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STRUCTURAL RELIABILITY ANALYSIS WITH POLYNOMIAL EXPANSION

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Abstract. *In general, the evaluation of the probability of structural failure generates a significant computational cost. Therefore, aiming to reduce the computational effort and maintaining the efficiency of the obtained results, the proposal of this study is to use the polynomial expansion in structural reliability analysis. For this purpose, the original functions are replaced by polynomials, resulting in a accurate analysis with low computational cost. Thus, numerical applications for truss and beam structures are presented and the results obtained show the effectiveness of the proposed formulation.*

Keywords: *reliability analysis, structural reliability, polynomial expansion.*

1. INTRODUCTION

In deterministic methods, it's known exactly all the values of the parameters involved in the problem, but due to the uncertainties of the problem and the influence of several external factors, these values must be evaluated as random variables with a probability of non-zero failure. Thus, during the last years the research in the field of reliability analysis has been intensified due to the need to take this factor into consideration in the analyzed problems (Melchers, 1999; Haldar and Mahadevan, 2000). It is important to incorporate these uncertainties into the designs, because the final engineering design must meet a number of criteria and some of these depend on a variety of sources (e.g. types of forces and loads).

The probability of failure can be evaluated by several methods. Two popular approaches are First and Second Order Reliability Methods (FORM and SORM, respectively) and Monte Carlo simulation (Haldar and Mahadevan, 2000). The FORM/SORM, although computationally efficient, can have its accuracy limited when the function g is nonlinear and in the cases of inclusion of non-Gaussian random variables. On the other hand, Monte Carlo simulation, although computationally costly, can be used for any type of function g (linear or nonlinear) and random variables (Gaussian, Lognormal, Betta and others), becoming one of the most used method to calculate the probability of failure of the system.

This work addresses structural reliability analysis with the use of polynomial expansion applied to different types of structures. The proposed formulation is based on replacement of the original functions of the problem by polynomial functions $g_k(x)$ through the collocation method. Polynomial expansion was oringinally proposed by Wiener (1938) and latter employed in engineering applications by Ghanem and Spanos (1991) and Xiu and Karniadakis (2002). Since then, several works on the subject were also presented. In the numerical examples, a nonlinear analysis was performed in order to verify the effectiveness of the proposed formulation.

2. STRUCTURAL RELIABILITY

2.1 Reliability Analysis

As presented in Melchers (1999); Haldar and Mahadevan (2000), in the evaluation of the probability of failure, we use a performance function $g(\mathbf{X})$, in which $\mathbf{X}$ is a vector of random variables related to the safety of the structure. We also assume that $g(\mathbf{x}) < 0$ indicates failure of the system, $g(\mathbf{x}) > 0$ indicates that the system is safe and $g(\mathbf{x}) = 0$ defines the limit state function of the reliability problem. The situation is illustrated in Fig. 1, where X_1 and X_2 being two random variables.

Thus, the probability of failure P_f is given by

$$P_f = P(g < 0) = \int_{g<0} f_x(\mathbf{x}) = \int_D f_x(\mathbf{x}), \tag{1}$$

in which $f_x(\mathbf{x})$ is the probability density function (PDF) and $D = \{\mathbf{x} \in \Re^n, g < 0\}$ is the failure domain of the system.

