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THE GENERALIZED FINITE ELEMENT METHOD APPLIED TO DYNAMIC TRANSIENT ANALYSIS

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Abstract. *The Finite Element Method (FEM), although widely used as an approximate solution method, has some limitations when applied in dynamic analysis. As the loads excite the high frequencies and modes, the method may lose precision. To improve the representation of these high frequency modes, the Generalized Finite Element Method (GFEM) is used to enrich the approach space with appropriate functions according to the problem under study. Although the GFEM has proven to be efficient in dynamic analysis of structures, there are still some issues to be analyzed. In this context, the present work presents an alternative to improve the numerical response of the structure obtained by GFEM, condensing the modal matrix, and eliminating from it the modes of vibration with bad approximation. The results showed that the technique improves the transient response as well as the computational efficiency of the analysis in question.*

Keywords: *dynamic analysis, FEM, GFEM.*

1. INTRODUCTION

In the context of analysis of structures, one of the numerical methods most used is the Finite Element Method (FEM), which provides accurate and efficient results. However, in the dynamic analysis the FEM presents good results only for the first frequencies of the structure. For the higher frequencies the precision can be improved with a high computational cost.

In order to overcome this problem, particular information is used to improve the analysis results. This is the main purpose of the enriched methods, in particular the Generalized Finite Element Method (GFEM). The GFEM uses enriching functions order to obtain better results for problems with singularities or other situations where the FEM needs a more refined mesh, and as a consequence, more computational effort.

The GFEM has been shown to be efficient in dynamic structural analysis (Arndt, 2009; Torii and Machado, 2012; Shang, Machado and Abdalla Filho, 2016; Weinhardt, 2015). However, there is still a need for a more detailed study of transient analysis. Thus, the present work intends to improve the numerical response obtained by the GFEM, using a condensed modal matrix where the vibration modes with unsatisfactory approximations are disregarded.

2. BAR TRANSIENT ANALYSIS

The vibration of the bar is a time dependent problem, and the equation of motion governing this problem is a partial differential equation. The problem is to find the axial displacement u satisfying:

$$\rho A \frac{\partial^2 \overline{u}}{\partial t^2} dx - \frac{\partial}{\partial x}\left(EA \frac{\partial \overline{u}}{\partial x} \right) = p(x,t) \qquad (1)$$

where $A = A(x)$ is the cross section area, $E = E(x)$ is the Young modulus, ρ is the specific mass, p is the externally applied axial force per unit length and t is the time. The solution $u = u(x, t)$ must satisfy the boundary and initial conditions defined in the problem.

The application of the standard Finite Element Method (FEM) procedure (i.e. multiplying the PDE by some trial function, integrating inside the domain and then applying Green's Theorem) (Bathe, 1996; Hughes, 1987; Zienkiewicz and Taylor, 2000) results in the following system of differential equations:

$$Ku + M\ddot{u} = F, \tag{2}$$

where K is the stiffness matrix, u is the vector of displacements, M is the mass matrix, $\ddot{u}$ is the vector of accelerations and F is the vector of applied forces. In matrix form from the contributions of each finite element is given by:

$$K_{ij}^{e} = EA\int_{\Omega^{e}} \frac{d\phi_i}{\partial x}\frac{d\phi_j}{\partial x}\, d\Omega^{e} \tag{3}$$

$$M_{ij}^{e} = \rho A\int_{\Omega^{e}} \phi_i\phi_j\, d\Omega^{e} \tag{4}$$

$$F_{i}^{e} = \int_{\Omega^{e}} p(x)\phi_i\, d\Omega + EA\left[\frac{\partial u}{\partial x}\phi_i\right]_{\partial\Omega} \tag{5}$$

The computation of Eq. (2) in displacement terms, can be done using a direct time integration method for each time step, although it is easily understood that for structural systems with a large number of degrees of freedom the computational costs starts to be very significant or even unsustainable. The modal superposition technique is computationally more efficient than the direct time integration (Bathe, 1996) once the global dynamic behavior of the structure can be properly reproduced considering the superposition of a limited number of vibration modes, being this possible if the structure is subjected to a linear analysis, as in this work. The decoupling of the system is done through the modal matrix, pre and post multiplying the system of Eq. (2). Using the modal superposition method, the system of $n \times n$ simultaneous equations is converted in n uncoupled equations:

$$\omega_i^2 u_i + \ddot{u}_i = F_i, \quad i = 1,2,...,n, \tag{6}$$

where ω_i is the ith natural frequency of the structure. That can be solved independently by a direct integration method, such as Newmark Method (Newmark, 1959), which was used in this work.

