw}(s) = P_{zw}(s) + P_{zu}(s)K(s)(I - P_{yu}(s)(K)(s))^{-1}P_{yw}(s). \quad (5)$$

The function $T_{zw}(s)$ is called inferior fractional linear transformation and corresponds to the transfer matrix between the disturbance input $W(s)$ and the performance output $Z(s)$.

3.1 $\mathcal{H}_\infty$ Norm and $\mathcal{H}_\infty$ Control

In the case of MIMO systems, the $\mathcal{H}_\infty$ norm corresponds to the maximum system singular value of the transfer matrix between the disturbance input and the performance output. The idea of the $\mathcal{H}_\infty$ control is to reduce the effects of external disturbances keeping the system performance. This control technique minimizes the $\mathcal{H}_\infty$ norm of the system, that corresponds to the peak of the amplitude of the frequency response in the case of the single-input and single-output (SISO) systems, or in the case of MIMO systems, to reduce the peak of the maximum singular value (Zhou and Doyle, 1996).

One possibility to express the relation between the input and the output of the system is using transfer functions. The $\mathcal{H}_\infty$ norm of the transfer matrix $T_{zw}(s)$ (Eq. (5)), in the case of a MIMO stable system, is defined as

$$||T_{zw}(s)||_\infty = \sup_w \sigma_{max}(T_{zw}(jw)); \quad (6)$$

where $\sigma(\cdot)$ is the maximum singular value of the matrix. In this case, the $\mathcal{H}_\infty$ norm corresponds to the maximum system singular value.

The $\mathcal{H}_\infty$ controller goal is to determine the controller $K(s)$ that minimizes the $\mathcal{H}_\infty$ norm from the disturbance to the performance, i.e.,

$$\min_{K(s)} ||T_{zw}(s)||_\infty; \quad (7)$$

which corresponds to the closed-loop system frequency response peak minimization. The controller $K(s)$ can be a dynamic linear model that has $y(t)$ as input and $u(t)$ as output (Figure 1), according to the state-space model

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t); \quad (8)$$
$$u(t) = C_c x(t) + D_c y(t); \quad (9)$$

where all the matrices have compatible dimensions. A_c is the dynamic matrix of the controller and $x_c(t)$ is the state-space vector of the controller.

3.2 $\mathcal{H}_2$ Norm and $\mathcal{H}_2$ Control

The $\mathcal{H}_2$ norm is the energy of the output of the system with relation to the disturbance input. Figure 1 shows the block diagram and Eq. (2), (3) and (4) represent the state-space model of $\mathcal{H}_2$ control.

The $\mathcal{H}_2$ norm in the frequency domain is given by (Doyle et. al, 1989)

$$||T_{zw}(s)||_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} trace[T_{zw}(jw)^* T_{zw}(jw)]\, dw \right)^{\frac{1}{2}}; \quad (10)$$

where $T_{zw}(s)$ is the inferior fractional linear transformation and corresponds to the transfer matrix between the disturbance input $W(s)$ and the performance output $Z(s)$, shown in Eq. (5).

The $\mathcal{H}_2$ controller goal is to determine the controller $K(s)$ that minimizes the $\mathcal{H}_2$ norm from the disturbance to the performance, i.e.,

$$\min_{K(s)} ||T_{zw}(s)||_2; \quad (11)$$

where controller $K(s)$ can be a dynamic linear model, according to the state-space model given in Eq. (8) and (9).

The minimization problems shown in (7) and (11) can be solved using techniques based on the Riccati algebraic equations (Gawronski, 2004) or using techniques based on Linear Matrix Inequalities (LMI) (Boyd et al., 1994).

According to Doyle et. al (1989), for computing the $\mathcal{H}_2$ norm is necessary just to solve a linear equation, and for computing the $\mathcal{H}_\infty$ norm is necessary to solve a optimization problem. Because of this, computing the $\mathcal{H}_\infty$ norm can be more expensive in computational terms. Moreover, to obtain the controller $\mathcal{H}_2$ is necessary to solve two Riccati equations and to obtain the controller $\mathcal{H}_\infty$ is necessary to solve the Riccati equations subject to a line search process.

3.3 Sensor and Actuator Placement Optimization Problem

The goal of this paper is to find the optimal positions of the actuators and sensors using discrete techniques. The objective functions that are used in this paper are the $\mathcal{H}_\infty$ norm of the system (Eq. (6)) and the $\mathcal{H}_2$ norm (Eq. (10)). Therefore, two optimization problems are investigated in this paper, that is,

$$\min_{p_a,p_s} ||T_{zw}||_\infty \qquad \text{and} \qquad \min_{p_a,p_s} ||T_{zw}||_2; \tag{12}$$

where p_a is the position of the actuator and p_s is the position of the sensor.