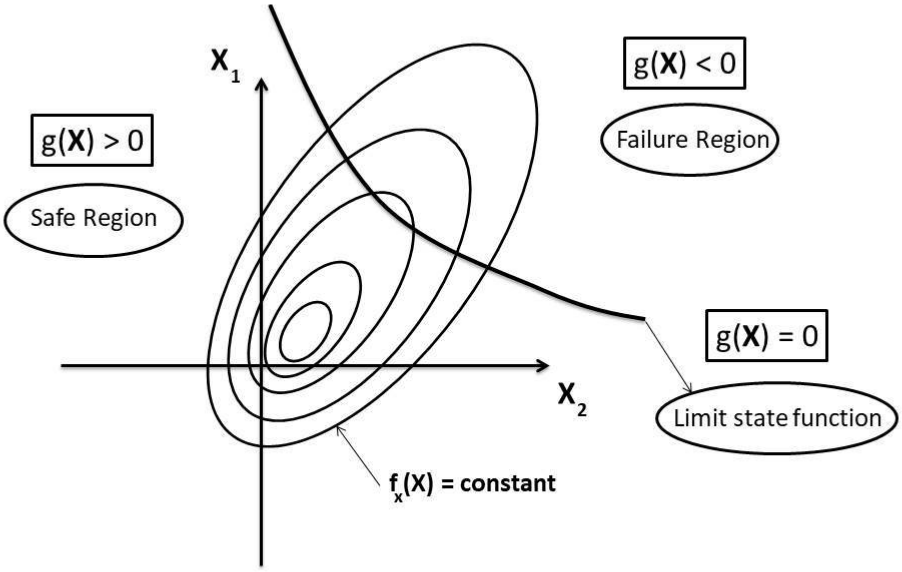

Figure 1. Performance function g(x).

Nevertheless, we usually evaluate the probability of failure of Eq. (1) by

$$P_f = \int_{\Omega} I(g(\mathbf{x})) f_x(\mathbf{x}), \tag{2}$$

where I is the indicator function and Ω is the support of function $f_x(\mathbf{x})$. The support Ω can be defined as

$$\Omega = \{\mathbf{x} \in \Re^n | f_x(\mathbf{x}) > 0\} \tag{3}$$

and the indicator function I is defined by

$$I(t) = \begin{cases} 0 & , \quad t \geq 0 \\ 1 & , \quad t < 0 \end{cases} \tag{4}$$

In reliability analysis, another important concept is the expected value $E[\,]$ (or mean value) of the random variable $\mathbf{X}$, that can be calculated as

$$E[X] = \int_{\Omega} \mathbf{x} f_x(\mathbf{x}) \tag{5}$$

Substituting Eq. (2) in Eq. (5), the probability of failure results

$$P_f = \int_{\Omega} I(g(\mathbf{x})) f_x(\mathbf{x}) = E[I(g)], \tag{6}$$

indicating that the probability of failure P_f is equal to the expected value $E[I(g)]$ of the indicator function $I(g)$.

2.2 Monte Carlo Simulation

As is a probabilistic method, the Monte Carlo simulation draws a huge amount of samples with possible results of the random variables $\mathbf{X}$. It is important to highlight that a greater number of samples results in a greater number of observations and, consequently, a better accuracy. Thus, to calculate the P_f, several samples are simulated and the values of the functions g and $I(g)$ are calculated.

Making a total number of simulations n_{si} by Monte Carlo, the probability of failure of Eq. (6) can be estimated as (Melchers, 1999; Haldar and Mahadevan, 2000)

$$P_f \approx \tilde{P}_f = \frac{1}{n_{si}} . \sum_{i=1}^{n_{si}} I(x_i) = \frac{n_f}{n_{si}} \tag{7}$$

in which n_f indicates the number of simulations where $g < 0$. In other words, the probability of failure can be calculated from the ratio of the number of simulations n_f where the system fails by the number total n_{si} of simulations performed.

