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UNCERTAINTY COMPENSATION IN DAMAGE IDENTIFICATION BY MEANS OF THE APPROXIMATION ERROR APPROACH

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Abstract.*This work presents an application of the approximation error approach (APA) proposed by Kaipio and Somersalo (2007) in the context of Structural Health and Monitoring (SHM). Based on the Bayesian framework of statistical inversion, this approach allows one to compensate for errors caused by incorrect modelling of a physical system while still providing a relatively simple mathematical formulation. The application of different prior distributions of the unknown parameters is investigated with the help of a toy problem. The APA is compared to a traditional least-squares approach consisting of a forward model (FM) unable to compensate for model related errors. To reduce the dimension of the parameter space, a Gaussian bell curve was used to model the damage field in the FM.*

Keywords: *approximation error model, damage identification, SHM*

1. INTRODUCTION

In this work, we chose to follow the definition given by Farrar and Worden (2007), that is to say *damage* can be defined as changes introduced into a system that adversely affect its performance. Implicit in this definition is the existence of a reference state of the system, often its undamaged one, to which comparisons can be drawn. Therefore, changes in material properties and/or geometry as well as boundary conditions fall into this definition.

Friswell (2007) presents a review of the inverse methods commonly used in damage identification. Most of the works are found in the framework of *Model Updating* (MU), most of which are based on *Finite Element Model Updating* (FEMU). MU can use either parametric or direct updating methods: in direct methods, the goal is to reproduce the measured data by controlled chances to the stiffness matrix; while in parametric based methods a set of physically meaningful parameters can be selected and then estimated by means of minimizing a cost function.

In practice, however, one must rely on the use of models to predict the behavior of the observed system, and this is one of the major problems concerning SHM. There are always errors related to measurement data and physical/mathematical modeling of a system, and so it happens that many times these modeling errors can cause changes to the predicted features that are of an equal or greater order than the changes caused by the damage that one is trying to detect, which in turn make the solution of the inverse problem more difficult and can even render the estimated results useless. Modeling errors include (but are not limited to) environmental and other non-stationary effects (such as temperature, humidity, etc), unmodeled nonlinear behavior, incorrect material properties, discretization errors, geometry approximation, etc.

There are works where different methods are utilized to compensate for some of these errors. Simoen *et al.* (2015) wrote an extensive review paper showcasing two most common approaches to compensate for modeling errors in the FEMU framework: the non-probabilistic fuzzy approach and the probabilistic one, based on the Bayesian framework. The sources of modeling errors are discussed and examples are worked out comparing both approaches to the traditional (deterministic) FEMU.

Most of the work concerning the Bayesian approach for FEMU stems from the works by Beck and Katafygiotis (1998) and by Vanik *et al.* (2000) who showed that the Bayesian approach allows for updated probabilities of model parameters and damage measures. To exemplify other methodologies for dealing with model uncertainty, consider the following works by Moaveni and Behmanesh (2012); Behmanesh *et al.* (2015); Behmanesh and Moaveni (2016), Nandan and Singh (2014a,b) and Lee *et al.* (2005).

This work's goal is to introduce to the SHM community a tool for dealing with modeling errors in general: the Approximation Error Approach (APA), which was developed originally by Kaipio and Somersalo (2007) to handle model reduction errors. It has seen use in clinical applications as tomography, where the use of approximate models is desirable to reduce the computational burden necessary and still give a reliable and fast diagnosis. Given the extent of this author's knowledge, thus far this approach hasn't been applied in SHM.

The APA was extensively tested in the context of Optical Tomography and Electrical Impedance Tomography to compensate for modeling errors due to simplifications in distributed parameters and/or discretization (Kolehmainen *et al.* (2011)), domain geometry (Nissinen *et al.* (2011); Mozumder *et al.* (2014)), mathematical modeling (Nissinen *et al.* (2009); Koulouri *et al.* (2016)) and even simplified physics (Tarvainen *et al.* (2010)).

The approximation error approach was shown to be a feasible way to compensate for discretization errors, uncertain boundary data and geometry, as well as physical modeling errors in a complex set of problems. Due to its theoretical simplicity and computational cost reduction when compared to other approaches to compensate for modeling errors, we were motivated to apply it to SHM.

2. THEORY

The APA is based on the Bayesian framework of inverse problems, in which all unknowns are modeled as random variables (Kaipio and Somersalo, 2006; Tarantola, 2004). Let $\boldsymbol{\theta}$ be a vector of unknowns and $\mathbf{y}$ a vector containing measurements of some feature related to the system being identified. Once probabilistic models for $\boldsymbol{\theta}$ and $\mathbf{y}$ are constructed, the *Posterior* probability distribution function (PDF) $\pi(\boldsymbol{\theta}|\mathbf{y})$ can be accessed through the use of Bayes' Formula, Eq. (1). The *posterior* PDF reflects the uncertainty of the unknowns $\boldsymbol{\theta}$ given the measurements $\mathbf{y}$, given any *prior information* available.