In this paper, the goal is to obtain the discrete positions of the active elements and the same time to obtain the controllers using the theory of $\mathcal{H}_\infty$ control and the theory of $\mathcal{H}_2$ control. The discrete optimization technique applied here to obtain the positions of the actuators and sensors is based on Genetic Algorithms.

4. GENETIC ALGORITMS

GA was developed by Holland (1975). According to Haupt and Haupt (2004), GA has some advantages, for example: it optimizes problems with continuous and discrete variables; it does not require informations about gradients of the objective function and of the constraints, and it works for problems that have a large number of variables.

Before starting the iterations of the GA, it is necessary to define a representation for the variables of the optimization problem. In the case of GA, the optimization variables are treated as chromosomes. After the definition of the chromosomes, the initial population that needs to be evaluated should be chosen. This choice can be done randomly.

Then, for all chromosomes of this population, the objective function of the problem is evaluated. In this way, fittest individuals are referred to the step of the genetic operations. These individuals can be chosen using elitist technique (Haupt and Haupt, 2004).

After selecting the most suitable individuals, pairs can be formed randomly, so that the genetic operators can be applied. The genetic operators used by GA are the crossover and the mutation. Usually, crossover of one point and mutation of one point are used (Haupt and Haupt, 2004).

As in other algorithms, for Genetic Algorithms it is necessary to determine stopping criteria. One of the usual criteria is the setting of a maximum number of generations or a processing time limit. Another criterion may be related to the number of repetitions of the best solution obtained.

4.1 GA Positioning Method (GAP)

The method used here was applied in Soubhia and Serpa (2017). In this case, the authors adopted the optimization variables as the possible positions to allocate the active elements, thus these variables are discrete.

The difference between GAP Method and the traditional GA is the verification of the generation of valid chromosomes after the mutation. In the case of not valid genes appeared, these are replaced by valid representations chosen randomly (Soubhia and Serpa, 2017).

This algorithm is described in the following (Algorithm 1) and runs until stop conditions are reached.

Algorithm 1 GAP Method

1. Generate randomly the initial population with N_{pop} individuals.
2. Design the controller ($\mathcal{H}_\infty$ or $\mathcal{H}_2$) and evaluate the norm ($\mathcal{H}_\infty$ or $\mathcal{H}_2$) for each individual of population.
3. Select from the elitist technique the N_{otim} fittest individuals.
4. Generate new parents randomly.
5. Perform the crossover of one point in the chromosome.
6. Perform the punctual mutation in the chromosome.
7. Feasibility test: if there is infeasible individuals, go to step 9. Otherwise, go to step 8.
8. Replace the non feasible individuals for feasible individuals and go to step 9.
9. If predefined conditions are achieved, END. Otherwise, go back to step 2.

5. NUMERICAL EXAMPLES

In this section, a system with 12 degrees of freedom is investigated. In this case, two problems were studied: the positioning problem of three actuators and three sensors and the allocating problem of two actuators and three sensors. In both cases, the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms were used as objective functions (Eq. (12)).

Then, a system with 50 degrees of freedom was investigated. In this case, just the $\mathcal{H}_2$ norm was used as objective function.

5.1 System with 12 degrees of freedom

The GAP Method was tested and evaluated in a mass-spring-damping system with 12 degrees of freedom (DOF) as the system presented in Figure 2, with $N = 12$. The values used for mass, damping coefficient and springs stiffness were defined as $m_1 = m_2 = \ldots = m_{12} = 1$ kg, $c_1 = c_2 = \ldots = c_{13} = 0.10$ Ns/m, and $k_1 = k_2 = \ldots = k_{13} = 10^4$ N/m.

Considering the states as displacements and velocities, the state-space model has 24 states. In this example, the disturbance was considered in all degrees of freedom of the system, i.e., in masses m_1, m_2,...,m_{12}. The performance outputs was considered as displacements of the masses m_3, m_6 and m_9.

