POLYNOMIAL APPROXIMATIONS FOR SHEAR STRESSES AND VELOCITY GRADIENTS OF FLOWS IN PIPES

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Abstract. *The flow in pipes for non-Newtonian fluids is considered using a truncated series relating the shear stresses whit the radial velocity gradient. This approximation is used as an alternative way for the quantification of velocity profiles, and a lower order theoretical solution for the velocity is presented. It is observed that the calculated profiles allow approximations to pseudoplastic and to dilatant behaviors, approaching the Newtonian (parabolic) profile from the "inner side" or the "outer side", respectively. This suggest that power series may be used to quantify aspects of non-Newtonian fluids. In the sequence, the possibility of turbulent flows in pipes was considered, and a qualitative view of the velocity profiles and the turbulent shear stresses is then presented here. This study was conducted theoretically, aiming the obtainance of solutions that allow verifying mathematical possibilities and impossibilities (such as discontinuities). The results suggest further numerical studies to evidence the possibilities of this kind of approximation.*

1. INTRODUCTION

The study of non-Newtonian fluids is a matter of continuous interest, considering the production of fluidic materials that show interesting behaviors due to the intrinsic nonlinear dependence between shear stresses and velocity gradients. Simple "show examples" of these curious behaviors are distributed to the general public (see, for example https://www.youtube.com/watch?v=D-wxnID2q4A, can you walk on water?), but evidently the main interest of the study of non-Newtonian fluids resides in their applications in the process industry (see, for example, Chhabra and Richardson, 1999). These applications induce the search of possible solutions that may allow a further understanding, quantification and application of the observed effects.

During its continuous development, the branch of non-Newtonian fluids is being enriched with concepts and adequate models, that already allow classifying fluids accordingly their main characteristics. A classical sketch used in the description of the different fluids is reproduced in Fig. 1.

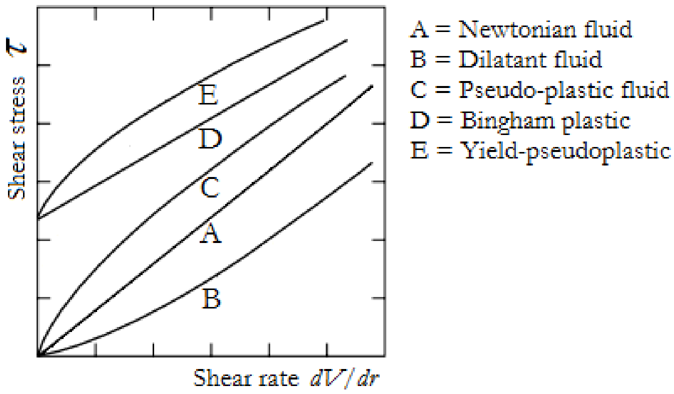

Figure 1. General definition of non-Newtonian fluids accordingly the behavior of the shear stress with the shear rates.

Chhabra and Richardson (1999) describe and exemplify the behavior of the different fluids mentioned in Fig. 1. Because of the multiple possibilities of describing similar behaviors, equations associated to specific characteristics are named accordingly these characteristics or accordingly the researchers that elucidated relevant aspects of that fluids. Perhaps the most classical example is the "Newtonian fluid", that follows a direct proportionality between shear stresses and shear rates (velocity gradients):

$$\tau = \mu\left(\frac{dV}{dr}\right) \tag{1}$$

τ is the shear stress, μ is the absolute viscosity, V is the velocity and r is the dimension perpendicular to the velocity. Along the decades, different names were included in the references, as the "power law fluid", also known as Ostwald – de Waele fluid, which follows the relationship:

$$\tau = K\left(\frac{dV}{dr}\right)^n \tag{2}$$

Now K is the consistency index, and n is the flow behavior index. If n<1 we have a pseudoplastic fluid, and if n>1 we have a dilatant fluid. A variation of equation (2) is the so called "Hershey-Buckley fluid", written as:

$$\tau = \tau_0 + K\left(\frac{dV}{dr}\right)^n \tag{3}$$

τ_0 is known as the yield stress. Other examples of fluid names, which consider the mathematical model followed to describe their properties, are Bogue (Prilutski et al., 1983), Carreau (Chhabra and Richardson), Prandtl-Eyring, Ellis, Reiner-Philippoff (Cavatorta and Tonini, 1987), Yasuda (Robertson, 2005), Cross (Cross, 1965), and Briant (Ippolito and Sabatino, 1985), among others. Table 1 shows five models for Non-Newtonian fluid with pseudoplastic and to dilatant behavior, and includes the dimensionless velocity profile. Some studies were devoted only to obtain analytical expressions for velocity profiles of different non-Newtonian fluids, and comparing the solutions with detailed experimental data (see, for example, Lucius and Roth, 1976). Further, time-independent and time-dependent properties are also described in the literature, and concepts like thixotropy and rheopexy are then introduced, related to the decresing and increasing of the apparent viscosity along time, respectively, when the fluid is subjected to a constant shear rate.