In Eq. (1), $\pi(\mathbf{y}|\boldsymbol{\theta})$ is the *Likelihood function* of the measurements $\mathbf{y}$ given the unknown parameters $\boldsymbol{\theta}$, this function associates a probability to the occurrence of a given measurement realization to a realization of the parameters. $\pi(\boldsymbol{\theta})$ is the *Prior* PDF of the unknowns $\boldsymbol{\theta}$, this function allows for any current knowledge regarding the possible values of $\boldsymbol{\theta}$ to be taken into account. $\pi(\mathbf{y})$ is the prior for the measurements, and once a realization of $\mathbf{y}$ is drawn, that is to say an experiment is made, the value of $\pi(\mathbf{y})$ is simply a scaling constant.

$$\pi(\boldsymbol{\theta}|\mathbf{y}) = \frac{\pi(\mathbf{y}|\mathbf{x})\pi(\boldsymbol{\theta})}{\pi(\mathbf{y})} \tag{1}$$

The *maximum a posteriori* (MAP) of $\pi(\boldsymbol{\theta}|\mathbf{y})$ is a commonly used estimate for the unknowns:

$$\boldsymbol{\theta}' = arg\max_{\boldsymbol{\theta}} \pi(\boldsymbol{\theta}|\mathbf{y}) \tag{2}$$

2.1 Approximation Error Approach and Premarginalization

Let $(\bar{\boldsymbol{\theta}}, \mathbf{z}, \boldsymbol{\xi}, \mathbf{e})$ be our set of unknowns. Where $\mathbf{e}$ represents additive errors, $\boldsymbol{\xi}$ represents auxiliary uncertainties such as unknown boundary data and/or geometry, and $(\bar{\boldsymbol{\theta}}, \mathbf{z})$ are two parameters of which only $\bar{\boldsymbol{\theta}}$ is of interest.

Let $\mathcal{A}_c(\bar{\boldsymbol{\theta}}, \mathbf{z}, \boldsymbol{\xi})$ be a deterministic *accurate* forward model (FM) without any uncertainties or other model errors, i.e., the map $(\bar{\boldsymbol{\theta}}, \mathbf{z}, \boldsymbol{\xi}) \mapsto \mathcal{A}_c(\bar{\boldsymbol{\theta}}, \mathbf{z}, \boldsymbol{\xi})$ provides accurate predictions whose differences from measured data are basically due to the measurement noise.

Let $\mathbf{e}$ be a vector of additive errors related to the measurement process. In the classical Bayesian approach, one marginalizes over the possible values of $\mathbf{e}$ in order to construct $\pi(\bar{\boldsymbol{\theta}}|\mathbf{y})$. The key feature of the APA is to perform premarginalization *approximately*, in a computationally feasible way, of the *other unknowns*. Let $\mathbf{e}$ be mutually independent with $(\bar{\boldsymbol{\theta}}, \mathbf{z}, \boldsymbol{\xi})$, thus the observation model can be written as:

$$\mathbf{y} = \mathcal{A}_c(\bar{\boldsymbol{\theta}}, \mathbf{z}, \boldsymbol{\xi}) + \mathbf{e} \tag{3}$$

In the following, let $\boldsymbol{\theta}$ be an approximation of the primary unknown $\bar{\boldsymbol{\theta}}$, where $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ can be related by some sort of projection operator $\mathbf{P}$, $\boldsymbol{\theta} = \mathbf{P}\bar{\boldsymbol{\theta}}$. Proceed by setting $(\mathbf{z}, \boldsymbol{\xi}) \rightarrow (\mathbf{z}_0, \boldsymbol{\xi}_0)$, and substituting the accurate FM by a much simpler one, $\boldsymbol{\theta} \mapsto A(\boldsymbol{\theta}, \mathbf{z}_0, \boldsymbol{\xi}_0)$. Thus, we rewrite the observation model as:

$$\mathbf{y} = \mathcal{A}_c(\bar{\boldsymbol{\theta}}, \mathbf{z}, \boldsymbol{\xi}) + \mathbf{e} = A(\boldsymbol{\theta}, \mathbf{z}_0, \boldsymbol{\xi}_0) + \boldsymbol{\epsilon} + \mathbf{e} \tag{4}$$

Where

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\boldsymbol{\theta}, \mathbf{z}, \boldsymbol{\xi}) = \mathcal{A}_c(\bar{\boldsymbol{\theta}}, \mathbf{z}, \boldsymbol{\xi}) - A(\boldsymbol{\theta}, \mathbf{z}_0, \boldsymbol{\xi}_0) \tag{5}$$

is defined to be the *approximation error* that arises when using a simplified model.