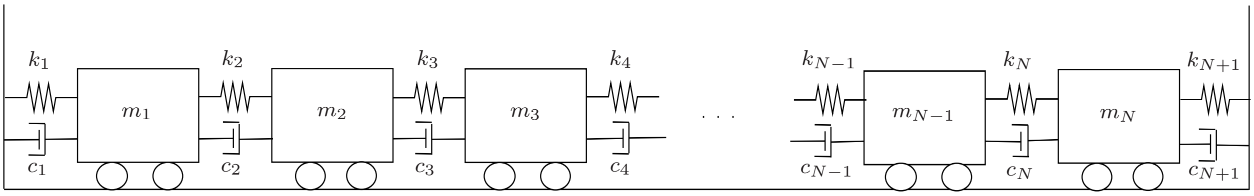

Figure 2. Mass-spring-damping system with N degrees of freedom

According to equations (2), (3) and (4), the matrices of the state-space model of this system are:

$$A = \begin{bmatrix} 0_{12\times12} & I_{12\times12} \\ M^{-1}K & M^{-1}C \end{bmatrix}_{24\times24}; \tag{13}$$

$$B_1 = \begin{bmatrix} 0_{12\times12} \\ M^{-1}I_{12\times12} \end{bmatrix}_{24\times12}; \tag{14}$$

$$C_1 = \begin{bmatrix} 0_{1\times2} & I_{1\times1} & 0_{1\times21} \\ 0_{1\times5} & I_{1\times1} & 0_{1\times18} \\ 0_{1\times8} & I_{1\times1} & 0_{1\times15} \\ - & 0_{3\times24} & - \end{bmatrix}_{6\times24}; \tag{15}$$

$$D_{12} = \begin{bmatrix} 0_{3\times n_u} \\ I_{n_u\times n_u} \\ 0_{(3-n_u)\times n_u} \end{bmatrix}_{6\times n_u}; \tag{16}$$

where n_u is the number of control inputs, and

$$D_{21} = [0]_{n_y\times12}; \tag{17}$$

where M, C and K are, respectively, the mass, damping and stiffness matrices, I is the identity matrix and n_y is the number of sensors that is used.

With these coefficients and matrices, the $\mathcal{H}_\infty$ norm and the $\mathcal{H}_2$ norm of the open-loop system are, respectively, 4.0594 and 0.2268. The function *norm* of Matlab was used to obtain the $\mathcal{H}_\infty$ norm and $\mathcal{H}_2$ norm of the system.

The matrix B_2 is obtained from the positioning of actuators and the matrix C_2 is related to the positioning of sensors. These matrices vary with the changes of the positioning of the active elements, i.e., these matrices can vary in each iteration of the algorithm, since they reflect the decision variables of the problem. In this case, the possible positions to allocate the actuators and sensors are all the degrees of freedom of the system.

In this paper, Matlab was used to implement the algorithm presented. The functions *hinfsyn* and *h2syn* were used to design the controllers $\mathcal{H}_\infty$ and $\mathcal{H}_2$. The method adopted to solve the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimization problems was based on the solution of Riccati equations. The computer used to obtain the results was an Intel Core $i7-4790$ @3.60 GHz with 32 GB RAM memory.

GAP Method was applied for two different examples. First, the positioning problem of 3 actuators and 3 sensors was analyzed. Then, the allocation problem of 2 actuators and 3 sensors was studied. In both cases, the complete combinatorial method (method in that all the possible positions are evaluated, called here as CO Method) was applied to compare the optimal solution with the results obtained from GAP Method.

Here, the amount of individuals in the population was the same during all the iterations of the algorithms. The adopted stopping criterion was the amount of repetitions of the best response obtained during the optimization process. For the tests presented here, 10 repetitions were considered. The mutation rate used was 20%. These initial parameters were defined after some trials of the algorithm.

5.1.1 Three actuators and three sensors

Using the CO Method, 48400 combinations need to be evaluated to allocate three actuators and three sensors. For GAP Method, the value adopted for the initial population was $N_{pop} = 60$ and the amount of individuals selected to go to parents generation was $N_{otim} = 60$.

The results obtained, using $\mathcal{H}_\infty$ norm as objective function, from the CO Method and GAP Method are shown in Tab. 1. It is possible to verify that the positioning obtained, from both methods, corresponds to actuators in the DOF 6, 7 and 8, and sensors in the DOF 10, 11 and 12. With this configuration, the optimal $\mathcal{H}_\infty$ norm of the closed-loop system is 1.4165. But, in the case of the CO Method, the computational processing time was 47.14 hours. When the GAP Method was applied, the computational processing time was 0.97 hours and 1020 combinations were analyzed.