Table 1. Dimensionless velocity profile for Non-Newtonian fluids, adapted from Cavatorta and Tonini (1987).

Model	Rate of shear: $\frac{dV}{dr}$	Dimensionless Velocity Profile: $\frac{V}{V_0}$
Ostwald –de Waele	$(\tau / K)^{1/n}$	$1 - r^{+(n+1)/n}$
Prandtl-Eyring	$B\sinh(\tau / K)$	$\frac{\cosh\Phi - \cosh(\Phi r^+)}{\cosh\Phi - 1}$, $\Phi = \frac{\Delta pR}{2KL}$
Ellis	$\tau / K + \tau^\alpha / K'$	$\frac{\xi(1-r^{+2}) + \psi(1-r^{+\alpha+1})}{\xi+\psi}$ $\xi = \frac{\Delta pR}{4KL}$, $\psi = \frac{1}{(\alpha+1)K'}\left(\frac{\Delta pR}{2L}\right)^\alpha$
Reiner-Philippoff	$\frac{\tau[1+(\tau/K)^2]}{\mu_0 + \mu_\infty(\tau/K)^2}$	$\frac{\Theta_1\left(1-r^{+2}\right) - \frac{\Theta_1-\Theta_2}{\mu_\infty\Theta_1\kappa}\ln\left(\frac{\Theta_1\kappa r^{+2}+\Theta_2}{\Theta_1\kappa+\Theta_2}\right)}{\Theta_1 - \frac{\Theta_1-\Theta_2}{\mu_\infty\Theta_1\kappa}\ln\left(\frac{\Theta_2}{\Theta_1\kappa+\Theta_2}\right)}$, $\Theta_1 = \frac{\mu_\infty}{R^2}$, $\Theta_2 = \frac{\mu_0}{R^2}$, $\kappa = \frac{\Delta p}{2KL}$

The different ways of treating non-Newtonian fluids, which viscosity changes accordingly the flow condition, induced some researchers to also relate these mathematical models to turbulent flows. This extension to turbulent flows is due also to the dependence of the so called turbulent viscosity (eddy viscosity) on the fluid flow conditions. In the turbulent case, however, mean velocities are considered. One of such studies was presented by Brodkey et al. (1961), in which the authors considered the linear variation of the shear stress of a flow in a tube, and that the velocity may be expressed by a three term power series of the radial dimension. Only three terms were used because of the limited number of known boundary conditions. This model was also described in Brodkey (1967) and shows that truncated series may be a way of treating the flow of non-Newtonian or turbulent flows in pipes. The theoretical connection "turbulent/non-Newtonian" is mentioned also when comparing calculated velocity profiles, as done, for example, by Eu (1988). Additionally, the joint study of non-Newtonian and turbulent flows was also conducted experimentally, as shown for example by Bogue and Metzner (1963).

Schulz (2012) used a variation of the proposed expansion in power series of Brodkey et al. (1961) as a way to explore the possibilities of describing characteristics of non-Newtonian fluids for turbulent flows, but using a generalized power series representation of the Newton Law for shear stresses, in the form:

$$\tau = \sum_{i=0}^{\infty} \mu_i \left(\frac{dV}{dr}\right)^i \tag{4}$$

That is, the power law is used for the shear rate, or the velocity gradient. The usual Newton law is obtained for $\mu_1 \neq 0$ and $\mu_i = 0$, for i= 2, …∞. Having presented predictions and discussions for the set $\mu_1 \neq 0$ and $\mu_2 \neq 0$, Schulz (2012) introduced de possibilities of higher order approximations, eventually to be conducted numerically.

The present study shows theoretical results obtained considering a third order truncation of equation (4), that is, using

$$\tau = \sum_{i=0}^{3} \mu_i \left(\frac{dV}{dr} \right)^3 \tag{5}$$

The obtained results suggest more studies for higher order approximations, considering that the observed trends are compatible with the expectations.