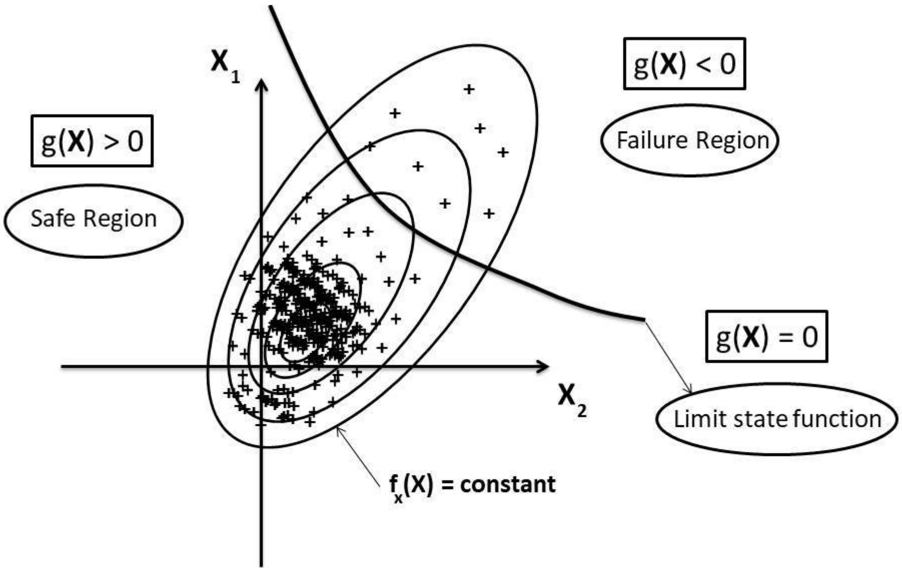

Figure 2. Monte Carlo Simulation

Graphically, in the case of two random variables (x_1, x_2) and representing each simulation by the symbol $+$, as shown in Fig. 2, the Monte Carlo simulation estimates the P_f as the ratio of the number of simulations that are in the failure region of the system $g < 0$ on the total number of simulations.

The accuracy and the computational cost of the Monte Carlo simulation are directly related to the number of simulations carried n_{si}, where a greater amount n_{si} indicates that more cases will be analyzed, generating a greater computational effort. In general, as in the engineering problems the probability of failure is between 10^{-3} and 10^{-4}, will be necessary around 10^5 and 10^6 simulations to estimate with precision the value of P_f. In this work was used $n_{si} = 10^6$ with a probability of failure in the order of 10^{-3} (Melchers, 1999).

3. STOCHASTIC POLYNOMIAL EXPANSION

In the polynomial expansion method, the original performance function $g(\mathbf{X})$ is replaced by a polynomial $g_k(\mathbf{X})$ of order k

$$g_k(\mathbf{X}) = \sum_{i=1}^{m} c_i \psi_i(\mathbf{X}) \tag{8}$$

where ψ_i form a basis $\mathcal{P}_k^n$ for the polynomial space of order k on n dimensions, c_i are coefficients to be determined and m is the number of basis functions employed.

A procedure for building the basis for the polynomial space of order k is described by Torii *et al.* (2017). First, we take a one dimensional basis of order k

$$\mathcal{P}_k(\mathbf{x}) = \{1, x, x^2, ..., x^{k-1}, x^k\}. \tag{9}$$

We then obtain the n dimensional basis $\mathcal{P}_k^n$ by full tensor product between the one dimensional basis, resulting in

$$\mathcal{P}_k^n = \mathcal{P}_k(x_1) \otimes \mathcal{P}_k(x_2) \otimes ... \otimes \mathcal{P}_k(x_n), \tag{10}$$

where the dyadic product $\otimes$ indicates a multiplication between all members of the two sets. Note that the number of basis functions in the set $\mathcal{P}_k^n$ is $(k+1)^n$.

Several methods can be used to find the coefficients c_i. In the Collocation Method, the coefficients c_i are obtained by imposing the conditions

$$g_k(x_j) = g(x_j),\ j = 1, 2, ..., n_c, \tag{11}$$

resulting in the system of linear equations

$$\sum_{i=1}^{m} c_i \psi_i(x_j) = g(x_j),\ j = 1, 2, ..., n_c \tag{12}$$

in which x_j are non repeated collocation points and n_c is the number of collocation points. Thus, this method ensure that the expansion g_k has the same value of the original function g in the collocation points x_j. The system of Eq. (12) has a unique solution when the number of coefficients m is equal to the number of collocation points n_c, that is the approach employed in this work. Consequently, the number of deterministic analysis necessary to obtain the expansion is equal to $(k+1)^n$.