Omitting some steps concerning the premarginalization of the random variables for conciseness and calculating the MAP estimate of the posterior, $\boldsymbol{\theta}$ can be obtained from the following problem:

$$\boldsymbol{\theta}' = arg\min_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) \tag{6a}$$

$$V(\boldsymbol{\theta}) = \|\boldsymbol{\Gamma}_{\nu|\theta}[\mathbf{y} - A(\boldsymbol{\theta}, \mathbf{z}_0, \xi_0) - (\mathbf{e}_* + \boldsymbol{\epsilon}_* + \boldsymbol{\Sigma}_{\epsilon\theta}\boldsymbol{\Sigma}_{\theta\theta}^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}_*))]\|^2 + \|\boldsymbol{\Gamma}_{\theta}(\boldsymbol{\theta} - \boldsymbol{\theta}_*)\|^2 \tag{6b}$$

$$\boldsymbol{\Sigma}_{\nu|\theta} = \boldsymbol{\Sigma}_e + \boldsymbol{\Sigma}_{\epsilon\epsilon} - \boldsymbol{\Sigma}_{\epsilon\theta}\boldsymbol{\Sigma}_{\theta\theta}^{-1}\boldsymbol{\Sigma}_{\theta\epsilon} \tag{6c}$$

Where $\boldsymbol{\Gamma}_{\nu|\theta}$ and $\boldsymbol{\Gamma}_{\theta}$ are Cholesky factors of $\boldsymbol{\Sigma}_{\nu|\theta}^{-1}$ and $\boldsymbol{\Sigma}_{\theta}^{-1}$, respectively, such that $\boldsymbol{\Gamma}_{\nu|\theta}^T\boldsymbol{\Gamma}_{\nu|\theta} = \boldsymbol{\Sigma}_{\nu|\theta}^{-1}$ and $\boldsymbol{\Gamma}_{\theta}^T\boldsymbol{\Gamma}_{\theta} = \boldsymbol{\Sigma}_{\theta}^{-1}$.

2.2 Computational Considerations

In this section it is discussed how one goes about gathering the necessary statistics in order to perform a MAP estimation using the Approxiation Error Approach. In practice a set of samples $(\boldsymbol{\theta}^{(l)}, \mathbf{z}^{(l)}, \boldsymbol{\xi}^{(l)})$ is to be drawn and the approximation error computed:

$$\boldsymbol{\epsilon}^{(l)} = \mathcal{A}_c(\bar{\boldsymbol{\theta}}^{(l)}, \mathbf{z}^{(l)}, \boldsymbol{\xi}^{(l)}) - A(\boldsymbol{\theta}^{(l)}, \mathbf{z}_0, \boldsymbol{\xi}_0),\ l = 1, 2, ..., N_{mc} \tag{7a}$$

$$\boldsymbol{\epsilon}^{(l)} = A(\boldsymbol{\theta}^{(l)}, \mathbf{z}^{(l)}, \boldsymbol{\xi}^{(l)}) - A(\boldsymbol{\theta}^{(l)}, \mathbf{z}_0, \boldsymbol{\xi}_0),\ l = 1, 2, ..., N_{mc} \tag{7b}$$

One should use Eq. (7a) if discretization errors are to be considered, and Eq. (7b) if not. Notice that, when no discretization errors are accounted for, the only distinction between models is how the distributed parameters are treated.

Finally, define $\mathbf{r}^{(l)} = [\boldsymbol{\epsilon}^{(l)}, \boldsymbol{\theta}^{(l)}]^T \in \mathbb{R}^{N_p+N_m}$, where N_p is the dimension of the parameter space and N_m is the total number of measurements. Then:

$$\mathbf{r}_* = \begin{pmatrix} \boldsymbol{\epsilon}_* \\ \boldsymbol{\theta}_* \end{pmatrix} = \mathbb{E}\left[\begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\theta} \end{pmatrix}\right] \approx \frac{1}{N_{mc}} \sum_{l=1}^{N_{mc}} \mathbf{r}^{(l)} \tag{8}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{\epsilon\epsilon} & \boldsymbol{\Sigma}_{\epsilon\theta} \\ \boldsymbol{\Sigma}_{\theta\epsilon} & \boldsymbol{\Sigma}_{\theta\theta} \end{pmatrix} = \mathbb{E}\left[(\mathbf{r} - \mathbf{r}_*)(\mathbf{r} - \mathbf{r}_*)^T\right] \approx \frac{1}{N_{mc} - 1} \sum_{l=1}^{N_{mc}} (\mathbf{r}^{(l)} - \mathbf{r}_*)(\mathbf{r}^{(l)} - \mathbf{r}_*)^T \tag{9}$$

Figure 1 depicts a flowchart for the general application of APA. The upper half of the flowchart, hereby denominated "*Training stage*", consists of draws of the random variables and computation of the corresponding statistics, Eq. (8 - 9). This stage is done prior to actual solution procedure and can take advantage from parallel computation.

