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# A BEM implementation for nonhomogeneous and anisotropic fracture mechanics

**Fernando Simionato**
**Carlos Henrique Daros**
State University of Campinas, School of Mechanics Engineering, Campinas - SP, Brazil
fersimionato@gmail.com; chdaros@fem.unicamp.br

***Abstract.*** *A traction based Boundary Element Method (BEM) formulation is presented for the mode III crack problem in exponentially nonhomogeneous anisotropic media. The traction Boundary Integral Equation (BIE) is strong singular and it is evaluated using the Shifted Point Method. The crack surfaces near the crack tips are discretized with special quarter-point boundary elements. The three main crack configurations, central crack, central slant crack, and edge crack are studied to determine the stress intensity factors (SIFs). Results for the homogeneous and isotropic case are compared with available solutions in the literature, showing excellent results. For the case of exponential nonhomogeneity, numerical results are obtained for weakly stiffer media and for softer media.*

***Keywords:*** *mode III crack, nonhomogeneous medium, anisotropic medium, Stress Intensity Factor*

## 1. INTRODUCTION

A functionally graded material is a material in which its composition and structure gradually change, resulting in man made materials with tailored nonhomogenous properties. Such materials have important technological applications as thermal barries for components of the aerospace industry.

Since the study of this kind of material has grown extensively, which includes crack analysis, numerical techniques are required. One such numerical method is the BEM.

(Rangelov *et al.*, 2005) obtained time harmonic fundamental solutions for some two-dimensional nonhomogeneous anisotropic media. The same material nonhomogeneity used in (Rangelov *et al.*, 2005) was applied by (Daros, 2008, 2010) to obtain a fundamental solution for anti-plane time-harmonic problems in anisotropic solids. These fundamentals solutions will be used in the present work.

It is well known that the displacement Boundary Integral Equation (BIE) formulation degenerates for crack analysis as shown by (Cruse, 1978; Sladek and Sladek, 1984). Therefore, there has been an effort for deriving numerically stable traction BIEs. An interesting study on the accuracy and stability of hyper-singular traction BIE was developed by (Rangelov *et al.*, 2003). Based on previous works developed by (Rangelov *et al.*, 2003) and (Daros, 2008, 2010) we extend the BEM to obtain the SIF mode III for exponentially nonhomogeneous anisotropic media with different crack configurations. Exponentially Functionally Graded Materials play a key role in several advanced technological applications, see e.g. (Suresh and Mortensen, 1998).

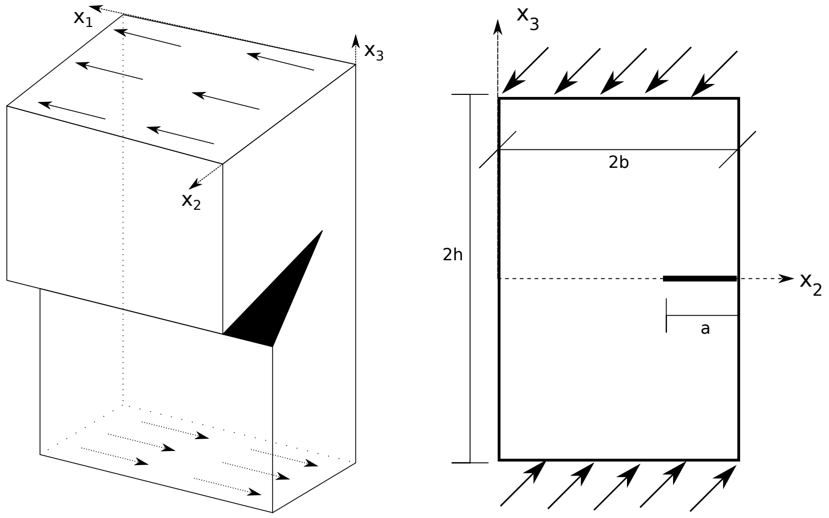
## 2. BOUNDARY ELEMENT METHOD FORMULATION

The aim of this work is to find SIFs at the tips of the cracks. We focus here on obtaining the mode III for the anti-plane problem in the special plane strain case. Figure 1 illustrates the mode III, tearing mode.