2. PRESENTATION OF THE EQUATIONS

Equation (5) is rewritten in extended form in equation (6). Of course, there are limitations to solve equation (4) theoretically obtaining V for any exponent "i", because only lower order polynomials have theoretical solutions. For higher truncation orders, computer solutions are needed. As mentioned, in the present study, the possibility of a polynomial dependence between the shear stress and the velocity gradient was limited to the third order polynomial,

$$\tau = \mu_1 \left(\frac{dV}{dr} \right) + \mu_2 \left(\frac{dV}{dr} \right)^2 + \mu_3 \left(\frac{dV}{dr} \right)^3 \tag{6}$$

For laminar flows, μ_1 is usually s the dynamic viscosity, while μ_2 and μ_3 are coefficients introduced to take into account the effect of higher velocity gradients. Considering fully developed profiles, τ varies linearly with r, so that, when expressing equation (1) in nondimensional form, we have:

$$\left(\frac{dV^+}{dr^+} \right)^3 + \frac{m_2}{m_3} \left(\frac{dV^+}{dr^+} \right)^2 + \frac{m_1}{m_3} \left(\frac{dV^+}{dr^+} \right) = \frac{a}{m_3} r^+, \quad m_1 = \frac{V_0 \mu_1}{R}, \quad m_2 = \frac{V_0^2 \mu_2}{R^2}, \quad m_3 = \frac{V_0^3 \mu_3}{R^3}, \quad a = \frac{R \Delta p}{2L} \tag{7}$$

L is the length of the tube, Δp is the pressure difference, R is the radius of the tube, V_0 is the velocity at r=0, $r^+ = r/R$, and $V^+ = V/V_0$. This algebraic cubic equation was solved to obtain a first order equation for dV^+/dr^+ (equation 8a):

$$\frac{dV^+}{dr^+} = -\frac{m_2}{3m_3} + \sqrt[3]{\frac{-q(r^+) + \sqrt{q^2(r^+) + \frac{4p^3}{27}}}{2}} + \sqrt[3]{\frac{-q(r^+) - \sqrt{q^2(r^+) + \frac{4p^3}{27}}}{2}} \tag{8a}$$

Where:

$$p = \left[\frac{m_1}{m_3} - \frac{1}{3} \left(\frac{m_2}{m_3} \right)^2 \right], \quad q = 2b - 2cr^+, \quad 2b = \frac{2m_2^3 - 9m_1 m_2 m_3}{(3m_3)^3}, \quad 2c = \frac{a}{m_3}, \quad \lambda^2 = \frac{p^3}{27} \tag{8b}$$

Three solutions exist for the cubic equation (7), but in the present study only equation (8a) was considered in detail. Equation (7) was obtained remembering that the local shear stress $\tau(r)$ is a linear function of the distance to the center of the transversal section, that is, it is a linear function of the radius r. Additionally, the coefficients a, m_1, m_2, and m_3 were obtained through the normalization of V^+ and r^+. These coefficients have proper dimensions (kg/ms^2), but m_1/m_3, m_2/m_3 and a/m_3 are nondimensional coefficients. The definitions shown in equations (8b), which were used to simplify the notation of equation (8a), involve only nondimensional parameters. The integration of equation (8a) furnishes:

$$V^{+}=-\frac{m_2}{3m_3}r^{+}-\frac{3(\lambda/c)^{2/3}}{4}c^{1/3}\left\{\begin{aligned}&\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^{+}\right)^2}+\left(\frac{b}{c}-r^{+}\right)\right]^{2/3}+\\&+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^{+}\right)^2}-\left(\frac{b}{c}-r^{+}\right)\right]^{2/3}\end{aligned}\right\}+$$
$$+\frac{3}{8}c^{1/3}\left\{\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^{+}\right)^2}+\left(\frac{b}{c}-r^{+}\right)\right]^{4/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^{+}\right)^2}-\left(\frac{b}{c}-r^{+}\right)\right]^{4/3}\right\}+K \tag{9a}$$

K is an integration constant. Its value may be obtained by applying the boundary condition $V^{+}=0$ for $r^{+}=1$.

$$K=\frac{m_2}{3m_3}+\frac{3(\lambda/c)^{2/3}}{4}c^{1/3}\left\{\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}+\left(\frac{b}{c}-1\right)\right]^{2/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}-\left(\frac{b}{c}-1\right)\right]^{2/3}\right\}+$$
$$-\frac{3}{8}c^{1/3}\left\{\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}+\left(\frac{b}{c}-1\right)\right]^{4/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}-\left(\frac{b}{c}-1\right)\right]^{4/3}\right\} \tag{9b}$$

From equations (9a) and (9b), the equation of the velocity assumes the form given by equation (10):

$$V^{+}=\frac{m_2}{3m_3}\left(1-r^{+}\right)+\frac{3(\lambda/c)^{2/3}c^{1/3}}{4}\left\{\begin{aligned}&\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}+\left(\frac{b}{c}-1\right)\right]^{2/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}-\left(\frac{b}{c}-1\right)\right]^{2/3}-\\&-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^{+}\right)^2}+\left(\frac{b}{c}-r^{+}\right)\right]^{2/3}-\\&-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^{+}\right)^2}-\left(\frac{b}{c}-r^{+}\right)\right]^{2/3}\end{aligned}\right\}+$$
$$+\frac{3}{8}c^{1/3}\left\{\begin{aligned}&\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^{+}\right)^2}+\left(\frac{b}{c}-r^{+}\right)\right]^{4/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^{+}\right)^2}-\left(\frac{b}{c}-r^{+}\right)\right]^{4/3}-\\&-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}+\left(\frac{b}{c}-1\right)\right]^{4/3}-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}-\left(\frac{b}{c}-1\right)\right]^{4/3}\end{aligned}\right\} \tag{10}$$

