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COBEM-2017-0263 PARAMETERS IDENTIFICATION WITH EVALUATION OF UNCERTAINTIES USING SENSITIVITY METHOD AND COVARIANCE MATRIX ANALYSIS

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Abstract. *An iterative sensitive-based procedure is presented in this work for the updating of some chosen input parameters of numerical or analytical models so that the predicted response approximates the measured data as closely as possible. At the same time, the uncertainty of these input parameters is estimated by matching, with the best approximation, the covariance matrices calculated for the predicted output vector and for the measured data. The two main sources of uncertainties, random and epistemic, are incorporated to the numerical model using the elicited uncertain parameters and sensitivities, i.e., a jacobian matrix defined as the variation of the output parameters to some small variation of the inputs, approximated by finite differences. The main steps used in the processes of model updating using the sensitivity method for a simple dynamic model and experimental modal data are delineated, and then an application of this methodology to a three degrees-of-freedom model is presented for which experimental data was acquired. The updated model presented good agreement with the statistical data (mean value and covariance) obtained in the experimental test campaign.*

Keywords: *Model Updating; Uncertainty Quantification; Experimental modal analysis*

1. INTRODUCTION

Model updating is a necessary phase in the development of any structural model that is meant to reliably represent real structures. Automotive, aerospace and aeronautical models need to match with high accuracy both the real static and dynamic behavior in order be practical to make predictions. This means that computational procedures, like finite element method, can only be used with confidence if numerical results lie close to experimental ones. So, in order to assess the quality of a structural mechanical model it is common practice, as a first approach, to compare the measured data against the predicted dynamic behaviour represented by some chosen parameters, most commonly the natural frequencies. Considering the significant levels of inaccuracy in the estimation of some important parameters, various methods have been proposed and are broadly available for identification/updating of dynamic models (Friswell and Mottershead, 1995; Maia *et al*., 1997). However, evaluation of uncertainties associated with the overall process (experimental errors as well as deviations introduced by mathematical model hypotheses) is not yet a common issue in this field, despite the great importance it has received in many areas of engineering research. Hence, it is the main goal of this work to study the application of a sensitivity-based model updating technique following a stochastic approach.
The former part of this work presents an application of the sensitivity method (Patelli *et al*., 2017) in the updating of equivalent transverse motion stiffnesses of a three-storey building model using experimental data from accelerometers. After that, the dispersion of these parameters is also fitted in such a way that the adjusted model can reproduce, with the best approximation, the covariance among the observed natural frequencies in the model measured data. Then, the second part presents the application of the proposed procedure considering mass, stiffness and damping matrices. A separate identification strategy is adopted in which the mass and the stiffness matrices are updated together in a first stage, taking the natural frequencies as the output parameters vector; and next the damping matrix is updated independently, with damping ratios only taken as outputs. This methodology follows the ideas presented by Chen, *et al.* (1996), and García-Palencia (2013), in a more recent work.

2. METHODOLOGY

The basic procedure relies on the minimization of the squared norm of a vector of differences between predicted and measured natural frequencies and damping ratios. In the next sections, further explanation is given on the numerical procedure and on the experimental data acquisition.

2.1 Sensitivity Method for Finite Element Model Updating

The sensitivity method is a Newton-Rahpson based scheme where the jacobian matrix contains the derivatives (or some approximation to them) of the output parameters with respect to the input ones (Friswell and Mottershead, 1995). Once the sensitivity matrix was calculated/estimated, this method returns the values of the input parameters that give outputs closest to the measured ones; these 'best approximation' values are treated here as mean values, and the same sensitivity matrix is used to calculate, in every iteration, the covariance matrix of the input parameters that produces the closest dispersion in the outputs (Patelli *et al*., 2017; Silva *et al*., 2016).

2.2 Mean values updating

Let the vector of unknown parameters $\boldsymbol{\theta} = \{\theta_1, \theta_2, \dots, \theta_{np}\}^T$ with dimension equal to the number of unknown parameters, *np*. Let the vector of the measured output parameters be $\boldsymbol{z}^m = \{z_1^m, z_2^m, \dots, z_{no}^m\}^T$ with dimension equal to the number of the measured outputs, *no*, and $\boldsymbol{z}^p = \{z_1^p, z_2^p, \dots, z_{no}^p\}^T$ be the vector of the predicted output parameters, with same dimension as the measured one, obtained by running the computational (numerical and/or analytical) procedure using the components of $\boldsymbol{\theta}$ as inputs. So, for every iteration it can be stated that:

$$\delta\boldsymbol{z} = \boldsymbol{z}^p - \boldsymbol{z}^m \tag{1}$$

and:

$$\boldsymbol{\varepsilon} = \delta\boldsymbol{z}^p - \boldsymbol{S}\delta\boldsymbol{\theta} \tag{2}$$

where $\boldsymbol{\varepsilon}$ is the (truncation) error function and the sensitivity matrix $\boldsymbol{S}$ is given by:

$$\boldsymbol{S} = \begin{bmatrix} \frac{\partial z_1^p}{\partial \theta_1} & \cdots & \frac{\partial z_1^p}{\partial \theta_{np}} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_{no}^p}{\partial \theta_1} & \cdots & \frac{\partial z_{no}^p}{\partial \theta_{np}} \end{bmatrix} \tag{3}$$