In this work, the collocation points are obtained from cartesian product of one-dimensional samples. We first generate a one dimensional sample

$$S_i = \{x_i^{(1)}, x_i^{(2)}, ..., x_i^{(l)}\}, \quad i = 1, 2, ..., n \tag{13}$$

where $\mathcal{S}_i$ is a set of l equally spaced points in the x_i axis. The points are defined as

$$\begin{array}{ll} x_i^{(1)} & = \mu_{X_i} - \alpha\sigma_{X_i} + \Delta x_i, \\ x_i^{(j+1)} & = x_i^{(j)} + \Delta x_i, \quad j = 1, 2, ..., l-1, \end{array} \tag{14}$$

with

$$\Delta x_i = \frac{2\alpha\sigma_{X_i}}{l+1}, \tag{15}$$

where μ_{X_i} and σ_{X_i} are the mean and the standard deviation of the random variable X_i, respectively. Note that the one dimensional sample S_i is composed by equally spaced points in the interval $[\mu_{X_i} - \alpha\sigma_{X_i}, \mu_{X_i} + \alpha\sigma_{X_i}]$. The size of the collocation interval is controlled by the parameter α. The collocation points can then be obtained by full tensor product between the one dimensional samples $S_1, S_2, ...$, by

$$S = S_1 \times S_2 \times ... \times S_n \tag{16}$$

where $\times$ represents the Cartesian product (i.e. combination of all elements of two sets).

4. EXAMPLES

In this section, some numerical examples are analyzed in order to show the effectiveness of the proposed approach to calculate structural reliability. The types of structures analyzed are a truss, a beam and a space truss, the nonlinear structural analysis are performed using the software MASTAN2, described in details by Mcguire *et al.* (2000). The software MASTAN2 was already employed by Torii *et al.* (2016) for structural reliability using Monte Carlo Simulation.

For all the numerical examples presented in this work, a second order inelastic analysis is performed, considering both physical nonlinearity (non linear constitutive law) and geometric nonlinearity (considering finite displacements), resulting in a more accurate analysis of the actual structural behavior. The nonlinear analysis method is performed by means of load fraction increments Δ. A load fraction increment of $\Delta = 0.1$, for example, indicates that each load increment is equal to 10% of the total load. The structural analysis ends when the total load is applied or when the iterative scheme fails to converge, likely indicating structural collapse. The reader can consult Mcguire *et al.* (2000) for more details on non-linear structural analysis.

The performance function is take as

$$g = \lambda - 1, \tag{17}$$

where λ is the collapse load multiplier. Note that in this case failure is assume to occur if the collapse load multiplier is smaller than 1, i.e. the structure collapses for loads smaller than the applied load.

In this work the safety level of the structure is measured using the reliability index β

$$\beta = \Phi^{-1}(1 - P_f), \tag{18}$$

where Φ^{-1} is the normal inverse cumulative density function (iCDF). In this case higher values of β indicate smaller values of P_f. In other words, the structure is safer for higher values of β. The reason of replacing P_f by β as a measure of structural safety is to avoid working with very small values of P_f that frequently occur in the case of structural reliability. The parameter α, the defines the collocation points, is taken as $\alpha = 3$.

4.1 Example 1: Truss

The first example is a planar truss with 6 bars subjected to concentrated load as presented in Fig. 3. Here we take $L = 1500mm$ and the cross-sectional area $A = 5420mm^2$. In this example, the random variables adopted are material elastic modulus E, the material yielding stress f_y and concentrated load P. In the structural analysis, it was adopted a