Equations (8a) and (8b) allow writing the gradient of the velocity in the form shown in equation (11a), which is compatible with the form of equation (10) for the velocity.

$$\frac{dV^{+}}{dr^{+}}=-\frac{m_2}{3m_3}+c^{1/3}\left(\sqrt[3]{-\left(\frac{b}{c}-r^{+}\right)+\sqrt{\left(\frac{b}{c}-r^{+}\right)^2+(\lambda/c)^2}}-\sqrt[3]{\left(\frac{b}{c}-r^{+}\right)+\sqrt{\left(\frac{b}{c}-r^{+}\right)^2+(\lambda/c)^2}}\right) \tag{11a}$$

Applying the boundary condition $dV^{+}/dr^{+}=0$ for $r^{+}=0$ it its possible to substitute some set of variables. In the present case, $m_2/3m_3$ was chosen to be substituted. So, we have:

$$\frac{m_2}{3m_3}=c^{1/3}\left\{\left[-\left(\frac{b}{c}\right)+\sqrt{\left(\frac{b}{c}\right)^2+(\lambda/c)^2}\right]^{1/3}-\left[\left(\frac{b}{c}\right)+\sqrt{\left(\frac{b}{c}\right)^2+(\lambda/c)^2}\right]^{1/3}\right\} \tag{11b}$$

From equations (10), (11a), and (11b) we have, as solution for the velocity profile:

$$\begin{aligned}V^+ &= c^{1/3}\left\{\left[-\left(\frac{b}{c}\right)+\sqrt{\left(\frac{b}{c}\right)^2+(\lambda/c)^2}\right]^{1/3}-\left[\left(\frac{b}{c}\right)+\sqrt{\left(\frac{b}{c}\right)^2+(\lambda/c)^2}\right]^{1/3}\right\}\left(1-r^+\right)+\\
&+\frac{3(\lambda/c)^{2/3}c^{1/3}}{4}\left\{\begin{array}{l}\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}+\left(\frac{b}{c}-1\right)\right]^{2/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}-\left(\frac{b}{c}-1\right)\right]^{2/3}-\\
-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^+\right)^2}+\left(\frac{b}{c}-r^+\right)\right]^{2/3}-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^+\right)^2}-\left(\frac{b}{c}-r^+\right)\right]^{2/3}\end{array}\right\}+\\
&+\frac{3}{8}c^{1/3}\left\{\begin{array}{l}\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^+\right)^2}+\left(\frac{b}{c}-r^+\right)\right]^{4/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-r^+\right)^2}-\left(\frac{b}{c}-r^+\right)\right]^{4/3}-\\
-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}+\left(\frac{b}{c}-1\right)\right]^{4/3}-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}-\left(\frac{b}{c}-1\right)\right]^{4/3}\end{array}\right\}\end{aligned} \tag{12a}$$

Applying now the boundary condition $V^+=1$ for $r^+=0$, the constant $c^{1/3}$ was calculated, as shown in equation (12b). This parameter is replaced in the sequence in the equation for V^+ and also for dV^+/dr^+.

$$\begin{aligned}c^{-1/3} &= \left\{\left[-\left(\frac{b}{c}\right)+\sqrt{\left(\frac{b}{c}\right)^2+(\lambda/c)^2}\right]^{1/3}-\left[\left(\frac{b}{c}\right)+\sqrt{\left(\frac{b}{c}\right)^2+(\lambda/c)^2}\right]^{1/3}\right\}+\\
&+\frac{3(\lambda/c)^{2/3}}{4}\left\{\begin{array}{l}\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}+\left(\frac{b}{c}-1\right)\right]^{2/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}-\left(\frac{b}{c}-1\right)\right]^{2/3}-\\
-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}\right)^2}+\left(\frac{b}{c}\right)\right]^{2/3}-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}\right)^2}-\left(\frac{b}{c}\right)\right]^{2/3}\end{array}\right\}+\\
&+\frac{3}{8}\left\{\begin{array}{l}\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}\right)^2}+\left(\frac{b}{c}\right)\right]^{4/3}+\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}\right)^2}-\left(\frac{b}{c}\right)\right]^{4/3}-\\
-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}+\left(\frac{b}{c}-1\right)\right]^{4/3}-\left[\sqrt{(\lambda/c)^2+\left(\frac{b}{c}-1\right)^2}-\left(\frac{b}{c}-1\right)\right]^{4/3}\end{array}\right\}\end{aligned} \tag{12b}$$