The sensitivity (jacobian) matrix is approximated here by forward finite differences. Considering independent weights for the error function and for the variation of $\boldsymbol{\theta}$ between each iteration, the problem can be stated as below, where $\boldsymbol{W}_{\varepsilon\varepsilon}$ and $\boldsymbol{W}_{\theta\theta}$ are weighting matrices for the error vector and for the input parameters, respectively:

$$minimise\ J(\delta\boldsymbol{\theta}) = \boldsymbol{\varepsilon}^T \boldsymbol{W}_{\varepsilon\varepsilon} \boldsymbol{\varepsilon} + \delta\boldsymbol{\theta}^T \boldsymbol{W}_{\theta\theta} \delta\boldsymbol{\theta} \tag{4}$$

Solving this minimization problem, and defining the components of the output parameters vector, $\boldsymbol{z}^m = \{f_1^m, f_2^m, \dots f_{no}^m\}^T$, as the measured natural frequencies, after algebraic manipulation we arrive at a weighted Newton-Raphson scheme for the updating of the input parameters vector:

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n + \boldsymbol{S}_n^T (\boldsymbol{S}_n \boldsymbol{W}_{\varepsilon\varepsilon} \boldsymbol{S}_n^T + \boldsymbol{W}_{\theta\theta})^{-1} \boldsymbol{S}_n^T \boldsymbol{W}_{\varepsilon\varepsilon} \big(\boldsymbol{f}^m - \boldsymbol{f}^p(\boldsymbol{\theta}_n)\big) \tag{5}$$

2.3 Covariance analysis

Writing the error as the difference between the predicted natural frequencies and their measured expected values, and using the definition of covariance as a function of this expected vector, it is possible to obtain the following relationship to the covariance of the input parameters at step *n+1* as a function of the sensitivity matrix at step *n* and the covariance of the experimental output data, $cov(\boldsymbol{f}^m, \boldsymbol{f}^m)$:

$$cov(\boldsymbol{\theta}_{n+1}, \boldsymbol{\theta}_{n+1}) = \boldsymbol{S}_n{}^{-1} cov(\boldsymbol{f}^m, \boldsymbol{f}^m) \boldsymbol{S}_n{}^{-T} \tag{6}$$

So, the two processes (matching mean and covariance for the output vector) may be considered independent. The overall process can be viewed as a translation of the mean predicted output vector to a point as close as possible to the measured ones, and then rotating its covariance matrix to the experimental covariance function. This can be summarized in two steps:

(a) Find the mean input parameter values that make the best experimental-predicted output match using an optimisation algorithm to minimize the error – here it was adopted the weighted Newton-Rahpson presented earlier;
(b) Once defined the optimum parameters vector, calculate the sensitivity of the error function at this point. The covariance of the input parameters is defined using Equation (6).

2.4 Numerical and experimental model

The three storey building used in the experiments is composed of three polymer blocks rigidly attached to two metal beams, one at each side, and the beams fixed to another block serving as a base to whole structure. The figure below shows a sketch of the model with main dimensions and variables.