Table 1. Results $\mathcal{H}_\infty$ norm - 3 actuators and 3 sensors

Method	Actuator (DOF)	Sensor (DOF)	Time (h)	Combinations	$\|\|T_{zw}(s)\|\|_\infty$
CO	6, 7, 8	10, 11, 12	47.14	48400	1.4165
GAP	6, 7, 8	10, 11, 12	0.97	1020	1.4165

The results obtained using $\mathcal{H}_2$ norm as objective function are shown in Tab. 2. It is possible to verify that the positioning obtained, from both methods, corresponds to actuators in the DOF 5, 6 and 7, and sensors in the DOF 3, 6 and 9. With this configuration, the optimal $\mathcal{H}_2$ norm of the closed-loop system is 0.1683. In the case of the CO Method, the computational processing time was 0.33 hours. When the GAP Method was applied, the computational processing time was 0.01 hours and 1500 combinations were analyzed.

Table 2. Results $\mathcal{H}_2$ norm - 3 actuators and 3 sensors

Method	Actuator (DOF)	Sensor (DOF)	Time (h)	Combinations	$\|\|T_{zw}(s)\|\|_2$
CO	5, 6, 7	3, 6, 9	0.33	48400	0.1683
GAP	5, 6, 7	3, 6, 9	0.01	1500	0.1683

The GAP Method reached the exact solution of the problem, using the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms, with a smaller computational processing time compared with to CO Method. However, when the $\mathcal{H}_2$ norm is used as objective function, the exact solution is obtained in a significantly lower computational time.

5.1.2 Two actuators and three sensors

Using the CO Method, 14520 combinations need to be evaluated for positioning of three actuators and three sensors. For GAP Method, the value adopted for the initial population was $N_{pop} = 30$ and the amount of individuals selected to go to parents generation was $N_{otim} = 30$.

The results obtained, using $\mathcal{H}_\infty$ and $\mathcal{H}_2$, from the CO Method and GAP Method are shown in Tab. 3 and Tab. 4, respectively.

In the case of the $\mathcal{H}_\infty$ norm, from both methods, the optimal position corresponds to actuators in the DOF 6 and 7, and sensors in the DOF 1, 2 and 3. With this configuration, the optimal $\mathcal{H}_\infty$ norm of the closed-loop system is 1.6577. In the case of the CO Method, the computational processing time was 14.34 hours. When the GAP Method was applied, the computational processing time was 0.75 hours and 780 combinations were analyzed.

Table 3. Results $\mathcal{H}_\infty$ norm - 2 actuators and 3 sensors

Method	Actuator (DOF)	Sensor (DOF)	Time (h)	Combinations	$\|\|T_{zw}(s)\|\|_\infty$
CO	6, 7	1, 2, 3	14.34	14520	1.6577
GAP	6, 7	1, 2, 3	0.75	780	1.6577

When the $\mathcal{H}_2$ norm was evaluated, the optimal position obtained was the actuators allocated in the DOF 6 and 7, and sensors in the DOF 4, 7 and 10. With the CO Method, the computational processing time was 0.10 hours and with the GAP Method the computational processing time was 0.005 hours.

Table 4. Results $\mathcal{H}_2$ norm - 2 actuators and 3 sensors

Method	Actuator (DOF)	Sensor (DOF)	Time (h)	Combinations	$\|\|T_{zw}(s)\|\|_2$
CO	6, 7	4, 7, 10	0.10	14520	0.1770
GAP	6, 7	4, 7, 10	0.005	660	0.1770

As in the allocation problem of three actuators and three sensors, in the positioning problem of two actuators and three sensors with GAP Method was possible to achieve the optimal solution of the problem in a computational processing time lower than when the CO Method is applied. Furthermore, with the $\mathcal{H}_2$ norm as objective function, the computational time is smaller compared with the case when the $\mathcal{H}_\infty$ norm is used as the objective function.

As seen in the previous examples, when the $\mathcal{H}_2$ norm was used as objective function, the computational time was smaller compared with the $\mathcal{H}_\infty$ norm as objective function. These differences may occur because for obtaining the $\mathcal{H}_2$ controller is necessary just to solve two algebraic Riccati equations and to obtain the $\mathcal{H}_\infty$ controller is necessary to solve an optimization problem (Doyle et. al, 1989).

Therefore, the $\mathcal{H}_2$ norm was used to solve the positioning problem in a system with 50 degrees of freedom, whose the goal is to allocate two actuators and two sensors. This example is presented in the next section.