Equations (12a) and (12b) allow now obtaining equation (13) for V^+. In this equation $\Psi=(\lambda/c)^2$ and $\Omega=b/c$.

$$V^{+}=\frac{\left\{\begin{array}{l}\left\{\left[\sqrt{\Omega^{2}+\Psi}-\Omega\right]^{1/3}-\left[\sqrt{\Omega^{2}+\Psi}+\Omega\right]^{1/3}\right\}\left(1-r^{+}\right)+\\ \quad+\dfrac{3\Psi^{1/3}}{4}\left\{\begin{array}{l}\left[\sqrt{(\Omega-1)^{2}+\Psi}+(\Omega-1)\right]^{2/3}+\left[\sqrt{(\Omega-1)^{2}+\Psi}-(\Omega-1)\right]^{2/3}-\\ -\left[\sqrt{\left(\Omega-r^{+}\right)^{2}+\Psi}+\left(\Omega-r^{+}\right)\right]^{2/3}-\left[\sqrt{\left(\Omega-r^{+}\right)^{2}+\Psi}-\left(\Omega-r^{+}\right)\right]^{2/3}\end{array}\right\}+\\ \quad+\dfrac{3}{8}\left\{\begin{array}{l}\left[\sqrt{(\Omega-r)^{2}+\Psi}+\left(\Omega-r^{+}\right)\right]^{4/3}+\left[\sqrt{(\Omega-r)^{2}+\Psi}-\left(\Omega-r^{+}\right)\right]^{4/3}-\\ -\left[\sqrt{(\Omega-1)^{2}+\Psi}+(\Omega-1)\right]^{4/3}-\left[\sqrt{(\Omega-1)^{2}+\Psi}-(\Omega-1)\right]^{4/3}\end{array}\right\}\end{array}\right\}}{\left\{\begin{array}{l}\left\{\left[\sqrt{\Omega^{2}+\Psi}-\Omega\right]^{1/3}-\left[\sqrt{\Omega^{2}+\Psi}+\Omega\right]^{1/3}\right\}+\\ \quad+\dfrac{3\Psi^{1/3}}{4}\left\{\begin{array}{l}\left[\sqrt{(\Omega-1)^{2}+\Psi}+(\Omega-1)\right]^{2/3}+\left[\sqrt{(\Omega-1)^{2}+\Psi}-(\Omega-1)\right]^{2/3}-\\ -\left[\sqrt{\Omega^{2}+\Psi}+\Omega\right]^{2/3}-\left[\sqrt{\Omega^{2}+\Psi}-\Omega\right]^{2/3}\end{array}\right\}+\\ \quad+\dfrac{3}{8}\left\{\begin{array}{l}\left[\sqrt{\Omega^{2}+\Psi}+\Omega\right]^{4/3}+\left[\sqrt{\Omega^{2}+\Psi}-\Omega\right]^{4/3}-\\ -\left[\sqrt{(\Omega-1)^{2}+\Psi}+(\Omega-1)\right]^{4/3}-\left[\sqrt{(\Omega-1)^{2}+\Psi}-(\Omega-1)\right]^{4/3}\end{array}\right\}\end{array}\right\}} \tag{13}$$

Applying the same series of definitions and substitutions into equation (11a), for the velocity gradient, we have:

$$\frac{dV^{+}}{dr^{+}}=\frac{\left\{\begin{array}{l}\left\{\left[-\left(\Omega-r^{+}\right)+\sqrt{\left(\Omega-r^{+}\right)^{2}+\Psi}\right]^{1/3}-\left[\left(\Omega-r^{+}\right)+\sqrt{\left(\Omega-r^{+}\right)^{2}+\Psi}\right]^{1/3}\right\}-\\ -\left\{\left[-\Omega+\sqrt{\Omega^{2}+\Psi}\right]^{1/3}-\left[\Omega+\sqrt{\Omega^{2}+\Psi}\right]^{1/3}\right\}\end{array}\right\}}{\left\{\begin{array}{l}\left\{\left[\sqrt{\Omega^{2}+\Psi}-\Omega\right]^{1/3}-\left[\sqrt{\Omega^{2}+\Psi}+\Omega\right]^{1/3}\right\}+\\ \quad+\dfrac{3\Psi^{1/3}}{4}\left\{\begin{array}{l}\left[\sqrt{(\Omega-1)^{2}+\Psi}+(\Omega-1)\right]^{2/3}+\left[\sqrt{(\Omega-1)^{2}+\Psi}-(\Omega-1)\right]^{2/3}-\\ -\left[\sqrt{\Omega^{2}+\Psi}+\Omega\right]^{2/3}-\left[\sqrt{\Omega^{2}+\Psi}-\Omega\right]^{2/3}\end{array}\right\}+\\ \quad+\dfrac{3}{8}\left\{\begin{array}{l}\left[\sqrt{\Omega^{2}+\Psi}+\Omega\right]^{4/3}+\left[\sqrt{\Omega^{2}+\Psi}-\Omega\right]^{4/3}-\\ -\left[\sqrt{(\Omega-1)^{2}+\Psi}+(\Omega-1)\right]^{4/3}-\left[\sqrt{(\Omega-1)^{2}+\Psi}-(\Omega-1)\right]^{4/3}\end{array}\right\}\end{array}\right\}} \tag{14}$$