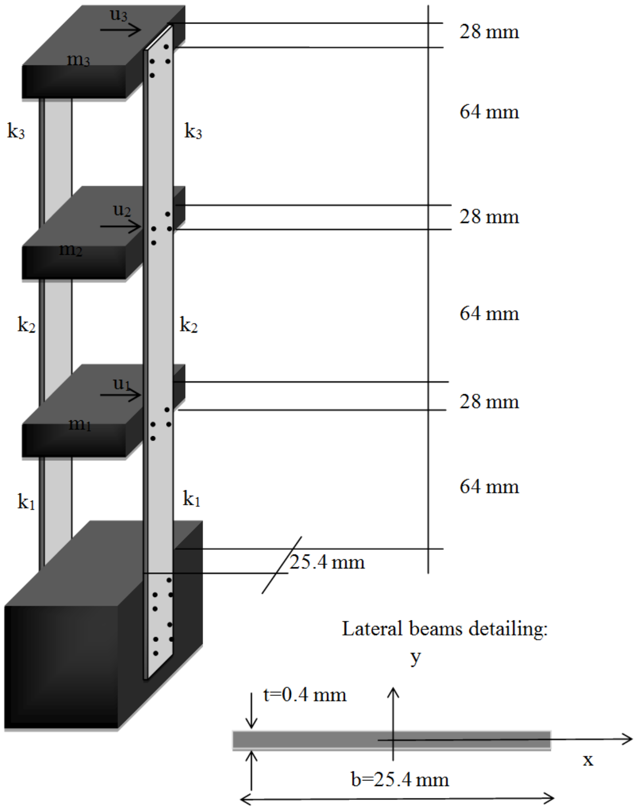

Figure 1. Main dimensions and variables for the three story building example

Due to the two lateral beams, the rotational degree of freedom is suppressed, remaining only the lateral one. The nominal values for the masses are m1=m2=m3=112 g. The lateral beams are made of steel and have a nominal Young's modulus E=196 GPa, and so the equivalent stiffness of the relative transverse motion between the blocks was calculated considering clamped-clamped Euler-Bernoulli beams, giving 1,215 N/m as a first estimate to be used in the iterative numerical procedure. With these values, mass and stiffness matrices were assembled, as written below, and the predicted natural frequencies were calculated considering the undamped case.

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m_1} & 0 & 0 \\ 0 & \boldsymbol{m_2} & 0 \\ 0 & 0 & \boldsymbol{m_3} \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 2\boldsymbol{k_1} + 2\boldsymbol{k_2} & -2\boldsymbol{k_2} & 0 \\ -2\boldsymbol{k_2} & 2\boldsymbol{k_2} + 2\boldsymbol{k_3} & -2\boldsymbol{k_3} \\ 0 & -2\boldsymbol{k_3} & 2\boldsymbol{k_3} \end{bmatrix} \tag{7}$$

The corresponding (undamped) eigenvalue/eigenvector system is defined, as usual, as:

$$\boldsymbol{K}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{M}\boldsymbol{\phi}_i \tag{8}$$

where $\boldsymbol{\phi}_i$ is the ith eigenvector and the ith eigenvalue, λ_i, is the square of the ith natural frequency, ω_i, measured in rad/s ($\lambda_i = \omega_i^2$).

Then, the nominal natural frequencies for the model are evaluated solving Eq. (7), giving: f_1=10.43 Hz, f_2=29.24 Hz, f_3=42.25 Hz.

2.5 Experimental Setup

The base of the block was rigidly fixed and each of the storeys was excited by manual tapping. The acceleration of each storey block was measured with an Analog Devices ADXL 203 accelerometer, with sensitivity of 970 mV/g and saturation of ±1.7 V, and the signals were acquired with a Measurement Computing 12 bit USB acquisition board at a sampling rate of 500 Hz. The period of acquisition was defined as 15 seconds, so that the frequency resolution of the discrete Fourier transforms was approximately 0.066 Hz, and the natural frequencies were obtained by simple peak picking. Applying this procedure, the following measured natural frequencies were obtained: 10.82 Hz, 28.86 Hz and 42.96 Hz.

3. RESULTS AND DISCUSSION

In this section the results obtained by applying the procedure delineated above are presented. First those concerning the simpler case in which the damping is neglected, i.e., uncertainties are assumed to be restricted to the stiffness matrix. Then, in the second case the components of the mass and damping matrices also are allowed to be adjusted.

3.1 First Case: model updating of stifness parameters only

The figure below shows the convergence of the parameters along iterations, where μ stands for the mean and σ for standard deviation. The vector θ of input parameters is composed by corrections for initial estimates (the factors by which these estimates must be multiplied to yield the updated values). The graphics show that convergence was achieved already at 3rd iteration. Within 2 decimal places, the mean values for the stiffnesses corrections were calculated as 1.107, 1.098 and 0.892, with standard deviations of 0.040, 0.011 and 0.015, respectively.

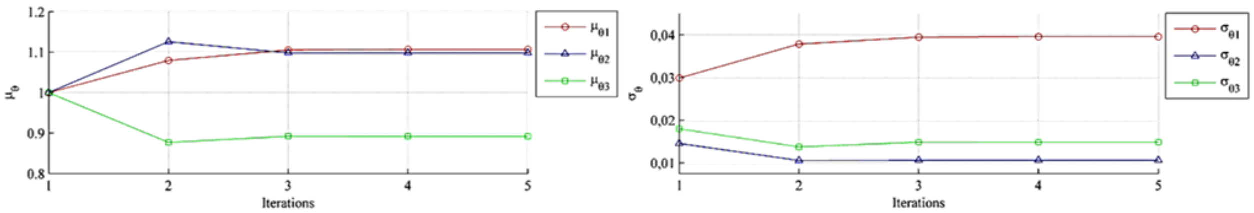

Figure 2. Convergence of the iterative numerical procedure for stiffness only

At this first stage, output parameters were chosen to be only the undamped natural frequencies. So, calculated values for these quantities were obtained, at each iteration, with the (mean) updated input parameters, and then compared with measured ones. The result is presented in the next figure, where dashed lines mark the average measured values. Up to the 5th iteration, within 2 decimal places the numerical procedure predicted exactly the same mean values as measured data, with standard deviations equal to 0.09 Hz, 0.05 Hz and 0.13 Hz, respectively.