To solve the problem using the BEM, we need a fundamental solution. Both solutions for displacement and traction are extracted from the work by (Daros, 2008) and with a few modifications, we obtain the kernels for the nonhomogeneous anisotropic case. We consider here the anti-plane displacement ($U_{x_1}$) of a transversely isotropic nonhomogeneous media in the cartesian plane $(x_2, x_3)$. The stiffness components $c_{44}(x_2, x_3)$ and $c_{66}(x_2, x_3)$ are defined as $c_{44}(x_2, x_3) = c^0_{44} h(x_2, x_3) = c^0_{44} e^{2(a_2x_2+a_3x_3)}$ and $c_{66}(x_2, x_3) = c^0_{66} h(x_2, x_3)$. With $h(x_2, x_3)$ defining the inhomogeneity function and $c^0_{44}$, $c^0_{66}$ the stiffness constants of a reference point.

The fundamental solution for displacement is represented by the Eq. 1 and the associated traction fundamental solution is given by Eq. 2.

$$U^*_{x_1}(x_2, x_3) = -\frac{h^{-1/2}(x^i_2, x^i_3)h^{-1/2}(x_2, x_3)}{2\pi c^0_{44}\sqrt{\delta}} K_0\left(ac\right) \tag{1}$$




(a) 3D model of a tearing mode (b) Mode III crack problem

Figure 1. Illustration of a crack in a mode III problem.

$$P^*_{x_1}(x_2,x_3) = h^{1/2}(x^i_2,x^i_3)[h^{1/2}(x_2,x_3)p^*_{x_1} - h_l U^*_{x_1}] \tag{2}$$

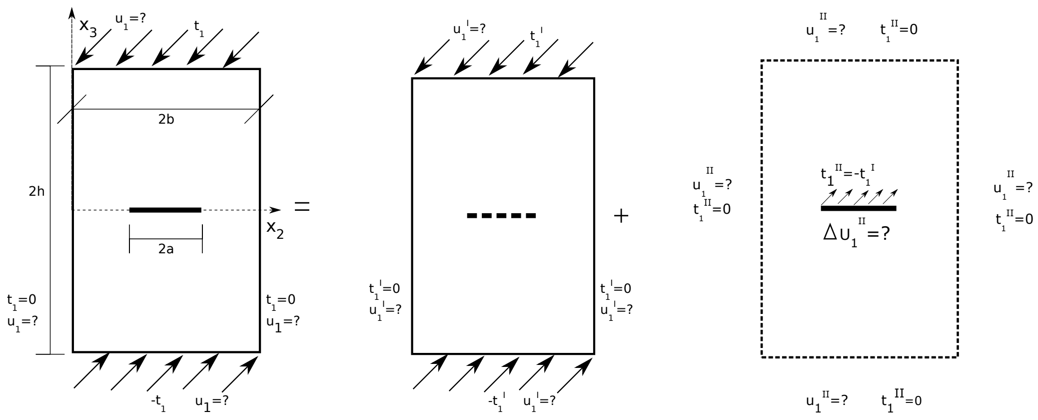
With $c = \sqrt{(\delta a_{22} + a_{33})}$, $a = \sqrt{(x_2 - x^i_2)^2/\delta + (x_3 + x^i_3)^2}$, $\alpha = ((x_2 - x^i_2)/a)n_{x_2} + ((x_3 - x^i_3)/a)n_{x_3}$, $\delta = c^0_{66}/c^0_{44}$ and $h_l = c^0_{66}(h^{1/2}(x_2,x_3))_{,x_2}n_{x_2} + c^0_{44}(h^{1/2}(x_2,x_3))_{,x_3}n_{x_3}$. $K_0$ and $K_1$ are the modified Bessel functions of order zero and one, respectively. The terms $a_{22} = a^2_2$ and $a_{33} = a^2_3$ are related to the nonhomogeneity of the material. The term $p_{x_1}$ is given by Eq. 3 (see (Clements and Azis, 2000)).

$$p^*_{x_1}(x_2,x3) = -\frac{ca}{2\pi\sqrt{\delta}}K_1(ac). \tag{3}$$

Notice that as $c \to 0$ and $h(x_2,x_3) \to 0$, we recover the fundamental solution for the homogeneous isotropic media

### 2.1 Non-hypersingular traction BIE formulation

(Cruse, 1978) and (Sladek and Sladek, 1984) showed that a displacement BIE formulation degenerates for the crack problem. Therefore (Rangelov *et al.*, 2003) and (Dineva *et al.*, 2010) use a traction BIE formulation to achieve a more accurate solution. Their derivation is based on the use of a non-hypersingular traction BIE and the Shifted Point Method (SPM). This method can be used in any arbitrary cracked domains using the well known superposition method. The problem is divided into two parts. Figure 2 illustrates how the problem is divided.