Equations (13) and (14) express the solution of the velocity profile of a cubic polynomial law for the shear stress as function of the velocity gradient. Two control parameters were left in these equations: Ω and Ψ. In the present tentative approximation eventual intrinsic restrictions of Ω and Ψ were not considered.

3. POSSIBLE EXTENSION TO TURBULENT FLOWS

When talking about turbulence, the usual goal is to express adequately the mean variables of the flow, such as the mean velocities and mean stresses. In this sense, equations (4) and (5) may also be used to verify if polynomial expansions are promising approaches for turbulent flows. These equations, and the subsequent solutions, were prospectively tested here in order to observe possible quantifications of mean turbulent velocities and turbulent shear stresses.

The total normalized shear stress is already given by equation (7), rewritten as a sum of turbulent and laminar parcels, is given by:

$$\frac{a}{m_3} r^+ = \tau_{LAMINAR} + \tau_{TURBULENT} = \left(\frac{d\overline{V}^+}{dr^+}\right)^3 + \frac{m_2}{m_3}\left(\frac{d\overline{V}^+}{dr^+}\right)^2 + \frac{m_1}{m_3}\left(\frac{d\overline{V}^+}{dr^+}\right) \tag{15}$$

In this case, the turbulent parcel is then:

$$\tau_{TURBULENT} = \left(\frac{d\overline{V}^+}{dr^+}\right)^3 + \frac{m_2}{m_3}\left(\frac{d\overline{V}^+}{dr^+}\right)^2 + \left(\frac{m_1}{m_3} - \frac{\mu_L V_o}{R m_3}\right)\left(\frac{d\overline{V}^+}{dr^+}\right) \tag{16a}$$

μ_L is the absolute viscosity of the fluid. For turbulent flows it was not imposed that μ_1 would be equal to μ_L. The bar over the velocity V^+ indicates that mean velocities are now considered. In order to obtain a view of the behavior of the turbulent stress expressed by equation (16a) along the nondimensional radius r^+, the nondimensional coefficients m_1/m_3, m_2/m_3, and $\frac{\mu_L V_o}{R m_3}$ =M must be known. The previous conclusions attained in this paper were then extended to turbulent flows, maintaining the obtained equations. From those conclusions and definitions (mainly equations (8b), (10), (11a), (11b), (12b), and (14)), following auxiliary equations for the coefficients m_2/m_3 and m_1/m_3 of equation (16a) are then obtained:

$$\frac{m_2}{m_3} = \frac{3\left\{\left[-\Omega + \sqrt{\Omega^2+\Psi}\right]^{1/3} - \left[\Omega + \sqrt{\Omega^2+\Psi}\right]^{1/3}\right\}}{\begin{gathered}\left\{\left[\sqrt{\Omega^2+\Psi} - \Omega\right]^{1/3} - \left[\sqrt{\Omega^2+\Psi} + \Omega\right]^{1/3}\right\} + \\ + \frac{3\Psi^{1/3}}{4}\left\{\begin{matrix}\left[\sqrt{(\Omega-1)^2+\Psi} + (\Omega-1)\right]^{2/3} + \left[\sqrt{(\Omega-1)^2+\Psi} - (\Omega-1)\right]^{2/3} - \\ -\left[\sqrt{\Omega^2+\Psi} + \Omega\right]^{2/3} - \left[\sqrt{\Omega^2+\Psi} - \Omega\right]^{2/3}\end{matrix}\right\} + \\ + \frac{3}{8}\left\{\begin{matrix}\left[\sqrt{\Omega^2+\Psi} + \Omega\right]^{4/3} + \left[\sqrt{\Omega^2+\Psi} - \Omega\right]^{4/3} - \\ -\left[\sqrt{(\Omega-1)^2+\Psi} + (\Omega-1)\right]^{4/3} - \left[\sqrt{(\Omega-1)^2+\Psi} - (\Omega-1)\right]^{4/3}\end{matrix}\right\}\end{gathered}} \tag{16b}$$