The Solution stage consists of constructing the Likelihood function, Eq. (6), and solving the optimization problem for $\boldsymbol{\theta}'$.

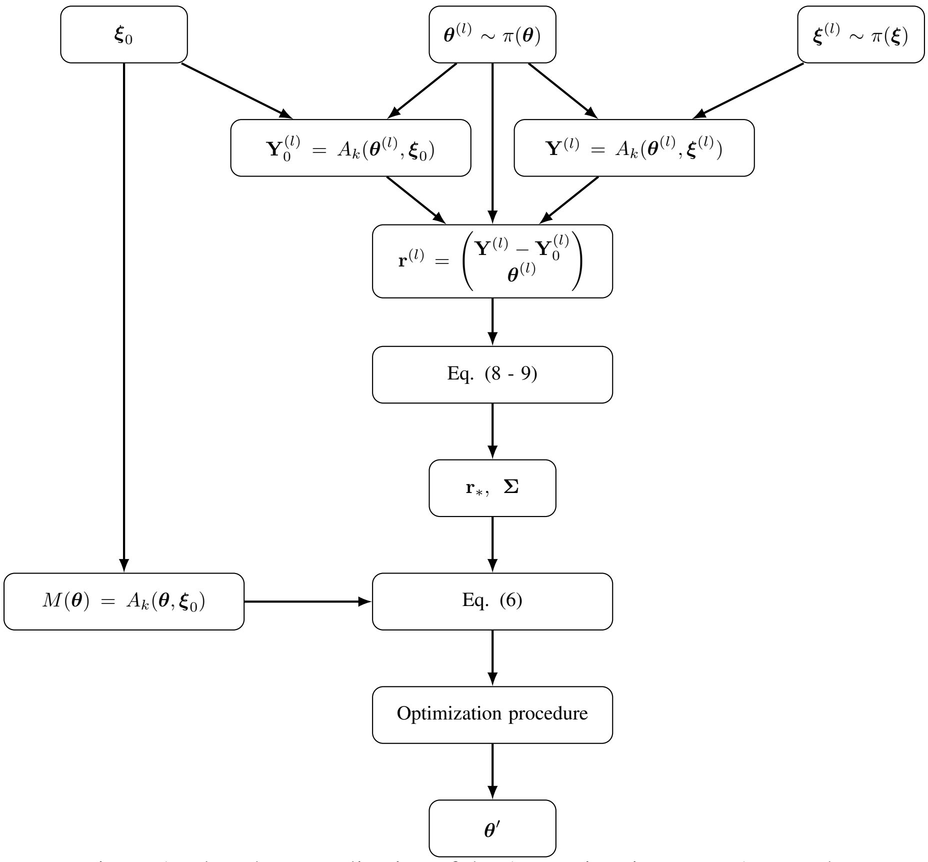


Figure 1: Flowchart: application of the Approximation Error Approach.

3. APPLICATION

In this chapter, the APA is applied to a toy problem: a clamped-free beam and the inverse problem consists of identifying the distributed damage given a set of scenarios with varying degrees of stiffness of the clamped side. We primarily chose the system proposed by Ritto *et al.* (2016) due to the availability of experimental results concerning the stiffness of the clamped side. In their paper, an experimental test rig was constructed where several rubber layers were added to the clamped side, between the beam and the support, in order to produce different stiffness values to be identified.

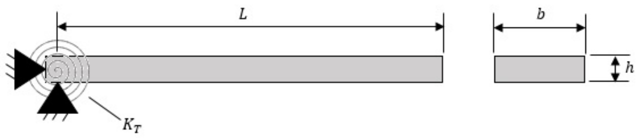


Figure 2: Physical model: Pinned-free beam with torsion spring.

3.1 The System

The system is modeled as shown in Fig. 2: assuming that one cannot ensure the clamped boundary condition in practice, one considers a beam pinned at the left side, with a torsion spring attached to this end in such a way that, when taking the limit $K_t \to \infty$, the system behaves like a clamped-free beam, K_t being the torsional stiffness constant of the spring.

The beam properties and the limit values of stiffness parameter K_t identified by Ritto *et al.* (2016) are given in Table 1.

Table 1: Beam Properties.