(a) The complete problem. (b) Problem I. (c) Problem II.

Figure 2. Method of superposition.

The non-hypersingular traction BIE for problem *I* and problem *II* are given by the Eq. 4 and Eq. 5, respectively.

$$\frac{1}{2}t^I_1(\mathbf{x^i}) \quad = \quad c_{\lambda 11\gamma}(\mathbf{x^i})n_\lambda(\mathbf{x^i})\left\{\int_\Gamma [\sigma^*_{1\alpha}(\mathbf{x},\mathbf{x^i})u^I_{1,\alpha}(\mathbf{x})\delta_{\beta\gamma} - \sigma^*_{1\beta}(\mathbf{x},\mathbf{x^i})u^I_{1,\gamma}(\mathbf{x})\delta_{\beta\gamma}]n_\beta(\mathbf{x})d\Gamma\right.$$

$$- \int_{\Gamma} U^*_{1,\gamma}(\mathbf{x},\mathbf{x^i}) t^I_1(\mathbf{x}) d\Gamma \Bigg\} , \tag{4}$$

$$\begin{aligned} t^c_1(\mathbf{x^i}) &= c_{\lambda 11\gamma}(\mathbf{x^i}) n_\lambda(\mathbf{x^i}) \left\{ \int_{\Gamma^+_{cr}} [\sigma^*_{1\alpha}(\mathbf{x},\mathbf{x^i}) \Delta U^{II}_{1,\alpha}(\mathbf{x}) \delta_{\beta\gamma} - \sigma^*_{1\beta}(\mathbf{x},\mathbf{x^i}) \Delta U^{II}_{1,\gamma}(\mathbf{x})] n_\beta(\mathbf{x}) d\Gamma \right\} \\ &+ c_{\lambda 11\gamma}(\mathbf{x^i}) n_\lambda(\mathbf{x^i}) \left\{ \int_{\Gamma} [\sigma^*_{1\alpha}(\mathbf{x},\mathbf{x^i}) u^{II}_{1,\alpha}(\mathbf{x}) \delta_{\beta\gamma} - \sigma^*_{1\beta}(\mathbf{x},\mathbf{x^i}) u^{II}_{1,\gamma}(\mathbf{x})] n_\beta(\mathbf{x}) d\Gamma \right\} \end{aligned} \tag{5}$$

With $t^c_1(\mathbf{x^i}) = \begin{cases} -t^I_1(\mathbf{x^i}) & \mathbf{x^i} \in \Gamma^+_{cr} \\ 0 & \mathbf{x^i} \in \Gamma \end{cases}$ ,

$c_{\lambda 11\gamma}(\mathbf{x^i}) = c^0_{\lambda 11\gamma} h(\mathbf{x^i})$, $\sigma^*_{1\alpha} = c_{\alpha 11\beta}(\mathbf{x^i}) U^*_{1,\beta}$, $\Delta U^{II}_{1,\alpha}(\mathbf{x^i}) = u^+_{1,\alpha}(\mathbf{x^i}) - u^-_{1,\alpha}(\mathbf{x^i})$, $u^+_{1,\alpha}(\mathbf{x^i})$ and $u^-_{1,\alpha}(\mathbf{x^i})$ are the displacement gradients of upper and lower surface of the crack, respectively. $u^I_{1,\alpha}$ and $u^{II}_{1,\alpha}$ represent the boundary's displacement gradients of the problem I and II, and $n_\lambda(\mathbf{x^i})$ is the normal vector of the source point. Greek symbols imply here variation from 2 to 3

Problems I and II are coupled by adding the boundary displacement and tractions ($t_1(\mathbf{x^i})$) as shown by Eq. 6.

$$t_1 = t^I_1 + t^{II}_1, u_1 = u^I_1 + u^{II}_1 \tag{6}$$

Since all variables are known it is now possible to obtain the traction ($t_1(\mathbf{x^i})$) field inside the domain, near the crack tip. By doing so we can numerically obtain the SIFs using a limiting process.