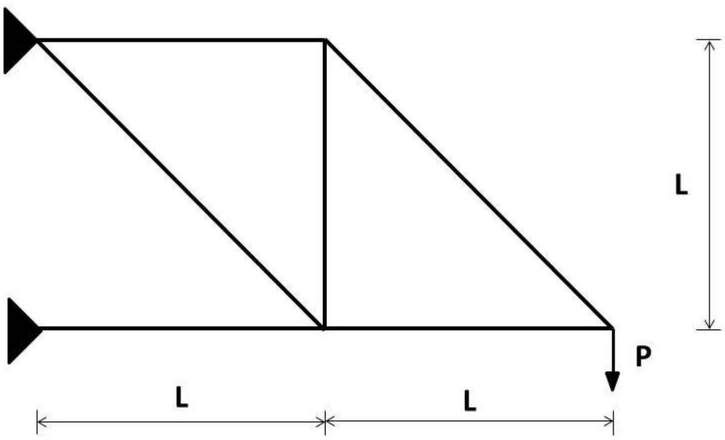

Figure 3. Planar Truss

Table 1. Random variables

Random variable	Probabilistic Distribution	Mean Value	Standard Deviation
$E(N/mm^2)$	Normal	210000	2000
$f_y(N/mm^2)$	Normal	250	25
$P(N)$	Normal	500000	60000

Table 2. Results of β

k	2	3	4
$\Delta = 0.1$	1.995	1.964	2.061
$\Delta = 0.01$	1.921	1.927	1.922
nfe*	27	64	125

*nfe: number of function evaluations

planar truss model with both physical and geometric nonlinearity. The type of probabilistic distribution, mean value and standard deviation adopted for these random variables are presented in Tab. 1.

This is a valuable example because the reliability index considering collapse can be evaluated analytically. From equilibrium, we find that the highest axial force inside the structure is $T = 2P$ (at the lower left bar). Consequently, the maximum stress inside the structure is given by $\sigma = 2P/A$. Since the structure is statically determinate, yielding of some bar leads to structural collapse. The reliability index considering collapse can then be evaluated analytically with the limit state function $g = f_y - 2P/A$ and gives $\beta = 1.9613$.

In order to compare the results, the degree k of the polynomials used in $g_k(\mathbf{X})$ and the increments of load Δ was varied in each analysis. The results obtained are shown in Tab. 2. From the results of β, presented in Tab. 2, it was verified the convergence of results with the analytical result. In addition, the use of a low order polynomial ($k = 2$) was sufficient to obtain the desired results.

The computational effort was measured by the number of function evaluations (nfe) of the performance function, that is the number of structural analyses required in each case. The nfe is also presented in Tab. 2, indicating that the computational effort is orders of magnitude that what would be required by Monte Carlo Simulation. A similar case studied using Monte Carlo Simulation by Torii *et al.* (2016), for example, required 10^5 simulations.

4.2 Example 2: Beam

In this example, the beam subjected uniform distributed load q, as presented in Fig. 4, was analyzed. The beam is composed by a steel profile with length $L = 5000mm$, cross-sectional area $A = 1910mm^2$, inertia $I = 12800000mm^4$ and plastic modulus $Z = 145000mm^3$. The random variables adopted are the material elastic modulus E, the material yielding stress f_y and the intensity of the distributed load q. The beam was modeled using planar beam model with 10 elements. The type of probabilistic distribution, mean value and deviation standard are shown in Tab. 3.

The degree k of the polynomials used in $g_k(\mathbf{X})$ and the increments of load Δ were varied in each analysis to compare the results. The results obtained are shown in Tab. 4. From the results of Tab. 4, the values of β do not converge when the increment of load $\Delta = 0.1$ is adopted, because the increment of load $\Delta = 0.1$ is big for the nonlinear analysis performed.

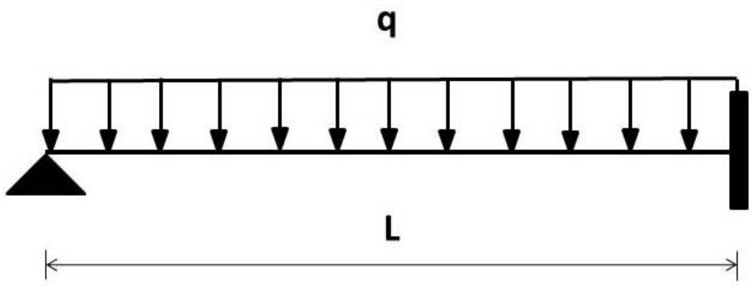

Figure 4. Beam

Table 3. Random Variables

Random variable	Probabilistic Distribution	Mean Value	Standard Deviation
$E(N/mm^2)$	Normal	210000	2000
$f_y(N/mm^2)$	Normal	250	25
$q(N/mm)$	Normal	15	1.25