3. THE FEM AND THE GFEM

The FEM uses polynomials shape functions (Bathe, 1996; Hughes, 2000) in the approximated solution which can be expressed in matrix form as:

$$u_h^{\,e}(\xi) = N^T q, \tag{7}$$

where N is the matrix of shape functions and q is the displacement vector. The polynomial functions may be of any order. For bar element with one degree of freedom per node, the terms of the approximated solution (Eq. 7) are defined in the master domain as:

$$N^T = [\psi_1 \quad \psi_2], \tag{8}$$

$$q^T = [u_1 \quad u_2], \tag{9}$$

where u_1 and u_2 are the nodal displacements, and ψ are the local shape functions.

The basic concepts of FEM can be extended to the GFEM (Babuska et al., 2004; Oden et al., 1998) which is a method based on the Partition of Unit Method, proposed by Melenk and Babuska (1996). It is a Galerkin method that aims to enrich the finite element by constructing a subspace of pre-established solution approximation functions. This subspace aims to improve local and global results when compared to conventional FEM.

The approximated solution proposed by the GFEM in the master element domain may be written as the sum of two components:

$$u_h^{\,e} = u_{FEM} + u_{ENRICHED} \tag{10}$$

where u_{FEM} is the Finite Element Method component based on nodal degrees of freedom and $u_{ENRICHED}$ is the enriched component by the partition of unity approach based on field degrees of freedom. In this sense, the bar approximated solution on a master element is (Arndt, 2009):

$$u^e_{MEF}(\xi) = \sum_{i=1}^{2} \eta_i(\xi) u_i , \tag{11}$$

$$u^e_{ENRICH}(\xi) = \sum_{i=1}^{2} \eta_i(\xi) \left[\sum_{j=1}^{n_l} \gamma_{ij}(\xi) a_{ij} + \varphi_{ij}(\xi) b_{ij} \right], \tag{12}$$

where η_i are the partition of unity (PU) functions, γ_{ij} and φ_{ij} are the enrichment functions, n_l is the number of enrichment levels, u_i are the nodal displacement (nodal degrees of freedom), and a_{ij} and b_{ij} are the field degrees of freedom related to the enrichment functions. Linear Lagrangian partition of unity (Eq. 13) were used by the works of Arndt (2009) and Torii (2012) as the basis for trigonometric GFEM and therefore will be considered here as well.

$$\begin{cases} \eta_1 = 1 - \xi \\ \eta_2 = \xi \end{cases} \tag{13}$$

In this work, the enrichment functions for the bar element are the trigonometric functions proposed by Arndt (2009). This group of functions consists in the construction of a pair of clouds, a sinusoidal and a cosine, subordinated to the covering of the enriched node. These clouds are written in the element domain as two pairs of sine and cosine functions (Eq. 14, 15, 16, 17). The elementary domain is considered for $\xi \in [0, +1]$.

Sine cloud:

$$\gamma_{1j} = \sin(\beta_j L_e \xi) \tag{14}$$

$$\gamma_{2j} = \sin(\beta_j L_e (\xi - 1)) \tag{15}$$

Cosine cloud:

$$\varphi_{1j} = \cos(\beta_j L_e \xi) - 1 \tag{16}$$

$$\varphi_{2j} = \cos(\beta_j L_e (\xi - 1)) - 1 \tag{17}$$

where L_e is the length of the element and $\beta_j = j\alpha\pi$ is a hierarchical enrichment parameter proposed by Arndt (2009) for function levels and α is an adjustable parameter that can involve the parameters of the material. In the work of Weinhardt (2015), in order to stabilize the method, a modification of the parameter β_j was proposed, being of form:

$$\beta_j = \left(2j - \frac{5}{4} \right) \pi . \tag{18}$$

The modification suggested by the author is used in this work.

4. GFEM FREQUENCY SPECTRUM BEHAVIOR

Transient analysis can be performed initially by decoupling the system of dynamic equilibrium equations of the structure. For this, the technique of modal superposition is used, where the transformation matrix that decouples the system is the modal matrix itself. With the equations decoupled, a direct time integration method can be used to solve the problem. In this work, the Newmark Method is used.

At first, it is sought to analyze the already consolidated application of the GFEM in the free vibration analysis of structures. Afterwards, the results of the frequency spectrum of the analyzed structures are evaluated in order to observe the amount of vibration modes with good approximation in comparison with the analytical solution. With these results, analyzes regarding this percentage can be performed.