$L(m)$	$b(mm)$	$h(mm)$	$E_0(GPa)$	$\rho(kg/m^3)$	$K_{t,min}(Nm)$	$K_{t,max}(Nm)$
0.511	30.7	3.04	200	7850	0.8×10^3	3.8×10^3

Damage is assumed to be the cause of a local reduction in stiffness, this is modeled as a change in the elastic parameter, E_0, at some position x along the length of the beam. It is further assumed that the damage state does not evolve during the vibration tests, thus it can be modeled as a field independent of time, Eq. (10).

$$E(x) = (1 - d(x))E_0 \tag{10}$$

Where $d(x)$ determines the intensity of the damage at the position x, and E_0, given in Table 1, is the *undamaged* value of the Young's Modulus.

3.2 Direct Problem

Generally speaking, the *Direct Problem* consists of predicting system response $\mathbf{Y}$ (displacements, accelerations, modal data, etc), given a set of *known* input variables $\mathbf{u}$ (prescribed displacement, external excitation, etc) and parameters/boundary conditions $\boldsymbol{\theta}$. This is accomplished through the use of a *forward model* (FM):

$$\mathbf{Y} = \mathcal{M}(\mathbf{u}, \boldsymbol{\theta}) \tag{11}$$

In our case, given the system in Fig. 2 modeled as a Euler-Bernoulli beam, with properties defined in Table 1, torsion spring with constant K_t fixed at some value $K_{t,True}$ or $K_{t,ref}$ and damaged given by Eq. (10), one wishes to predict the dynamic behavior of the system given an excitation.

In other words, one wishes to solve Eq. (12) subjected to the boundary conditions Eq. (13) for the displacement field $w(x,t)$ and/or its derivatives at given positions along the beam (Géradin and Rixen, 2014). Where $\omega_0 = 1500\,Hz$, $\Delta\omega = 500\,Hz$, $F_0 = 1\,N$ and $T_f = 0.025\,s$. In Eq. (12a), inertial effects concerning rotation are not considered.

$$\rho A \ddot{w} + (EIw'')'' = F(t)\delta(x - x_F), \quad \forall t > 0, x \in (0, L) \tag{12a}$$

$$F(t) = F_0 sin\left(2\pi \left(\frac{t}{T_f}\Delta\omega + \omega_0\right) t\right) \tag{12b}$$

$$w(0,t) = 0,\ -EIw''(0,t) = -K_t w'(0,t),\ -EIw''(L,t) = 0, \quad \forall t > 0 \tag{13a}$$

$$w(x,0) = 0, \quad \forall x \in [0, L] \tag{13b}$$

The equations of motion, Eq. (12,13), are discretized using the Finite Element Method (See Reddy (1993); Fish and Belytschko (2007)). The resulting system of ordinary differential equations is integrated using a Newmark Method with appropriate parameters (See Géradin and Rixen (2014)). The output of any FM is the vector $\mathbf{Y}$, which contains the responses collected from the $N_s = 8$ sensors concatenated vertically such that $\mathbf{Y} \in \mathbb{R}^{N_m \times 1}$, $N_m = \sum_{i=1}^{N_s} N_i$. Where $N_m = 2008$ results from measurements taken with sampling frequency is $f_s = 10^4 \ Hz$. The sensors are uniformly spaced along the length of the beam.

3.3 Inverse Problem

As for the *Inverse Problem*, given a set of measurement data $\mathbf{y}$ and input variables $\mathbf{u}$, one seeks information about *unknown* parameters and/or boundary conditions $\boldsymbol{\theta}$. Given the ill-posedness of the inverse problem, one usually seeks to find the set estimate $\boldsymbol{\theta}'$ which minimizes the discrepancy between the experimental data and the predicted response.

In our case, the measurement set $\mathbf{y}$ consists of acceleration data. Given this data, we seek to estimate the damage field $d(x)$ given by Eq. (10) using a FM that is *not necessarily correct* in terms of its boundary conditions.

The field $d(x)$ is a function defined along the length of the beam, $d(x) : [0, L] \mapsto [0, 1]$. In this text, however, a *localized damage* field is estimated and for such field a specific parametrization is proposed, Eq. (14), in order to reduce the dimension of the parameter space. The proposed bell shaped curve is a continuous unimodal damage field. We are well aware of the limitations imposed by this parametrization in regard to damage estimation, however this text's goal is to present the Approximation Error Approach in the SHM context. This strategy was proved amenable for our applications.

$$d(x) = \mathcal{D}_m \exp\left\{ -\left(\frac{x_{adm} - x_c}{2s} \right)^2 \right\}; \ x_{adm} = x/L; \tag{14}$$

Where the parameters to be estimated are

- x_c, position at which the damage intensity is maximum
- $d\ell = 6s$, the support of the field such that $d(x_c \pm 3s) \approx 0.1\mathcal{D}_m$
- $\mathcal{D}_m$, maximum intensity of damage