### 2.2 Stress Intensity Factor

The SIFs at the left and right crack tips are defined by

$$K^l_{III} = \lim_{x_2 \to l} \sqrt{2\pi(l - x_2)} t_1(x_2, 0) \tag{7}$$

$$K^r_{III} = \lim_{x_2 \to r} \sqrt{2\pi(x_2 - r)} t_1(x_2, 0) \tag{8}$$

respectively (Fig. 2(a)). It is worth point out that all obtained SIFs in the present work were normalized by $K_{III} = t_1\sqrt{a\pi}$, which is the stress intensity factor for a homogeneous media.

## 3. NUMERICAL RESULTS

The material used in this work is cadmium with the following properties, $c^0_{44} = 1,95.10^6 N/m^2$ and $c^0_{66} = 3,96.10^6 N/m^2$. In order to obtain the SIFs by Eq. 7 and Eq. 8, we have to determine the optimal radius that leads to accurate results.

### 3.1 Homogeneous isotropic case

This optimal radius was calculated using a homogeneous isotropic case as a benchmark. (Ma, 1988) studied a central crack in a rectangular sheet subjected to an anti-plane load. For this specific configuration with $2h = 40.10^{-3}\ m$, $2b = 20.10^{-3}\ m$, $t_0 = 100 N/m^2$, $c^0_{44} = c^0_{66} = 1,95.10^6 N/m^2$ and $2a = 10.10^{-3}\ m$, we have $K_{III} = 1.128$. $2a$ represents the length of the crack.

The boundary ($\Gamma$) and the crack ($\Gamma^+_{cr}$) are discretized with 70 and 21 quadratic elements, respectively. Figure 3 illustrates the behavior of the results for different radii. We will use a distance of $0.0025$ % to calculate the SIFs for next cases.

#### 3.1.1 Central crack

Table 1 shows the values of $K_{III}$ for both left and right sides of the crack.

Table 1. Numerical results of SIF in a homogeneous isotropic case
on both sides of the crack

| | (Ma, 1988) | traction BIE |
|---|---|---|
| $K_{III}$(l) | 1.128 | 1.1284 |
| $K_{III}$(r) | 1.128 | 1.1284 |

Singular quarter-point elements are traditionally used for obtaining SIFs in homogeneous, isotropic media (see (Brebbia *et al.*, 2012)). In the latter, special interpolation functions are used to obtain the SIFs directly from boundary traction

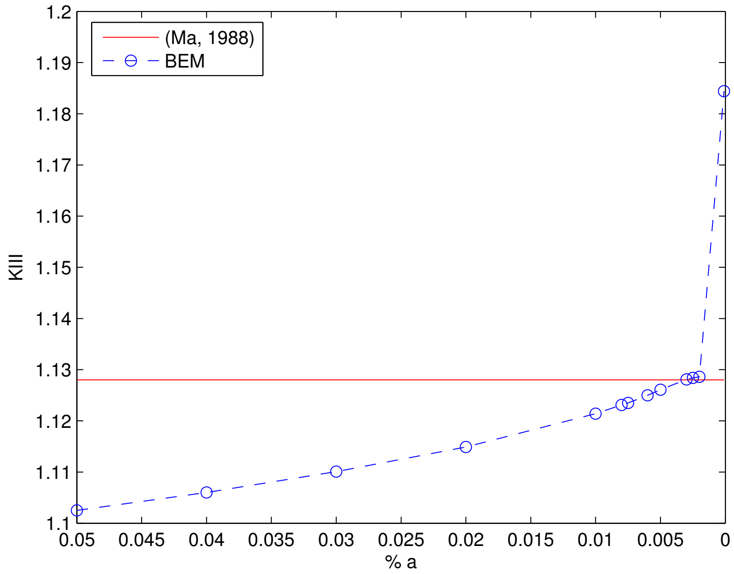
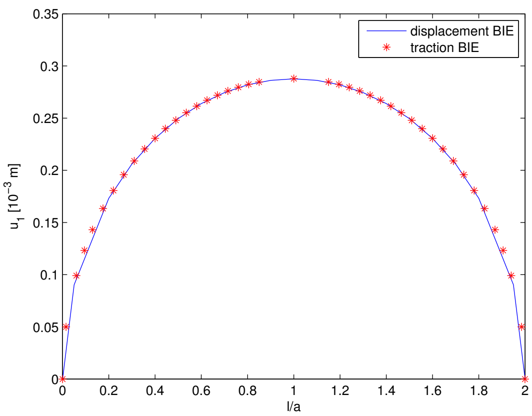