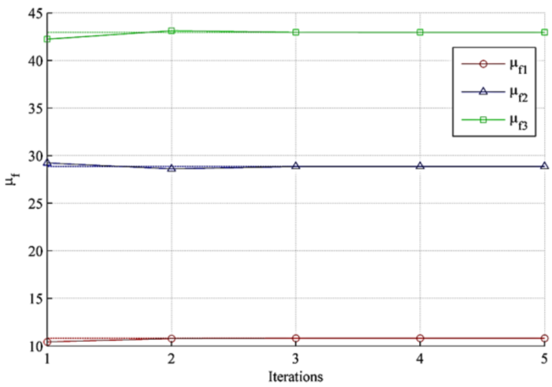

Figure 3. Predicted vs. measured natural frequencies at each iteration (dotted: measured data)

The relative (dimensionless) mean squared error – MSE – was also calculated, at each iteration *n*, using the vector $\delta\boldsymbol{z}$ of differences between measured and predicted values:

$$\boldsymbol{MSE} = \sqrt{\left(\frac{\boldsymbol{z}^p-\boldsymbol{z}^m}{\boldsymbol{z}^m}\right)^{\boldsymbol{T}}\left(\frac{\boldsymbol{z}^p-\boldsymbol{z}^m}{\boldsymbol{z}^m}\right)} = \left\|\frac{\delta\boldsymbol{z}}{\boldsymbol{z}^m}\right\| \tag{9}$$

Convergence of the MSE is shown below, where the final value achieved is 6.08×10^{-20}.

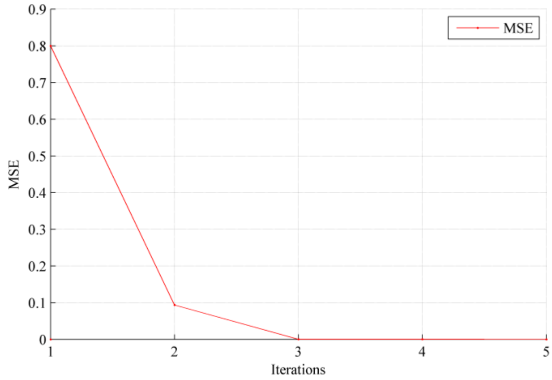

Figure 4. Convergence of the mean squared error

The dispersion of the data can be visualized by plotting two-by-two the measured output parameters obtained in each experiment, along with a sufficient large number of samples generated with the input parameters mean values and covariance matrix. The figure below shows an example for 1st vs. 2nd natural frequencies, using the same number of experiments (150) for the generated samples.

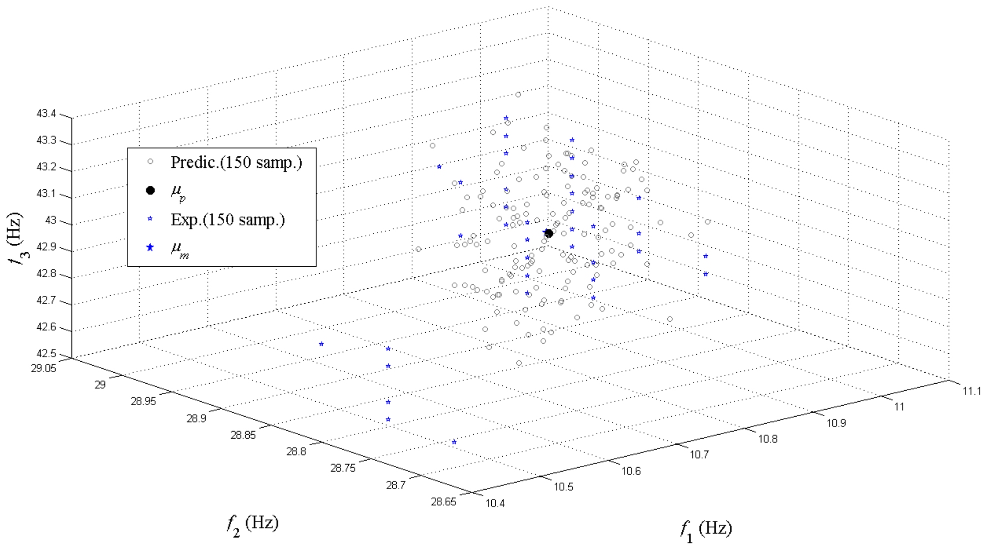

Figure 5. Dispersion of measured and predicted data for f_1, f_2 and f_3

Summarizing the results of this first part, in which only the stiffness matrix was adjusted, the figure below presents the dispersion of updated predicted vs. measured natural frequencies, along with confidence ellipses for σ, 2σ and 3σ.

