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THE INTERVAL EIGENVALUE PROBLEM FOR VIBRATING SYSTEMS

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Abstract. *In this paper, the interval eigenvalue problem with uncertainties in the mass matrix and stiffness matrix of the system is presented, and a method to solve this problem using the Interval Analysis and the Finite Element Method (FEM) is proposed. In order to support the solution of the problems, the basic fundamentals of the eigenvalue problem and the interval analysis were presented. Comparisons were made between the proposed method and two other methods, namely the parameter vertex solution theorem and the eigenvalue inclusion principle. Numerical examples have been presented to show the applicability and reliability of the proposed method. To compare the results, simulations were performed using the Monte Carlo method, with results presented in the form of tables and graphs.*

Keywords: *Interval Analysis, Vibrating Systems, Eigenvalue Problem, Uncertainties.*

1. INTRODUCTION

Errors are inherent in all known processes. They can be caused by measurement errors, approximations, rounding, among others. Thus, the purpose of an interval analysis is to provide an interval with lower and upper limits where the exact value of the variable under analysis is contained, that is, interval in which the errors present in the system are computed. In (Moore et al., 2009) can be seen several examples of the use of interval mathematics in the treatment of uncertainties.

The approach is important both in the design and in the rehabilitation of truss structures, as they are widely used in stadium coverings, transmission towers and general application sheds. The uncertainties must be quantified to ensure a longer life of the structure, avoiding sudden failures, and thus ensuring the safety of facilities and especially of people.

To ensure robustness and agility in structural analysis, computational techniques are increasingly employed. A widely used technique is the Finite Element Method (FEM), which allows good implementation and malleability in solving often complex problems. In conjunction with the interval analysis, FEM can be applied to quantify uncertainties in structural systems. A fairly complete approach to FEM can be seen in (Hutton, 2004).

Parameters of great importance to be determined in a dynamic analysis of structures are the natural frequencies, mainly in the design phase, whose objective is to move away the natural frequencies from those of excitation, thus avoiding the unwanted phenomenon of resonance. Theoretical studies, showing the invariance of the eigenvector signals, were performed by (Deif, 1991) in order to determine the eigenvalues for the standard interval eigenvalue problem for symmetric and asymmetric dynamic matrices. Later, (Qiu et al., 1995) presented the generalized interval eigenvalue problem, using the (Deif, 1991) method and iterative process with Rayleigh quotient. Another method of solution called the inclusion principle using FEM was presented by (Qiu et al., 2005) with the solution of several numerical examples. Modal analysis using the (Deif, 1991) method was used in (Sim et al., 2007) to solve the interval eigenvalue problem, being performed simulations using the Monte Carlo method and presenting graphical results using the Frequency Response Function. For the solution of the standard problem, in (Hladík et al., 2011) several algorithms were presented for the case of symmetric matrices. Using the matrix perturbation theory and FEM, (Albuquerque, 2015) studied the interval eigenvalue problem. Further, (Li et al., 2017) used the Taylor series of the second order to compute the eigenvalues of space truss and plate, with uncertain parameters.

2. REVIEW OF THE EIGENVALUE PROBLEM

It was spoken by (Chapra and Canale, 2010) that, eigenvalue problems are a special class of boundary-value problems that are common in engineering problem contexts involving vibrations, elasticity, and other oscillating systems. They are also used in a wide variety of engineering problems, in addition to the boundary-value problems.

The linear differential equation of motion with constant coefficients for non-damped mechanical systems with n degrees of freedom is presented by Eq. (1).

$$[M]\{\ddot{u}(t)\} + [K]\{u(t)\} = \{f(t)\} \tag{1}$$

where $[M]$ and $[K]$ are the symmetric matrices $n \times n$ of global mass and stiffness of the system, respectively, $\{u(t)\}$ is the column vector of dimension n with the generalized coordinates, and $\{f(t)\}$ is the external excitation.