In this context, figures 1 and 2 show the frequency spectrum graphs where the approximate frequencies are related to the analytic ones (ω/ω_h) versus the ratio of the nth vibration mode to the total number of modes (*n/N*).

In the work of Shang, Machado and Abdalla Filho (2016) (Fig. 1a), an elastoplastic dynamic analysis was performed using trigonometric enrichment and the accuracy of GFEM with 2, 3 and 4 enrichment levels was compared to the linear FEM. In the work of Weinhardt (2015), (Fig. 1b) the application was particularized for free vibration analysis of a bar element. The author applied the GFEM with trigonometric enrichment, and compared it with GFEM with Partition of Unit defined from the Shepard functions, and also with the linear FEM.

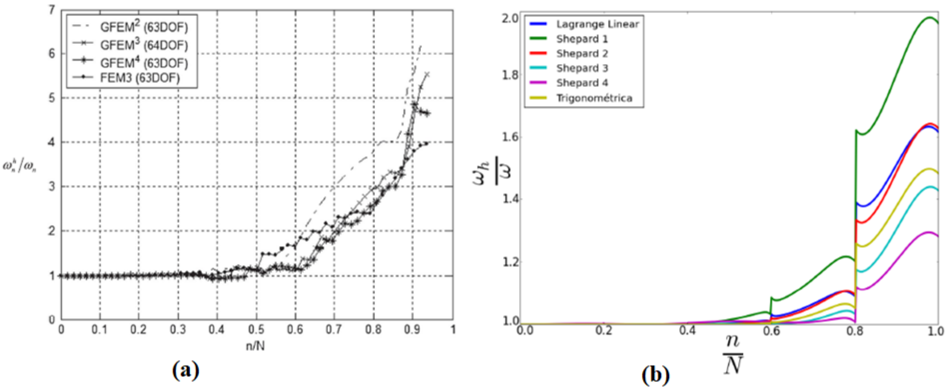

Figure 1: Frequency spectrum of Shang, Machado and Abdalla Filho (2016), and Weinhardt (2015).

The work of Garcia, Rossi and Linzmaier (2010) (Fig. 2), presented also an example of free vibration analysis of bar, where GFEM was applied with Partition of Unit defined from rational functions and polynomial enrichment, with 3 levels of enrichment, and then compared with the linear FEM.

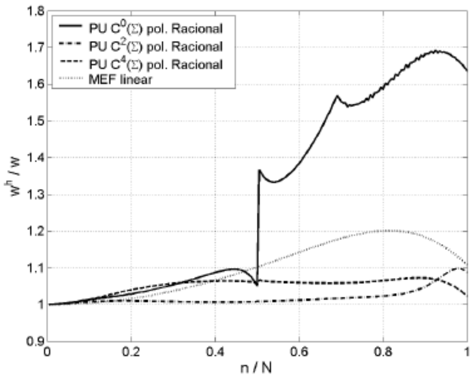

Figure 2: Frequency spectrum of Garcia, Rossi and Linzmaier (2010).

From Figures 1 and 2, it can be concluded that through the GFEM, the range of modes with good approximation is about 50%. Thus, it is expected that with the presence of a few modes, the transient response obtained through the GFEM may be as accurate as the analyzes made with the full modal matrix. Looking still at Figure 1, of which both authors use trigonometric enrichment, the range of acceptable modes is about 60%. This percentage is used as a parameter to study the efficiency of GFEM in the analysis of this work.

5. APPLICATION

To analyze the behavior of GFEM in transient dynamic analysis, one example of cantilever bar is discussed. The application presented below is simple due to the initial state of the research, but are useful to compare the GFEM performance and, once the proposal of this work is verified, it can be applied in more complex structures.

For the example, the transient response to displacement, velocity and acceleration is initially analyzed. A first analysis by applying *p - refinement* of the GFEM with 3 elements is done with 1, 2, 4 and 8 levels of enrichment and the responses are compared with an analytical reference solution given by Nowacki (1963) and Shang (2014), and still with the analysis performed by FEM. From this, the problem is tested with a 40% reduction in modal matrix modes, as discussed above. Discussions are also made about the most influential modes in the transient response of the structure.

For the solution with FEM to be reliable, a convergence test was performed analyzing the problem with 50, 100, 300, 600, and 1000 degrees of freedom. The method demonstrated a certain instability in the convergence of results with the lowest amount of degrees of freedom. Only from 600 degrees of freedom did the displacement values stabilize. Thus, the results obtained with 1000 degrees of freedom were used as reference.