5.2 System with 50 degrees of freedom

The GAP Method was also tested and evaluated in a mass-spring-damping system with 50 degrees of freedom as presented in Figure 2 ($N = 50$). The values used for mass, damping coefficient and springs stiffness were defined as $m_1 = m_2 = \ldots = m_{50} = 1$ kg, $c_1 = c_2 = \ldots = c_{51} = 0.10$ Ns/m, and $k_1 = k_2 = \ldots = k_{51} = 10^4$ N/m.

Here the goal is to allocate two actuators and two sensors. The performance outputs was considered as displacements of the masses m_{10}, m_{15}, m_{20}, m_{25}, m_{30}, m_{35} and m_{40}. The disturbance was considered in all degrees of freedom of the system. According to equations (13), (14), (15), (16) and (17), the matrices of the state-space model of the system with 50 DOF are

$$A = \begin{bmatrix} 0_{50\times50} & I_{50\times50} \\ M^{-1}K & M^{-1}C \end{bmatrix}_{100\times100};$$

$$B_1 = \begin{bmatrix} 0_{50\times50} \\ M^{-1}I_{50\times50} \end{bmatrix}_{100\times50};$$

$$C_1 = \begin{bmatrix} 0_{1\times9} & I_{1\times1} & 0_{1\times40} \\ 0_{1\times14} & I_{1\times1} & 0_{1\times35} \\ 0_{1\times19} & I_{1\times1} & 0_{1\times30} \\ 0_{1\times24} & I_{1\times1} & 0_{1\times25} \\ 0_{1\times29} & I_{1\times1} & 0_{1\times20} \\ 0_{1\times34} & I_{1\times1} & 0_{1\times15} \\ 0_{1\times39} & I_{1\times1} & 0_{1\times10} \\ - & 0_{7\times50} & - \end{bmatrix}_{14\times100};$$

$$D_{12} = \begin{bmatrix} 0_{7\times2} \\ I_{3\times2} \\ 0_{4\times2} \end{bmatrix}_{14\times2};$$

$D_{21} = [0]_{2\times 50}$.

With these matrices, the $\mathcal{H}_2$ norm of the open-loop system is 2.6646.

Using the CO Method, 1500625 combinations were evaluated to allocate two actuators and two sensors. For GAP Method, the value adopted for the initial population was $N_{pop} = 500$ and the amount of individuals to go to parents generation was $N_{otim} = 500$.

The results obtained using $\mathcal{H}_2$ norm as objective function are shown in Tab. 5. It is possible to verify that the positioning obtained, from both methods, corresponds to actuators in the DOF 16 and 21, and sensors in the DOF 15 and 28. With this configuration, the optimal H_2 norm of the closed-loop system is 0.6997. In the case of the CO Method, the computational processing time was 80.70 hours. When the GAP Method was applied, the computational processing time was 0.87 hours and 16000 combinations were analyzed.

Table 5. Results $\mathcal{H}_2$ norm - 2 actuators and 2 sensors

Method	Actuator (DOF)	Sensor (DOF)	Time (h)	Combinations	$\|\|T_{zw}(s)\|\|_2$
CO	16, 21	15, 28	80.70	1500625	0.6997
GAP	16, 21	15, 28	0.87	16000	0.6997

6. CONCLUSIONS

In this paper, the behavior of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms as objective functions in the positioning problem of active elements applied in a mechanical system is analyzed.

First, a mechanical system with 12 degrees of freedom was studied. In this case, the combinatorial method and the GAP method was applied in the problem to allocate three actuators and three sensors. Then, the problem to allocate two actuators and three sensors was solved. In both cases, when the $\mathcal{H}_2$ norm was used as objective function the computational time was smaller compared with the $\mathcal{H}_\infty$ norm as objective function. This may occur because to design the $\mathcal{H}_\infty$ controller is necessary to solve an optimization problem. This optimization problem can turns the algorithm more expensive in computational terms compared with the design of $\mathcal{H}_2$ control, that depends on the solution of two Riccati equations.

In the allocation problems for the system of 12 degrees of freedom, when the $\mathcal{H}_\infty$ norm was used as objective function the computational time was higher than when the $\mathcal{H}_2$ norm was used as objective function. Due this the $\mathcal{H}_2$ norm was used as objective function in the problem to allocate two actuators and two sensors in a mechanical system with 50 degrees of freedom. In this case, the CO Method and GAP Method were applied and it was possible to see that the GAP Method found the optimal solution in smallest computational processing time.

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