Following these definitions, the parameter space is the subset of $\mathbb{R}^3$ given by the cube $\Omega_P = [0, 1] \times [0, 1] \times [0, 1]$. The inverse problem is then solved by finding the set of parameters $\boldsymbol{\theta}' = (x'_c, d\ell', \mathcal{D}'_m)^T$ which is the solution of the minimization problem:

$$\boldsymbol{\theta}' = arg \min_{\boldsymbol{\theta} \in \Omega_P} \ V(\boldsymbol{\theta}) \tag{15}$$

Where $V(\boldsymbol{\theta})$ is some functional, given either by equations (6b-6c) or by the traditional least squares Eq. (16):

$$V_T(\boldsymbol{\theta}) = \|\boldsymbol{\Gamma}_e(\mathbf{y} - \mathcal{M}(\mathbf{u}, \boldsymbol{\theta}) - \mathbf{e}_*)\|^2 \tag{16}$$

With $\boldsymbol{\Gamma}_e^T \boldsymbol{\Gamma}_e = \boldsymbol{\Sigma}_e^{-1}$, and $\boldsymbol{\Sigma}_e$ being the measurement covariance matrix.

Concerning the *synthetic* measurement data, $\mathbf{y}$ is generated by adding uncorrelated gaussian noise with zero mean to the ideal response $\mathbf{Y}_i^{ideal}$ of each sensor. $\mathbf{Y}_i^{ideal}$ is obtained from a reference model with a known damage profile, finer discretization than the one used in optimization and either $K_t = K_{t,True}$ or $x_F = x_{F,True}$ specified by different scenarios.

$$\mathbf{y}_i = \mathbf{Y}_i^{ideal} + \mathbf{e}_i, \ \mathbf{e}_i \sim \mathcal{N}(\mathbf{0}, \sigma_i^2 \mathbb{1}_{N_i \times N_i}), \ \sigma_i = 0.05 \max(\mathbf{Y}_i^{ideal}) \tag{17}$$

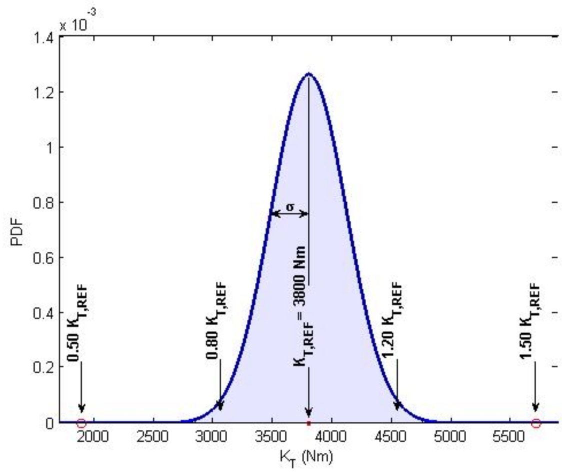


Figure 3: PDF of the torsional stiffness. The area corresponding to 99% of probability is painted blue. The values of $K_{t,True}$ tested in Section 4.1are indicated by arrows, low probability values are marked with red circles.

4. RESULTS

In order to apply the APA, the necessary statistics of the approximation error have to be computed. To that end, consider $N_{mc} = 10^4$ samples of $\boldsymbol{\theta}^{(l)} = (X_c^{(l)}, dL^{(l)}, D^{(l)})^T$ are drawn from the following distributions:

- $X_c \sim \mathcal{U}\,[0.1, 0.9]$
- $dL \sim \mathcal{U}\,[0.1, 0.25]$
- With 20% probability: generate an undamaged structure, $D = 0$. Otherwise $D \neq 0$, drawn D from $D \sim \mathcal{U}\,[0.05, 0.4]$.

Statistics concerning the parameter $\boldsymbol{\xi}^{(l)}$ to be premarginalized, whether it represents either $K_t^{(l)}$ or $X_F^{(l)}$ or both, must also be computed. In Section 4.1we consider the effects of errors due only to incorrect modeling of K_t, in Section 4.2 only errors due to the position of applied excitation are considered. Reference values for K_t and x_F are $K_{t,REF} = 3.8 \times 10^3\,Nm$ and $x_{F,REF} = 0.95L$, respectively. Samples from the corresponding random variables are drawn from:

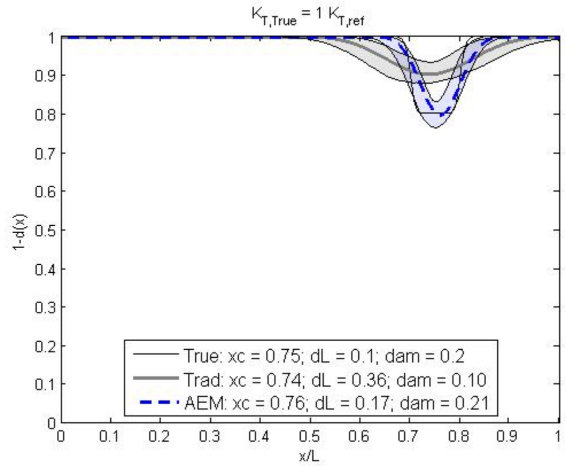
- $K_t \sim \mathcal{N}(\mu_K = K_{t,REF}, \sigma_K^2) \times \mathbb{1}_{\mathbb{R}+}[Nm]$
- $X_F \sim \mathcal{N}(\mu_F = x_{F,REF}, \sigma_F^2) \times \mathbb{1}_{[0,L]}[m]$

Where $\sigma_K = 0.0834\,\mu_K$ is such that $P\,(0.75\mu_K < K_t < 1.25\mu_K) > 99\%$. Figure 3 shows the proposed gaussian PDF for K_t as well as the tested values of Section 4.1

Analogously, $P\,(|X_F - \mu_F| < 3\sigma_F) > 99\%$. That is, there is an envelope of $6\sigma_F \approx 5.2mm$ in length ($\approx 1\%L$) in which the chirp, Eq. (12b), can be applied. Equivalently, $\sigma_F = 0.0017L = 0.87mm$.

The optimization problems given either by equations (15-16) or (6) are solved by means of Particle Swarm Optimization. Figure 4 shows the estimated damage field using both approaches without any sort of modeling errors associated. Notice how both approaches essentially give the same result, which is the best possible estimate given our current parametric approach for the field.

The 99% percentile envelope shown in all results is calculated using a gaussian approximation of the MAP estimate (Tarantola, 2004).