For the purposes of this example, the geometric and material properties (cross section area *A*, elasticity modulus *E*, specific mass ρ and length of bar *L*) are adopted such that that $EA/\rho L = 1$, without loss of generality. The number of degrees of freedom considered in each analysis is the total number of effective degrees of freedom after introduction of boundary conditions. Also, the time interval considered was 20 seconds, with $\Delta t = 1,0 \times 10^{-2}$.

Figure 3 shows the cantilever bar used in this example. The force applied at the free end is a heaviside force of *F = 1N.*

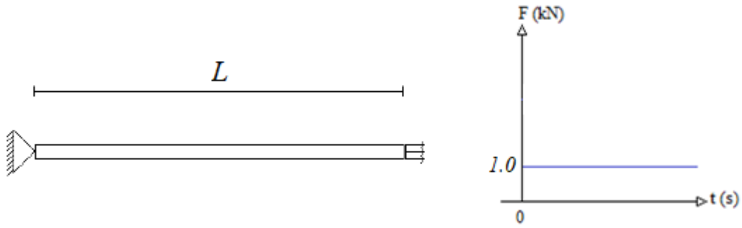

Figure 3: Cantilever bar and heaviside force

The displacement, velocity and acceleration results for the different levels of enrichment made are shown in Figures 4-7. The number in parentheses in the legend symbolizes the number of enrichment levels.

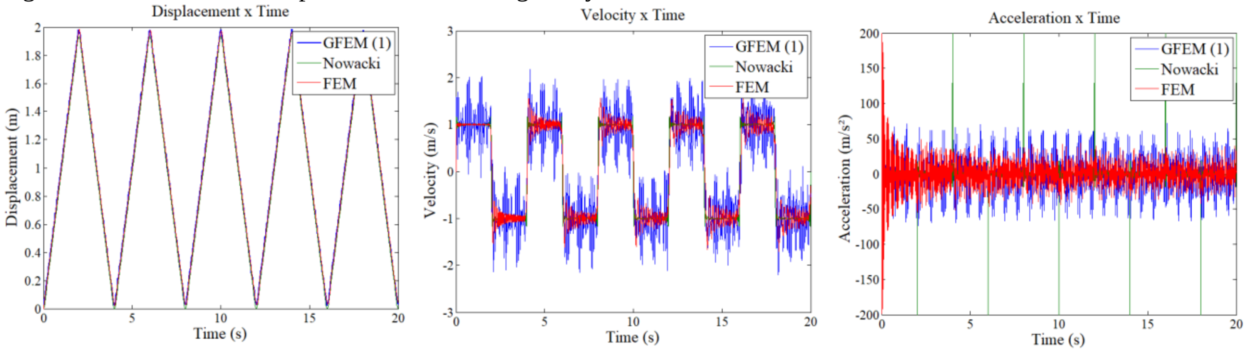

Figure 4: Displacements, velocities and accelerations for GFEM with one level of enrichment

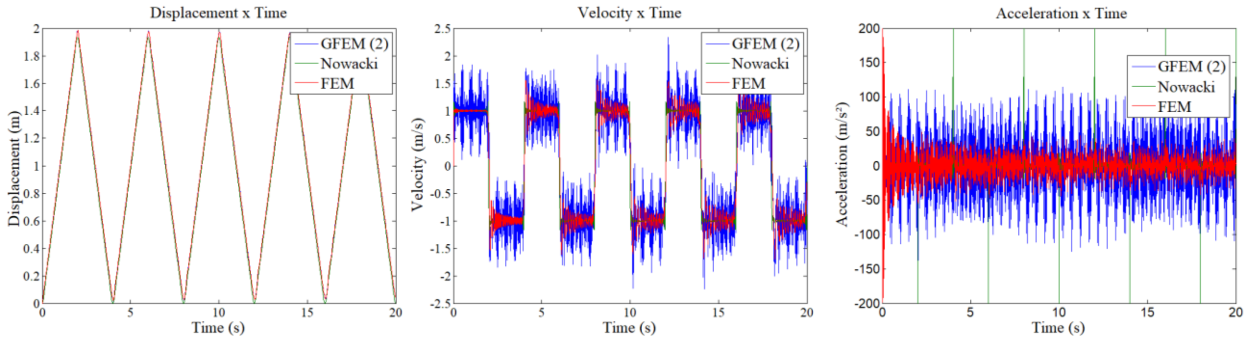

Figure 5: Displacements, velocities and accelerations for GFEM with two level of enrichment