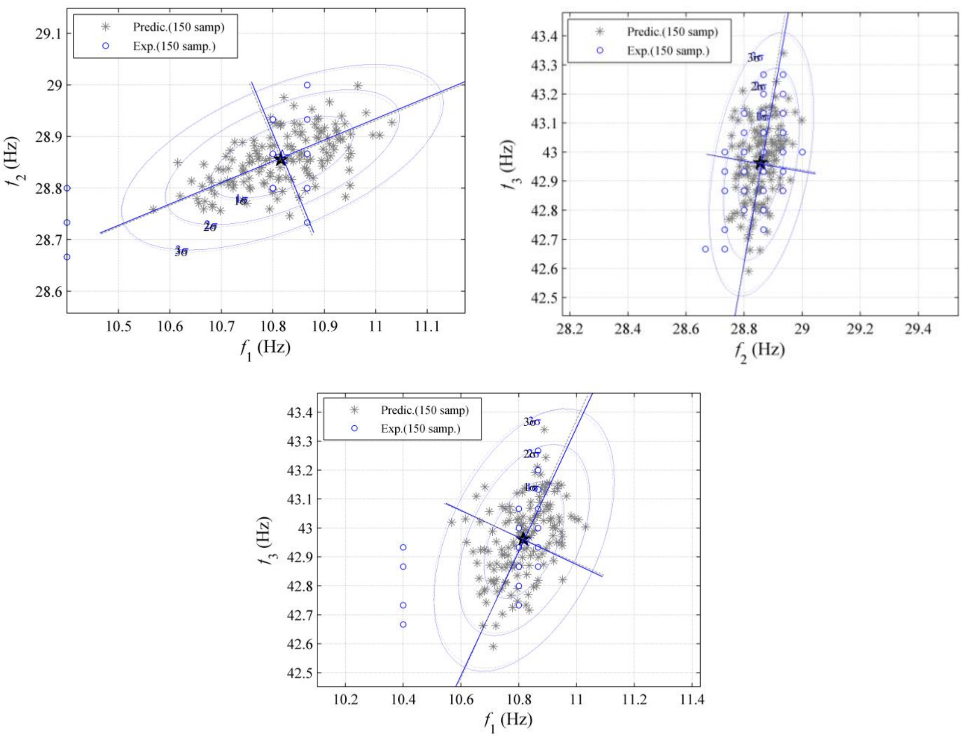

Figure 6. Dispersion of fitted vs. measured output data with confidence ellipses (dotted: predicted)

3.2 Second Case: model updating of stifness, mass and damping parameters

In order to also update the mass and damping matrices, and at the same time evaluate the uncertainties in these parameters, the procedure was performed again in two sequential steps: (a) first with the components of the mass and stiffness matrices taken as input parameters and the undamped natural frequencies as outputs; and (b) with only the components of the damping matrix as the input parameters and damping ratios as outputs (mass and stiffness matrices remaining fixed in the values achieved in the previous step). In this last step the predicted values of the damping ratios were obtained from polynomial eigensolution (Tisseur and Meerbergen, 2001), while experimental ones were extracted from measured time-domain data using exponential decrement technique and Hilbert transform to obtain response envelopes (Feldman, 2011).

Some additional treatments were performed on the experimental data before running this second case. The pick peaking process for natural frequencies determination was improved by tracing splines over some few points around the picks. Moreover, Chauvenet's criterion was applied to detect and remove experimental recordings most likely contaminated with gross human errors during the manual tapping process. The experimental data (resonance frequencies and damping ratios) were then determined again from the set of remaining observations, and the obtained mean values and standard deviations are presented further, together with the outputs predicted by the updated model.

For the identification of the damping parameters, a viscous non-proportional model was assumed and the physical system was idealized as a translational mass-spring-damper oscillator, as shown in the figure below.

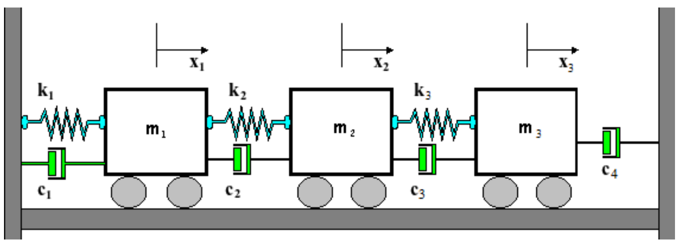

Figure 7. Schematic representation of the idealized 3 DOF model

One dashpot (viscous dissipation element) was placed side-by-side to every flexible element (springs), except that one extra dashpot was positioned between the highest storey and the ground in such a way that it could take into account the viscous dissipation for the still air. Thus, the resulting damping matrix depends on four parameters according to the following matrix written in Equation (9).

$$\boldsymbol{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix} \tag{9}$$

A first estimate for these parameters was evaluated based on the physical mass of the blocks and on the measured natural frequencies and damping ratios. The initial estimates for mass and stiffness parameters were reset to one.

Then, the next figure shows the updating of mass and stiffness parameters considering the undamped system, where θ_1 to θ_3 refer to k_1, k_2 and k_3, and θ_4 to θ_6 refer to m_1, m_2 and m_3.