For systems that are not subject to external excitation, the equation of motion is given by Eq. (2).

$$[M]\{\ddot{u}(t)\} + [K]\{u(t)\} = \{0\} \tag{2}$$

As the system is conservative, the solutions of Eq. (2) are expected to be periodic, of the type of Eq. (3),

$$\{u(t)\} = \{x\}e^{j\omega_n t} \tag{3}$$

where $\{x\}$ is a column vector of constants and ω_n is the natural frequency of the system.
Substituting Eq. (3) into Eq. (2), leads to

$$\left(-\lambda[M]\{x\} + [K]\{x\}\right)e^{j\omega_n t} = \{0\} \tag{4}$$

with $\lambda = \omega_n^2$, where λ are the eigenvalues of the system.

As $e^{j\omega_n t} \neq 0$ for every instant of time, arrives to

$$\left([K] - \lambda[M]\right)\{x\} = \{0\} \tag{5}$$

For the non-trivial solution, it is required that

$$\det\left([K] - \lambda[M]\right) = \{0\} \tag{6}$$

For the determination of the eigenvectors or vibration modes $\{x\}$, associated with eigenvalues, one must use Eq. (7), which is known as the Generalized Eigenvalue Problem.

$$[K]\{x\} = \lambda[M]\{x\} \tag{7}$$

3. DYNAMIC STRUCTURAL ANALYSIS

With application of the FEM, it is possible to solve complex problems involving structures, with high number of bars. Thus, it is necessary to assemble the global mass and stiffness matrices of the system in the global coordinate, which is the linear sum of the contribution of the mass matrix $[M]_e$ and stiffness $[K]_e$ of each element in the global coordinate. To perform the sum, use the degrees of freedom between the connectivities of the nodes. This way,

$$[K] = [K]_0 + \sum_{i=1}^{m}[K]_{e_i} \tag{8}$$

$$[M] = [M]_0 + \sum_{i=1}^{m}[M]_{e_i} \tag{9}$$

where m is the number of elements, $[K]_0$ and $[M]_0$ are the null $n \times n$ matrices.

Mass and stiffness matrices are functions of the vector $\{a\}$, which contains the geometric and structural parameters, in this way

$$[K] = [K(a)] \tag{10}$$

$$[M] = [M(a)] \tag{11}$$

with
$\{\underline{a}\} \leq \{a\} \leq \{\overline{a}\}, \quad a_i^I = [\underline{a}_i, \overline{a}_i] \quad$ ou $\quad \underline{a}_i \leq a_i \leq \overline{a}_i \quad i = 1, 2, ..., m$
where $\{\underline{a}\} = (\underline{a}_i)$ and $\{\overline{a}\} = (\overline{a}_i)$ are, respectively, the lower and upper limits of the structural parameter $\{a\}$, with $\{a\} \in \{a\}^I$.

Thus, Eq. (8) and Eq. (9) can be rewritten as functions of the uncertain structural parameters.

$$[K]^I = [\underline{K}, \overline{K}] = [K]_0 + \sum_{i=1}^{m} a_i^I [K]_{e_i} \tag{12}$$

$$[M]^I = [\underline{M}, \overline{M}] = [M]_0 + \sum_{i=1}^{m} a_i^I [M]_{e_i} \tag{13}$$

where

$$[\underline{K}] = [K]_0 + \sum_{i=1}^{m} \underline{a}_i [K]_{e_i} \tag{14}$$

$$[\overline{K}] = [K]_0 + \sum_{i=1}^{m} \overline{a}_i [K]_{e_i} \tag{15}$$

$$[\underline{M}] = [M]_0 + \sum_{i=1}^{m} \underline{a}_i [M]_{e_i} \tag{16}$$

$$[\overline{M}] = [M]_0 + \sum_{i=1}^{m} \overline{a}_i [M]_{e_i} \tag{17}$$

Thus, with uncertainties present in the matrices of mass and stiffness of the system, it leads to the generalized interval eigenvalue problem represented by Eq. (18).

$$[K]^I \{x\} = \lambda [M]^I \{x\} \tag{18}$$

The bounds for eigenvalues $\lambda^I = [\underline{\lambda}, \overline{\lambda}]$ can be computed through Eq. (19) and Eq. (20).