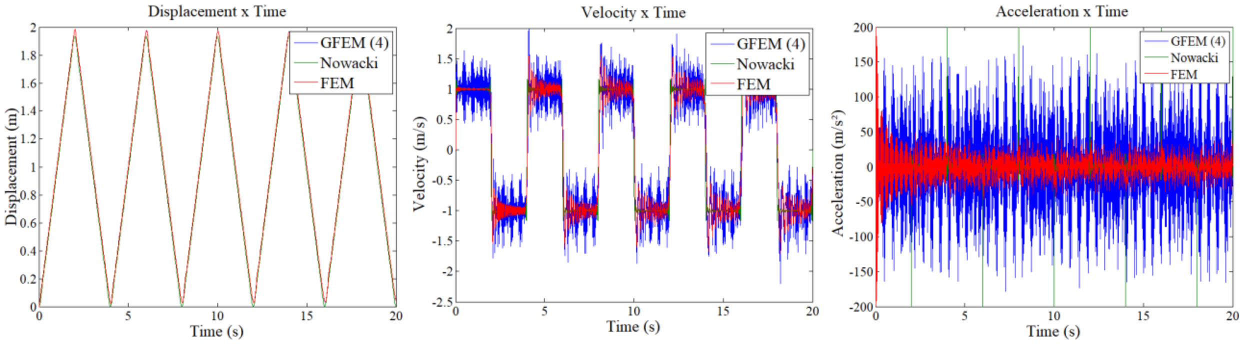

Figure 6: Displacements, velocities and accelerations for GFEM with four level of enrichment

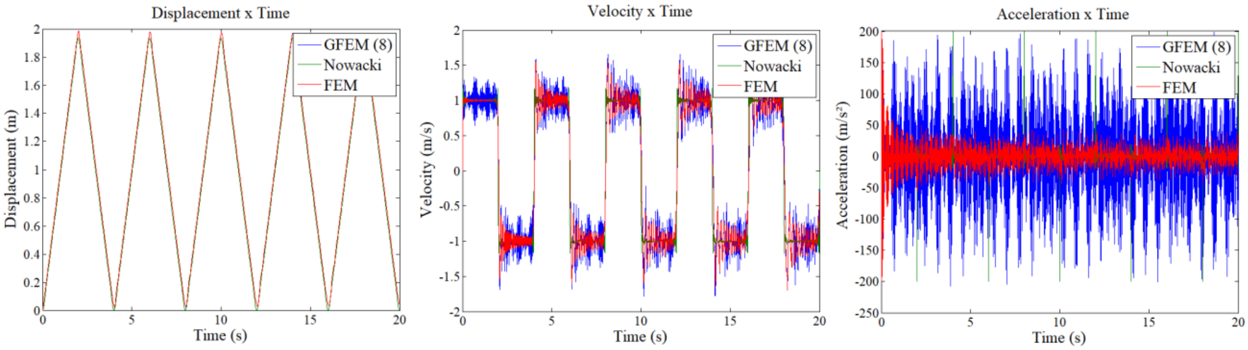

Figure 7: Displacements, velocities and accelerations for GFEM with eight level of enrichment

Analyzing Figures 4-7, we can see that, independently of the proposed level of enrichment, the results have a good approximation. This behavior in relation to the displacements was already expected, due to the results presented in the literature (Weinhardt, 2015; Shang, 2014; Torii, 2012).

With regard to velocities and accelerations, the method proved to be more unstable. In the accelerations, as the level of enrichment increases, the response showed to be convergent at the points of discontinuity, and strongly disturbed outside these points. The velocity has a similar behavior, but less pronounced.

To evaluate the transient response through GFEM using the modal matrix with 60% of the modes, the example with 4 levels of enrichment is again analyzed. As the displacement response proved to be efficient, the following analyzes focus on velocities and accelerations. Thus, Fig. 8 shows the velocities and accelerations when using a percentage of 60% of modes in the modal matrix.