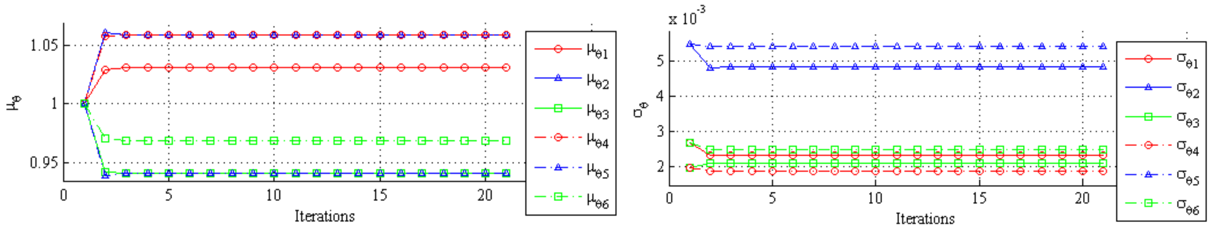

Figure 8. Convergence of the iterative numerical procedure for mass and stiffness only

The mean values and standard deviations thus obtained for the correction factors are presented in the following table:

Table 1. Updating factors for input parameters

Parameters	Mean (dimensionless)	Std deviation (dimensionless)
k_1	1.03108	0.00234
k_2	1.05863	0.00485
k_3	0.94154	0.00209
m_1	1.05838	0.00187
m_2	0.94159	0.00544
m_3	0.96880	0.00249

Output data obtained with these updated correction factors are presented in the next table, where it can be seen that the updated analytical model yielded exactly the same experimentally determined resonant frequencies within 12 decimal places. The full set of double precision decimal places (from the average calculations) were maintained in the experimental set for comparison with predicted ones, although being physically meaningful only up to the 4th significant digit.

Table 2. Measured vs. predicted outputs

	Experimental		Predicted	
Resonances	Mean (Hz)	Std. deviation (Hz)	Mean (Hz)	Std. deviation (Hz)
f_1	10.754102352282080	0.01729793500555	10.754102352282080	0.017492518475072
f_2	28.806120082016122	0.03524667215187	28.806120082016118	0.034791895099829
f_3	42.909364278544466	0.12919502257661	42.909364278544459	0.127873233961879

Convergence of these predicted natural frequencies towards measured ones is shown in the next figure.

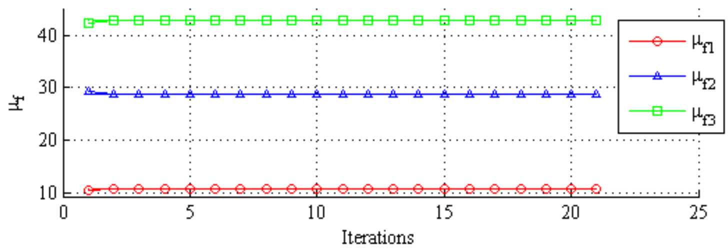

Figure 9. Predicted vs. measured natural frequencies at each iteration (dotted: measured data)

The MSE value at each iteration is presented below. It can be seen that it was a little more difficult to meet the stopping criterion in this case because MSE oscillated around the minimum found. Nevertheless, this residue was very low and close to the final value already at the 8th iteration.

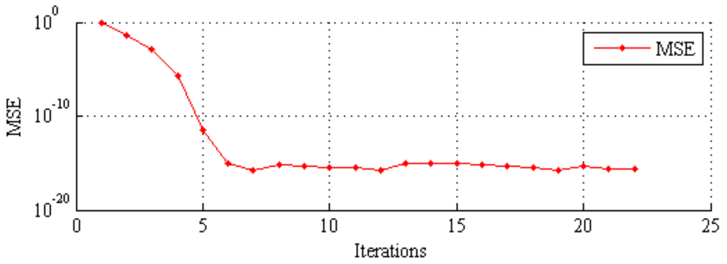

Figure 10. Convergence of the mean squared error

The clouds of dispersions for predicted and measured natural frequencies are presented next. After the application of the Chauvenet's criterion, 20 observations were rejected, thus remaining 130. Then, an equal number of predicted samples were generated with updated means and standard deviations given by updated covariance matrix.

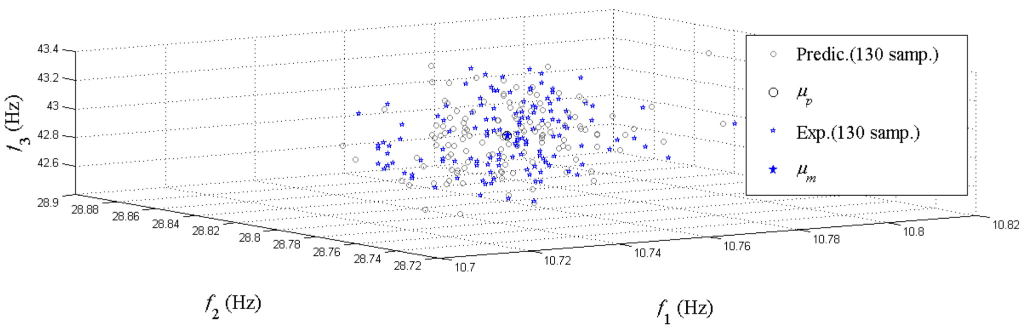

Figure 11. Dispersion (clouds) of measured and predicted data for f_1, f_2 and f_3

Next, the clouds are presented in the 3 orthogonal planes (2D) along with the confidence ellipses.