$$\frac{m_1}{m_3}=\frac{3\left\{\left[-\Omega+\sqrt{\Omega^2+\Psi}\right]^{1/3}-\left[\Omega+\sqrt{\Omega^2+\Psi}\right]^{1/3}\right\}^2+3\Psi^{2/3}}{\left\{\begin{array}{l}\left\{\left[\sqrt{\Omega^2+\Psi}-\Omega\right]^{1/3}-\left[\sqrt{\Omega^2+\Psi}+\Omega\right]^{1/3}\right\}+\\ +\dfrac{3\Psi^{1/3}}{4}\left\{\begin{array}{l}\left[\sqrt{(\Omega-1)^2+\Psi}+(\Omega-1)\right]^{2/3}+\left[\sqrt{(\Omega-1)^2+\Psi}-(\Omega-1)\right]^{2/3}-\\ -\left[\sqrt{\Omega^2+\Psi}+\Omega\right]^{2/3}-\left[\sqrt{\Omega^2+\Psi}-\Omega\right]^{2/3}\end{array}\right\}+\\ +\dfrac{3}{8}\left\{\begin{array}{l}\left[\sqrt{\Omega^2+\Psi}+\Omega\right]^{4/3}+\left[\sqrt{\Omega^2+\Psi}-\Omega\right]^{4/3}-\\ -\left[\sqrt{(\Omega-1)^2+\Psi}+(\Omega-1)\right]^{4/3}-\left[\sqrt{(\Omega-1)^2+\Psi}-(\Omega-1)\right]^{4/3}\end{array}\right\}\end{array}\right\}^2} \tag{16c}$$

The parcel M of equation (16a) can be adjusted by knowing that the turbulent stress is zero for r=1. Thus having all parcels of the equation for the turbulent stresses, it is possible to verify if the form of the function that describes the stresses using polynomial approximations for the velocity gradient behave in accordance with observed profiles. As already mentioned, it must be noted that this is a prospective analysis, because the series was truncated in a lower order term (third order), and no eventual intrinsic restrictions of Ψ, Ω, and M were considered. These restrictions are understood as specific ranges of values of the mentioned parameters when applying boundary conditions to the calculated flows. In the present study the value of M was adjusted after imposing values for Ψ and Ω, independently.

4. COMPARISONS OF THE THIRD ORDER POLINOMIAL APPROXIMATION WITH MODELS AND RESULTS OF THE LITERATURE

Figure 2 presents sketches of shear stresses obtained with some of the different models cited in Table 1. These graphs were generated to verify the possibility of describing dilatant and pseudoplastic fluids using the present approximation, and to compare the results with the existing models. As can be seen, the present approximation tends to the Newtonian behavior for low dV/dr, and departs to pseudoplastic or dilatant behaviors for higher dV/dr, accordingly the positive or negative sign of the coefficients of the equation.

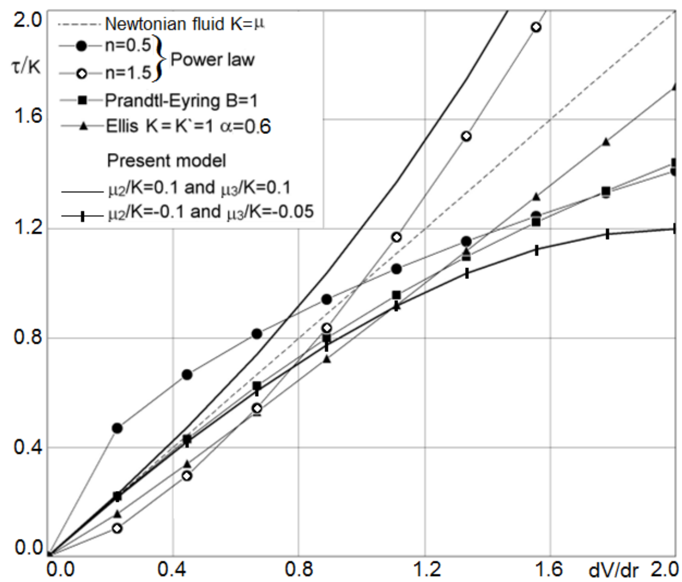

Figure 2. Sketches of the models presented in Table 1 and the present approximation, showing that the present model allows representing pseudoplastic and dilatant fluids for a tange of dV/dr values.

Figure 3 shows a comparison of velocity profiles of the different models of Table 1 and the present study. All profiles follow the expected behavior and boundary conditions. The present model shows that different values of Ω may lead to different evolutions of the velocity with the radius r^+. In general, the present approximation allows representing non-Newtonian fluids.