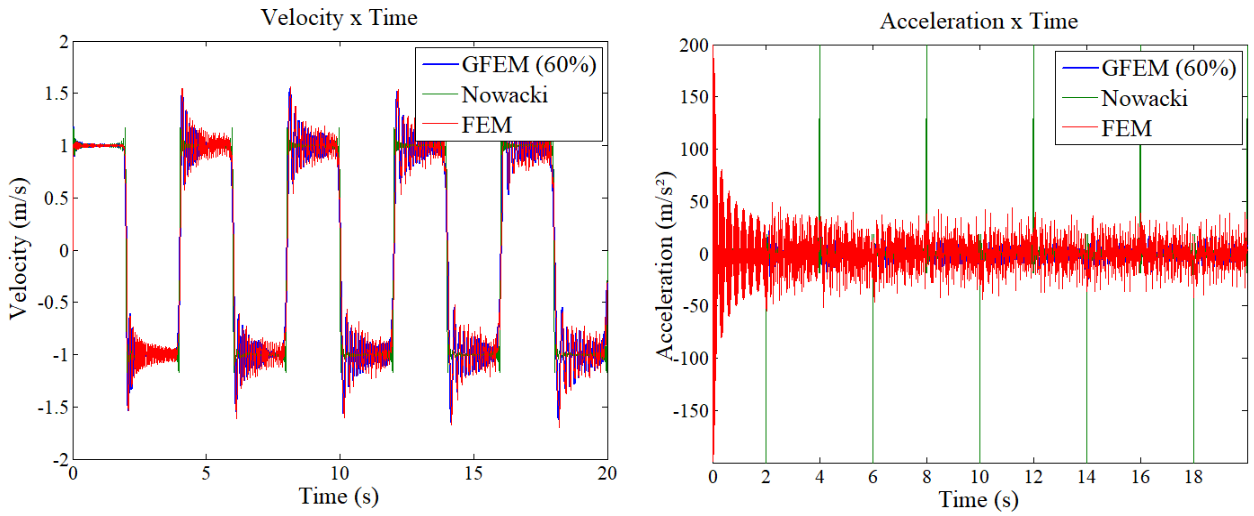

Figure 8: Velocities and accelerations for GFEM with 60% of modes in the modal matrix

It can be seen from Fig. 8 that the response of accelerations and velocities has a significant change in its behavior. Compared with the results obtained with the complete modal matrix, both velocities and accelerations were less disturbed and more convergent to the analytical solution. In order to show the results in more details, Fig. 9 shows a zoom of the graphs of Fig. 8, where the attenuation of the response perturbation can be perceived more clearly.

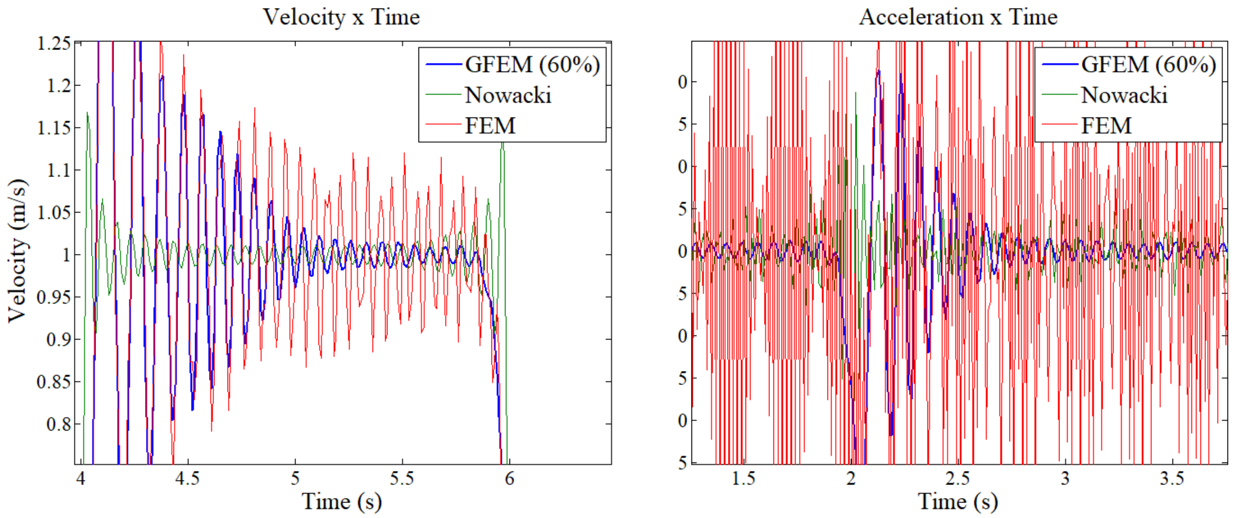

Figure 9: Zoon in velocities and accelerations for GFEM with 60% of modes in the modal matrix

Parallel to the analysis of the transient response with 60% of the modes, the influence of each mode on this response can be analyzed. For this, the generalized coordinates, direct results of the Newmark method, are shown in Fig. 10.