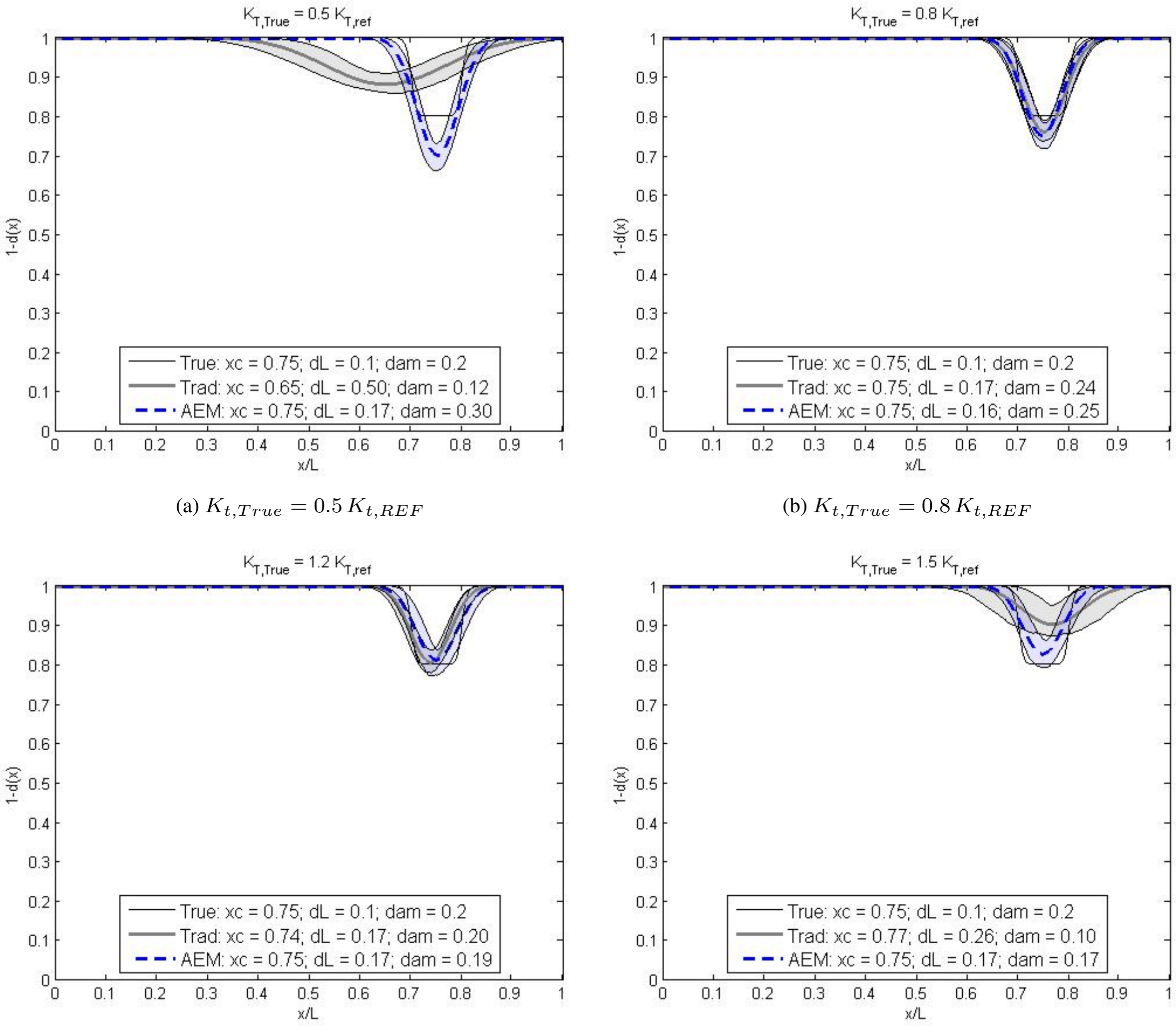
(a) $K_{t,True} = K_{t,REF}$

Figure 4: Damage field estimated both the APA and least-squares approach when no sources of error are included.

4.1 Uncertainty at Fixation Boundary Condition

This section presents results concerning the solution of the inverse problem, that is, the estimated damage field, in a scenario with unknown boundary conditions corresponding to the quality of “clamped” side – simulated by different values of $K_{t,True}$.

Figure 5 shows the estimated damage field with increasing values of $K_{t,True}$ compared to $K_{t,REF}$. There is a series of results in the range $K_{T,True} \in [0.9, 1.1] \times K_{T,REF}$ considering a fine discretization in values of $K_{T,True}$, however the results are not shown due to great similarity with Fig. 4.



(a) $K_{t,True} = 0.5\,K_{t,REF}$

(b) $K_{t,True} = 0.8\,K_{t,REF}$

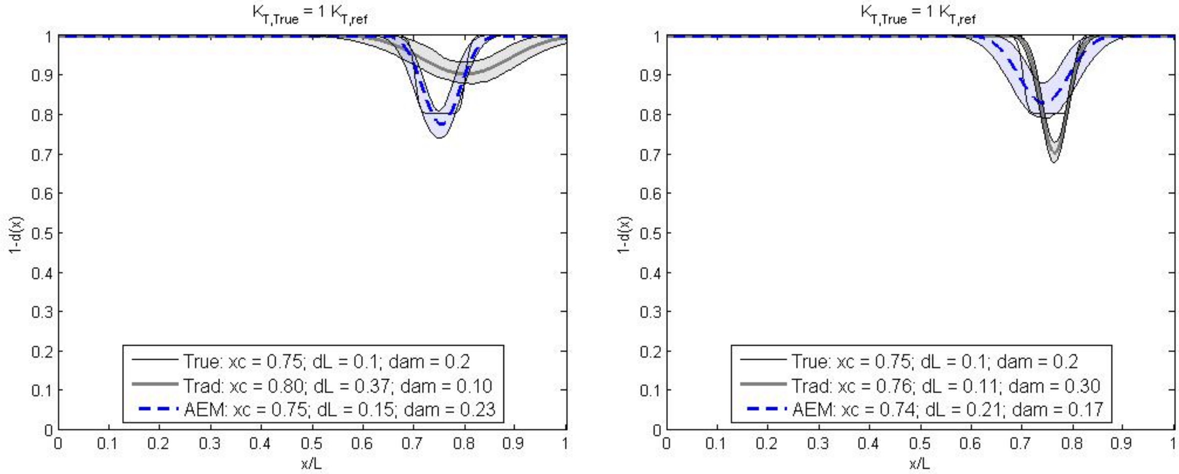
(c) $K_{t,True} = 1.2\,K_{t,REF}$

(d) $K_{t,True} = 1.5\,K_{t,REF}$

Figure 5: Damage field estimated for different values of $K_{t,True}$ using the same FM with $K_t = K_{t,REF}$ for both the APA and least-squares approach. All cases consider $x_F = x_{F,REF}$.

4.2 Uncertainty at Force Boundary Condition

This section presents results concerning the solution of the inverse problem, that is, the estimated damage field, in a scenario with unknown boundary conditions corresponding to the position in which the excitation is applied to the structure.



(a) $x_{F,True} = 0.945L$ (b) $x_{F,True} = 0.955L$

Figure 6: Damage field estimated for different values of $x_{F,True}$ using the same FM with $x_F = x_{F,REF}$ for both the APA and least-squares approach. All cases consider $K_{T,True} = K_{T,REF}$.

When dealing with uncertainties associated with the application of the excitation, the APA shows more consistent results when compared to the traditional approach.

5. CONCLUSIONS

This work has showed that, even though both approaches can be useful for estimate the damage field given the limitations of the parametrization selected, in certain scenarios the APA was able to produce better results when compared to the traditional approach while still not having to estimate the uncertain boundary conditions.

6. ACKNOWLEDGEMENTS

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