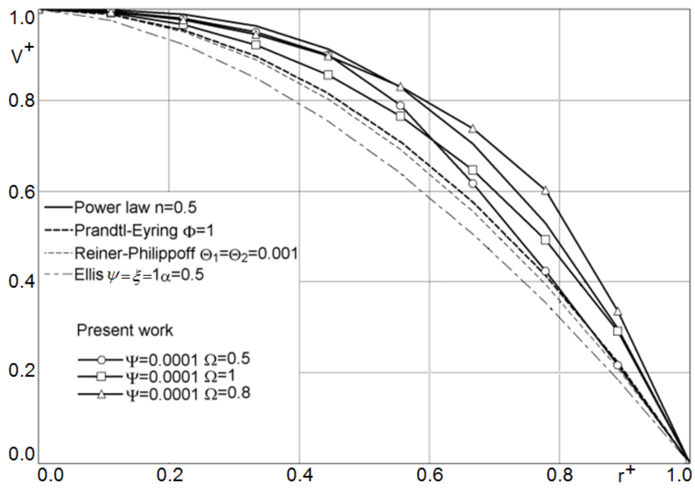

Figure 3. Comparison of the dimensionless velocity profiles and shear stress between the non-Newtonian models of Table 1 and the present approximation.

For the turbulent case, Fig. 4a presents in dashed lines some velocity profiles obtained by varying independently Ψ and Ω of the present formulation. As can be seen, the obtained profiles are located in the region between the laminar (parabolic) profile and measurements of turbulent profiles. This trend to the turbulence profiles suggests that polynomial approximations based on equation (1) may be used to represent turbulent situations. Additionally, Fig. 4b presents corresponding turbulent stress profiles, obtained by adjusting M to the previous imposed Ψ and Ω. The results show that the profiles for turbulent stresses tend to the form of known experimental profiles for values of Ψ around 0.95 and higher (closer to 1.0). On the other hand, for values of Ψ around 0,7 the velocity profiles are closer to the turbulent profiles, but the profiles for shear stresses depart from those observed. The general form of the calculated profiles points to the convenience of testing higher order truncations of equation (4).

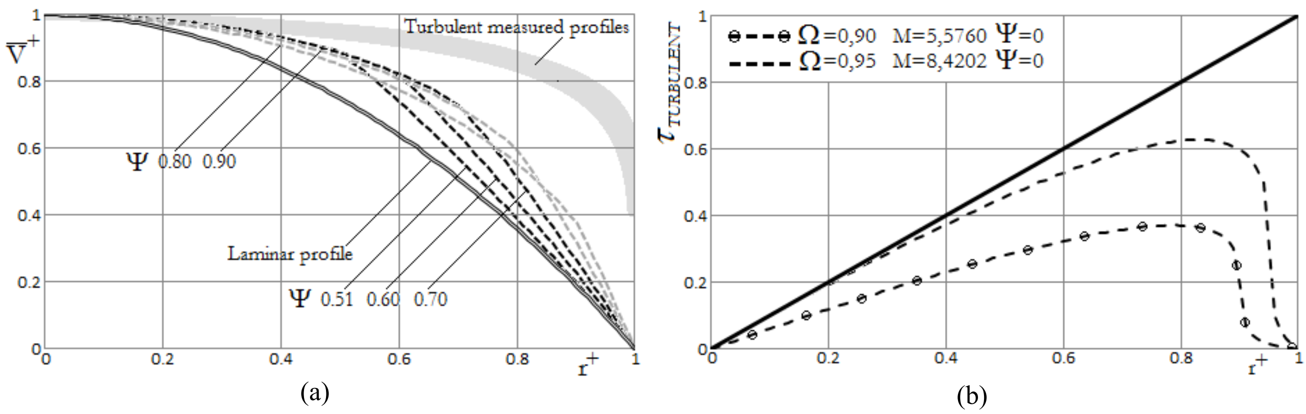

Figure 4. (a) Continuous line: parabolic laminar velocity profile. Dashed lines: profiles of equation (13) for Ω=0 and Ψ indicated in the figure. Gray cloud: envelope region of turbulent profiles, according to Hughes and Brighton (1967); (b) Continuous line: total shear stress. Dashed lines: “turbulent” shear stresses calculated with equations (16a, b, c). The values of the parameters Ψ, Ω and M are indicated in the figure.

5. CONCLUSIONS

It is shown that truncated series expansions for the relation between the shear stress and the shear rate in fluids may be used as alternative formulations for the prediction of velocity profiles of non-Newtonian laminar flows. The comparisons made in the present study show that similar behaviors are obtained for velocity profiles for some ranges of the involved control parameters. In this sense, pseudoplastic and dilatant behaviors may be represented by this approximation. Following previous comparative studies of the literature, the series truncation formulation was also used to prospectively verify potential uses for turbulent pipe flows. The results show that qualitatively the calculated results tend to forms of profiles observed in turbulent flows, for velocity and for shear stresses in pipes. In this sense, the results allow suggesting more studies for higher order truncations of the power series.

6. ACKNOWLEDGEMENTS

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5. RESPONSIBILITY NOTICE