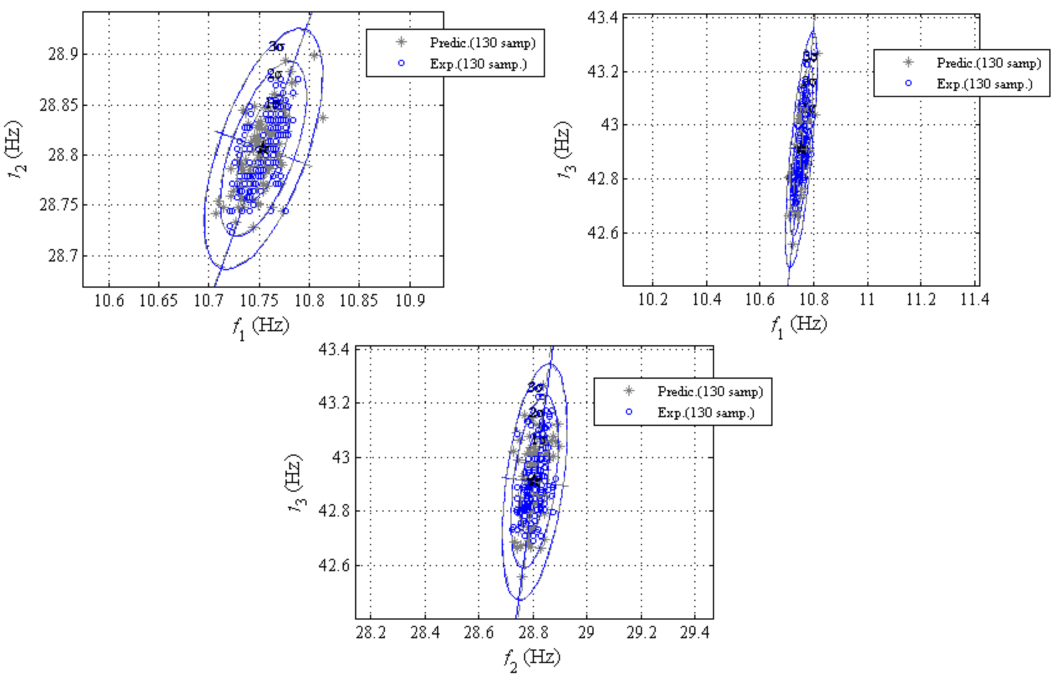

Figure 12. Dispersion of fitted vs. measured output data with confidence ellipses (dotted: predicted)

Then, for the updating of the damping matrix, the mass and stiffness parameters were fixed in the updated values presented right above, and the procedure was run again for the four independent quantities that compose the considered damping matrix, Eq. (1). Convergence for these parameters followed the same pattern as for the those of mass and stiffness, approximating very fast to the final values already at the first few iterations, and then oscillating around these values till meet the stopping criterion, as can be seen in the graphics below, where θ_1 to θ_4 refer to c_1 to c_4, respectively

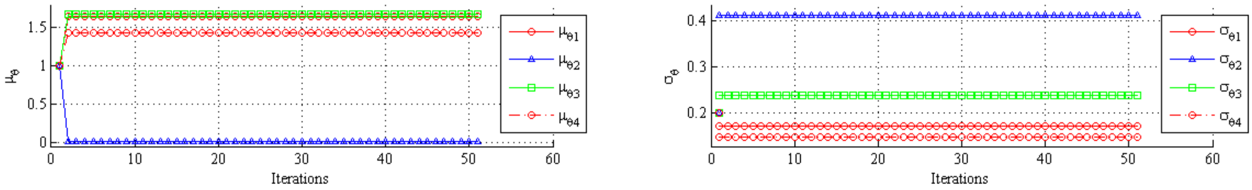

Figure 13. Convergence of (left) input parameters and (right) standard deviations

The mean updating factors thus obtained were: 1.63559, 0.01548, 1.66917 and 1.42906, with standard deviations of 0.17116, 0.41118, 0.23719 and 0.14540.

Next figure shows the updating of the mean damping ratios ζ_1, ζ_2 and ζ_3 as output parameters, beside of MSE convergence. As for the case of the natural frequencies, these values match exactly with measured ones within 12 decimal places.