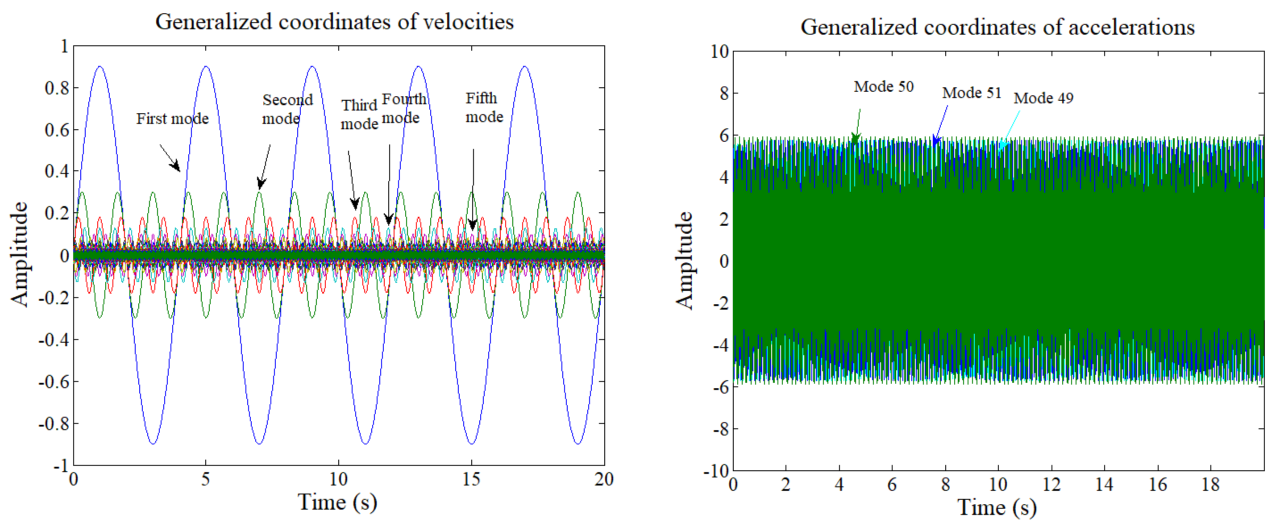

Figure 10: Generalized coordinates of velocities and accelerations

According to Fig. 10, the modes with the most influence on the transient response of the velocities are the first 5. In terms of accelerations, the most prevailing modes are the last 3. Thus, the problem is analyzed using the modal matrix initially with the first 5 modes (Fig. 11), and after, with the last 3 (Fig. 12). The graphs show the results in a range of 0 to 5 seconds, in order to show in more detail the results of the analysis.

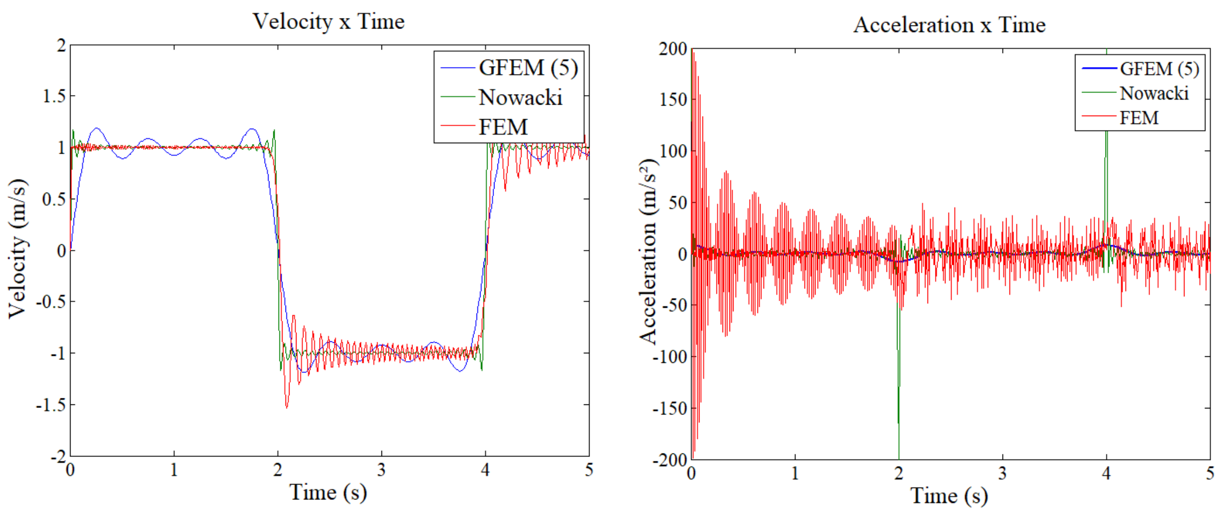

Figure 11: Velocities and accelerations for GFEM with first 5 modes in the modal matrix