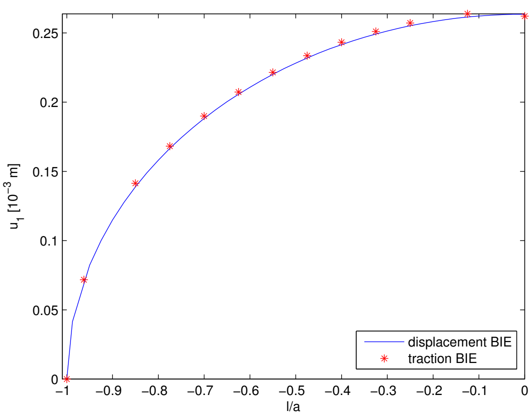

Figure 3. Numerical results of SIF in a homogeneous isotropic case for different radii to both sides of the crack

values, using the traditional displacement BIE. The SIF and COD for an edge crack in mode III can be easily obtained for symmetric geometries.

The COD for a central crack in a tearing mode is shown by Fig.4(a).




(a) Crack opening displacement of upper surface.




(b) Crack opening displacement of upper surface for an edge crack.

Figure 4. Crack Opening Displacement

### 3.1.2 Edge crack

The same standard displacement BIE can be extended for an edge crack. Here the boundary and the crack are discretized with 30 and 6 quadratic elements, respectively. The same radius is used here to calculate the SIF. Table 2 shows that the traction BIE produces excellent results when compared to the traditional BIE. Figure. 4(b) illustrates the COD for an edge crack.

Table 2. Numerical results of SIF in a homogeneous isotropic case for an edge crack

| | BIE | traction BIE |
|---|---|---|
| $K_{III}$ | 1.0269 | 1.0269 |

#### 3.1.3 Slant crack

(Wang, 1993) obtained several SIF for different angles in different geometries. For the case where $2a = 10.10^{-3}\ m$ and $2b = 2h = 20.10^{-3}\ m$, we have $K_{III}$ for different angles. Table 3 presents the numerical results obtained with the traction BIE.

Table 3. Numerical results of SIF in a homogeneous isotropic case for a slant crack

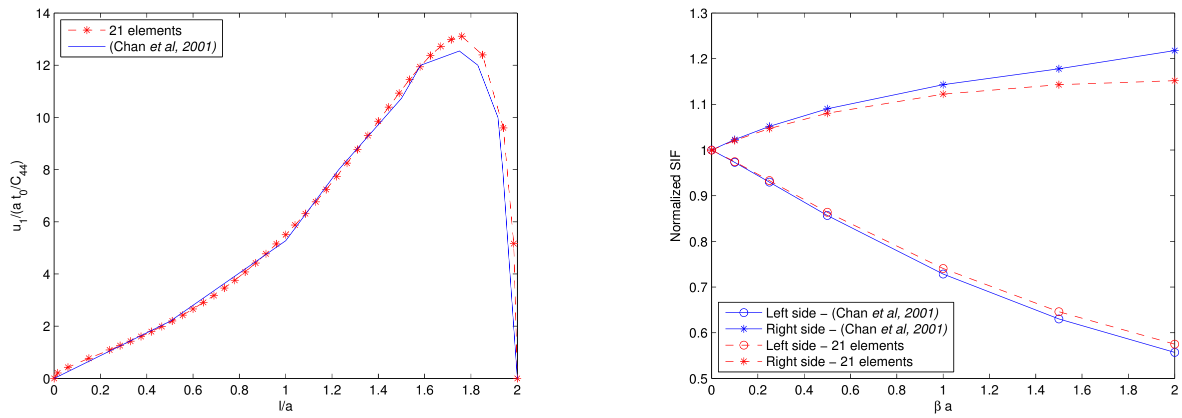
| | 20° | 40° | 60° | 80° |
|---|---|---|---|---|
| $K_{III}$ (Wang, 1993) | 1.0562 | 0.8543 | 0.5594 | 0.1960 |
| $K^l_{III}$ | 1.0539 | 0.8521 | 0.5578 | 0.1956 |
| $K^r_{III}$ | 1.0540 | 0.8523 | 0.5581 | 0.1955 |