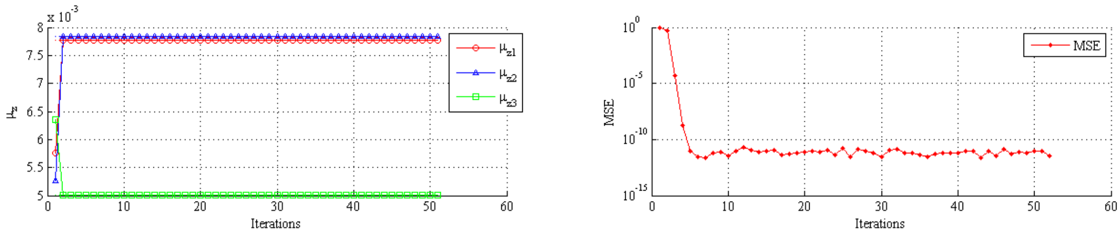

Figure 14. Convergence of the damping ratios as output parameters (left), and the MSE (right)

Finally, the experimentally observed data dispersion is compared with the clouds generated with the updated model, and it is shown in the figure below.

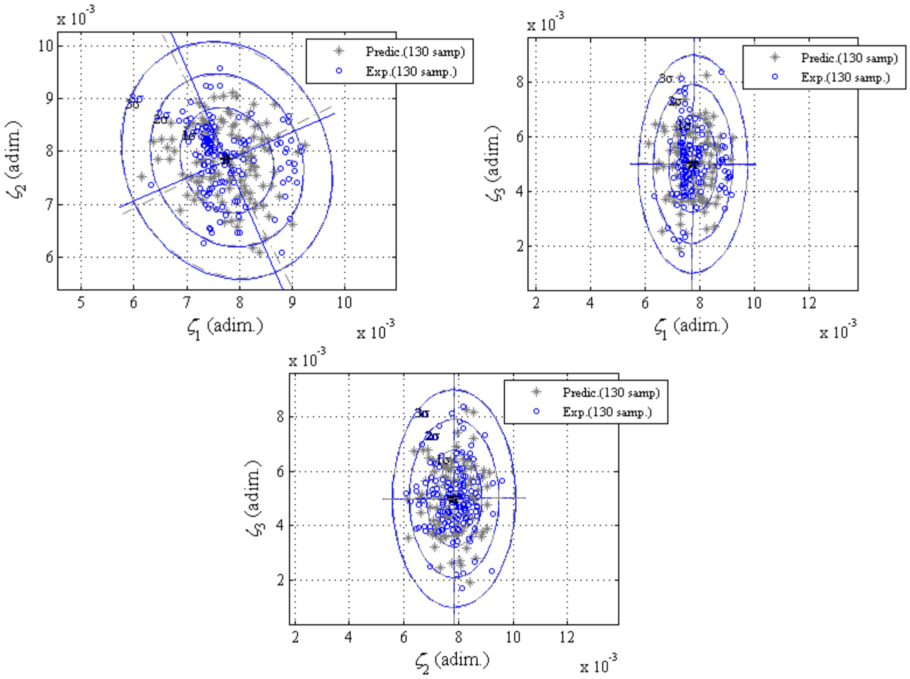

Figure 15. Dispersion of fitted vs. measured output data with confidence ellipses (dotted: predicted)

4. CONCLUSIONS

Sensitivity method-based model updating with uncertainty evaluation was applied to a shear frame three DOF-like structure for which a large set of experimental acceleration observations was available. The procedure was first applied in the case that the whole uncertainty of the system is considered as concentrated only in the stiffness matrix parameters. Fine agreement was achieved for mean values, as can be observed by the extremely small value of the MSE, with obtained correction factors in the order of ±10%. Confidence ellipses also suggest a good agreement, as well in the size of their semi-axes as in the in-plane rotation.

Nevertheless, damping is known to have the largest uncertainty levels among the parameters that define the dynamic behaviour of a structure. Then, although the fitted model can predict natural frequencies well, it is much more likely that the initial assumption about the origin of the uncertainties is not correct.

Thus, with the purpose of comparing the results obtained with different allocation of uncertainties, the procedure was run again, this time first considering the uncertainties distributed among stiffness and mass parameters only. A well fitted model was obtained for this case too, with correction factors decreasing to around ±5%, which corroborates the hypothesis that uncertainty is more realistically distributed in the case that mass parameters are also assumed free to be updated. Latter, fixing the mass and stiffness matrices in the updated values, damping parameters were updated by matching predicted damping ratios with measured ones. Extremely low MSE values and well fitted confidence ellipses suggest that the resultant fitted model can reliably be used to predict dynamic behaviour.

Finally, even though fairly well fitted models were obtained in both cases, an inverse relation between sensitivity and uncertainty level for the damping parameters could be perceived with the comparison of the results. It was noted that natural frequencies are almost insensitive to damping perturbations, but the uncertainty levels observed in measured damping ratios, and consequently in the fitted damping parameters too, are much greater than those of mass and stiffness components. Then, it was noticed that a more robust way of prescribing uncertainty allocation among the parameters that describe the physical system is needed for further developments of uncertainty quantification with sensitivity analysis.

5. ACKNOWLEDGEMENTS

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