Table 4. Results of β

k	2	3	4
$\Delta = 0.1$	1.113	1.345	0.352
$\Delta = 0.01$	0.963	0.965	0.958
nfe*	27	64	125

*nfe: number of function evaluations

However, the values of β converge to the increment of load $\Delta = 0.01$, because in this case the analysis is more accuracy and the structure fails to a total load closer to reality. In addition, the use of a low order polynomial ($k = 2$) was sufficient to obtain the desired results, resulting in smaller computational cost.

4.3 Example 3: Space Truss

In the last example, a space truss with 18 bars was analysed, as shown in Fig. 5. The cross-setional area of all the bars is $A = 596903mm^2$, and the lenghts $L_x = 400mm$ and $L_z = 100mm$. The material elastic modulus E, the material yielding stress f_y and concentrated load P were considered random variables, their mean value and deviation standard are presented in Tab. 5. In the structural analysis, a space truss model was adopted.

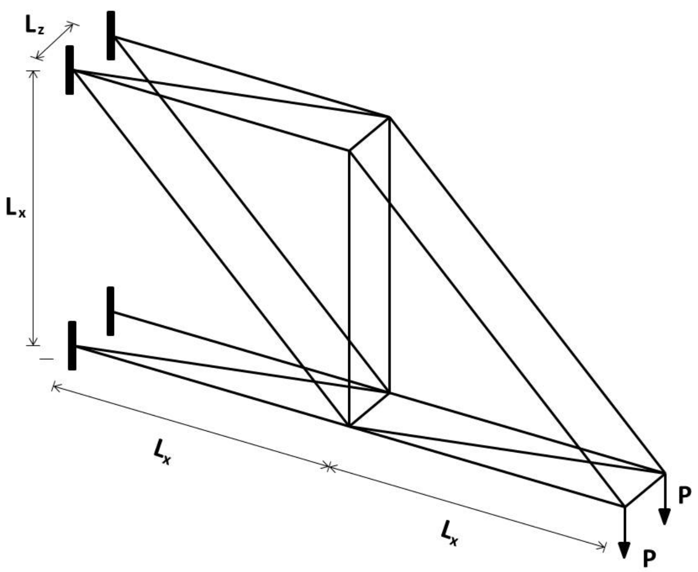

Figure 5. Space Truss

The results obtained of β are presented in Tab. 6. In each analysis, the degree k of the polynomials used in $g_k(\mathbf{X})$ and the increments of load Δ were varied. From the results of β, presented in Tab. 6, it was verified the convergence of results and the use of a low order polynomial ($k = 2$) was sufficient to obtain the desired results, resulting in smaller computational cost.

Table 5. Random Variables

Random variable	Probabilistic Distribution	Mean Value	Standard Deviation
$E(KN/cm^2)$	Normal	210000	2000
$f_y(KN/cm^2)$	Normal	35	3
$P(KN)$	Normal	750	100

Table 6. Results of β

k	2	3	4
$\Delta = 0.1$	1.933	1.881	1.912
$\Delta = 0.01$	1.879	1.841	1.867
nfe*	27	64	125

*nfe: number of function evaluations

5. CONCLUSIONS

This work referred to structural reliability analysis with the use of polynomial expansion applied to a truss, spacial truss and beam. The proposed formulation was based on replacement the original functions of the problem by polynomials through the collocation method. Structural analysis was made using MASTAN2.

The results indicate that accurate results can be obtained with much smaller computational effort in comparison to Monte Carlo Simulation. This is an important aspect, since non-linear structural analysis is a computationally demanding problem. Consequently, reducing the number of structural analyses necessary to evaluate the probability of failure leads to a great save in computational effort.

6. ACKNOWLEDGEMENTS

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