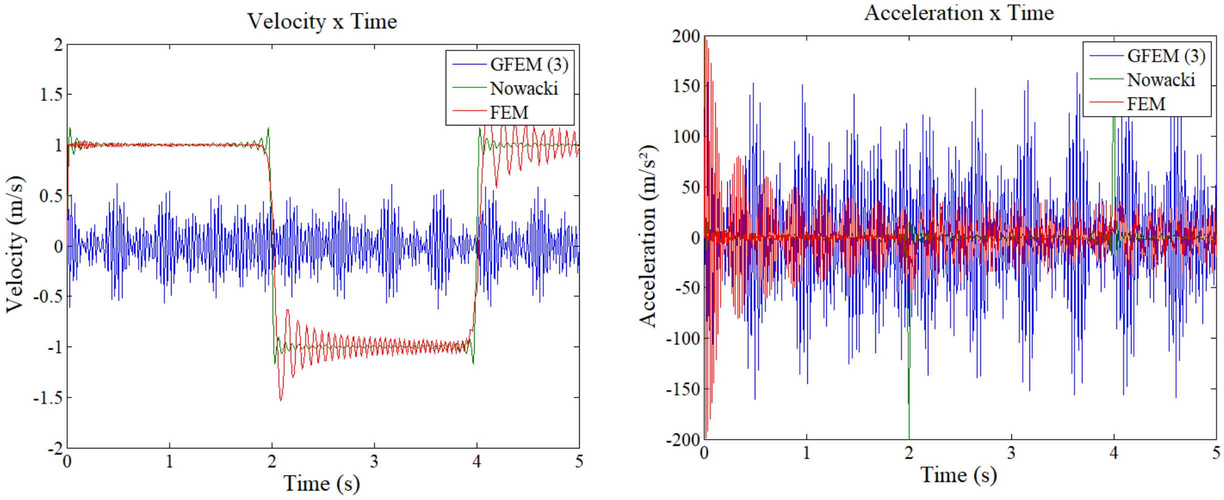

Figure 12: Velocities and accelerations for GFEM with last 3 modes in the modal matrix

Analyzing figures 11 and 12 it is noted that the response in terms of velocities and acceleration are best approximated when the first 5 modes are used in the modal matrix. In analyzing Figure 12, which shows the results using the last 3 modes in the modal matrix, it is noted that the velocity response differs greatly from the reference solutions. Accelerations also showed greater disturbance. By the results obtained with the accelerations, it is understood that the last 3 modes are the most responsible for the perturbations of the response, since, when using the first 5 modes in the modal matrix, the response has a more stable behavior.

In relation to velocities, the response was similar to that obtained with 60% of the modes. Both were more stable when compared with the analyzes made with the complete modal matrix. This reduction in the number of modes from 51 (degrees of freedom of the problem) to 5, represents a gain in the computational cost of the analysis, since the modal matrix size has been reduced to approximately 90.2% of its original size.

6. CONCLUSIONS

The main contribution of the present study consists in the analysis of the behavior of the GFEM in the dynamic analysis of bar element.

This study assumed, based on literature, that only a part of the modes of vibration obtained through GFEM have a good approximation. Thus, these modes with bad approximations were removed from the modal matrix, and the response in terms of acceleration and velocity were analyzed.

The results showed that the transient response of a bar element improves considerably using the modal matrix with the presence of only the modes with good approximation. In general, velocities were more stable. Also, the accelerations were shown with less disturbances outside the points of discontinuity.

In addition to removing the last 40% of modal matrix modes, as previously seen in the literature, the present work analyzed the influence of modes on the transient response of the bar element. From this analysis, it was possible to perceive that the first 5 modes are the most preponderant in the transient response of the bar. Thus, for the example tested, using only 5 modes in the modal matrix, the response in terms of velocity and acceleration proved to be as satisfactory as when using 60% of the modes.

The reduction of approximately 90.2% in the modal matrix size represents, in addition to more precision in the analysis, a gain in terms of computational cost. This efficiency gain can be more significant when analyzing is a larger structure with more elements and loads.

The bar example tested in this work is relatively simple, however, it has been shown that when using GFEM with a very small number of degrees of freedom, and with a small number of modes present in the modal matrix, the results are as accurate as those obtained with reference solutions. The research will continue to study the behavior of GFEM in the transient response, extending the analysis to other types of external excitation, and also to more complex structures such as trusses, frames, shells and plates.

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