### 3.2 Nonhomogeneous isotropic case

Once the homogeneous problem has been successfully used as a benchmark for the traction BIE, we consider now the nonhomogeneous medium. However, results for the nonhomogeneous case are very scarce, one exception is a central crack in an infinite exponentially nonhomogeneous, isotropic medium. The crack problem for nonhomogeneous materials was studied by (Chan *et al.*, 2001) in a infinite medium, with an exponential function $G(x_2) = G_0 exp(\beta x_2)$. Our exponential function is defined by $G(x_2) = G_0 exp(2a_2 x_2)$, therefore we have $\beta = 2a_2$. In this particular case, we use $G_0 = c^0_{44} = 1,95.10^6 N/m^2$ and $\beta a = -2$ for Fig. 5(a) (see(Chan *et al.*, 2001)). The reference system is located at the left crack tip. For every case in a finite medium, the reference system is located at the middle of the left side (see Fig. 1(b)).

#### 3.2.1 Central crack in an infinite medium

Figure 5(a) and fig. 5(b) show a comparison between (Chan *et al.*, 2001) results and the ones obtained using the traction BIE.




(a) COD for a central crack in a nonhomogeneous infinite medium..

(b) SIFs for a central crack in a nonhomogeneous infinite medium.

Figure 5. Crack in a nonhomogeneous infinite medium with radius 0.0025% of half crack's size.

(Chan *et al.*, 2001) results were obtained via numerical integrations involving a series of Chebyshev polynomials. We can see the traction BIE is only accurate for weakly stiffer nonhomogeneous media ($\beta a < 0.5$). For softer media the results are in good agreement with (Chan *et al.*, 2001) results.

#### 3.2.2 Central crack in a finite medium

We used here different factors for assessing the effects of nonhomogeneity in a crack embedded in a rectangular sheet. For this case and the next we use $2b = 20.10^{-3}\ m$, $2h = 40.10^{-3}\ m$ and $2a = 10.10^{-3}\ m$, except when we are dealing with edge crack. In this particular problem $a = 5.10^{-3}\ m$. The results are shown in Tab. 4.

Table 4. Numerical results for SIF in a nonhomogeneous isotropic case for a central crack

| $\beta$a | 0.0 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|
| $K^l_{III}$ | 1.1252 | 0.6608 | 0.2856 | 0.1035 |
| $K^r_{III}$ | 1.1255 | 1.3386 | - | - |

### 3.2.3 Edge crack in a finite case

The same procedure used in 3.2.2 is repeated here. The results are shown in Tab. 5.

Table 5. Numerical results for SIF in a nonhomogeneous isotropic case for an edge crack

| $\beta a$ | -1.5 | -1.0 | -0.5 | 0.0 | 0.5 |
|---|---|---|---|---|---|
| $K^l_{III}$ | 0.0543 | 0.1675 | 0.4510 | 1.0253 | 1.8931 |

## 3.3 Nonhomogeneous anisotropic case

This section presents the results for a crack in a nonhomogeneus anisotropic material (Cadmium).

### 3.3.1 Central crack

The results for a central crack are shown in Tab. 6.

Table 6. Numerical results of SIF in a nonhomogeneous anisotropic case for a central crack

| $\beta$a | 0.0 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|
| $K^l_{III}$ | 1.1217 | 0.6388 | 0.2613 | 0.0875 |
| $K^r_{III}$ | 1.1235 | 1.3409 | - | - |

### 3.3.2 Edge crack

For an edge crack, we have the following results indicated by Tab. 7.

Table 7. Numerical results of SIF in a nonhomogeneous anisotropic case for an edge crack.

| $\beta$a | -1.5 | -1.0 | -0.5 | 0.0 | 0.5 |
|---|---|---|---|---|---|
| $K^l_{III}$ | 0.0401 | 0.1425 | 0.4244 | 1.0269 | 1.9457 |

# 4. CONCLUSION

The two dimensional anti-plane crack problem of a functionally graded material is solved by non-hypersingular traction BIE. The material properties vary exponentially with two spatial variables. Quadratic shape functions and quarter-point boundary elements at the crack tips are applied. Numerical examples for center cracks, edge cracks and slant crack are solved using the Shifted Point Method. Numerical results demonstrate that SIFs are sensitive to direction and magnitude of the material nonhomogeneity, the radius for limiting process and the geometry of the crack system. Results for the homogeneous and isotropic case were compared with available solutions in the literature, showing excellent results. For the case of exponential nonhomogeneity, good numerical results were obtained for weakly stiffer media and for softer media.

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