$$\underline{\lambda} = \min\left([\underline{K}]\{\underline{x}\} - \underline{\lambda}[\underline{M}]\{\underline{x}\} = 0, [\overline{K}]\{\overline{x}\} - \overline{\lambda}[\overline{M}]\{\overline{x}\} = 0\right) \tag{19}$$

$$\overline{\lambda} = \max\left([\underline{K}]\{\underline{x}\} - \underline{\lambda}[\underline{M}]\{\underline{x}\} = 0, [\overline{K}]\{\overline{x}\} - \overline{\lambda}[\overline{M}]\{\overline{x}\} = 0\right) \tag{20}$$

where $\{\underline{x}\}$ is the lower bound and $\{\overline{x}\}$ the upper bound for eigenvectors, with $\{x\}^I = [\underline{x}, \overline{x}]$.

4. NUMERICAL EXAMPLES

Two examples are used to demonstrate the possibility of applying the proposed method. The first is a stepped beam with uncertainties in the cross-sectional areas and moments of inertia of the area. The second is a flat truss with eight bars, where the uncertain parameter is related to the areas of the cross sections of the bars. In both examples the uncertain parameters affect the mass matrix and stiffness simultaneously. Also, simulations will be presented using the Monte Carlo method for comparison of results.

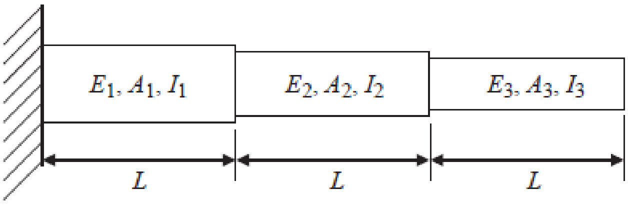

Figure 1: A three element stepped beam.

Table 1: Uncertain cross-sectional area and moment of inertia for stepped beam.

Cross-sectional area	Moments of inertia
$A_1^I = [1.426 \times 10^{-2}, 1.454 \times 10^{-2}]\ \mathrm{m}^2$	$I_1^I = [0.199\,80 \times 10^{-4}, 0.200\,20 \times 10^{-4}]\ \mathrm{m}^4$
$A_2^I = [0.990 \times 10^{-2}, 1.010 \times 10^{-2}]\ \mathrm{m}^2$	$I_2^I = [0.099\,90 \times 10^{-4}, 0.100\,10 \times 10^{-4}]\ \mathrm{m}^4$
$A_3^I = [0.634 \times 10^{-2}, 0.646 \times 10^{-2}]\ \mathrm{m}^2$	$I_3^I = [0.049\,95 \times 10^{-4}, 0.050\,05 \times 10^{-4}]\ \mathrm{m}^4$

4.1 A three element stepped beam

In this example, the fixed-free stepped beam shown by Fig. 1, presented by (Qiu et al., 2005), is analyzed, comparing the results obtained using the proposed method, parameter vertex solution theorem and the simulations using the Monte Carlo method. The specific mass is $\rho_i = 7.8 \times 10^3\ \mathrm{kg/m^3}$, $L_i = 0.4\ \mathrm{m}$ and Young's modulus is $E_i = 200 \times 10^9\ \mathrm{N/m^2}$, for $i = 1, 2, 3$. The areas of the cross sections and the moments of inertia of the area are the uncertain parameters that cause uncertainties in the dynamic response of the structure under analysis, and can be observed by means of Tab. 1.

Table 2 shows the lower and upper bounds for the eigenvalues, calculated using the proposed method, the parameter vertex solution theorem and the Monte Carlo method. It can be observed that, in general, the answers obtained through the proposed method are closer to the answers obtained with the Monte Carlo method. The graphical results of the simulations using the Monte Carlo method can be seen by means of Fig. 2 for 2,000 samples.

4.2 Eight bars truss

By means of Fig. 3, presented in (Qiu et al., 2005), it can be see a truss with eight bars, where all the joints are pinned, which will be used to compare the results obtained by means of the proposed method, the eigenvalue inclusion principle and the Monte Carlo method. The cross-sectional areas of bars 1, 2, 3, 4 and 6 are considered with uncertainties, according to $A_i^I = [A^c - \beta A^c, A^c + \beta A^c]$, with i=1, 2, 3, 4, 6, where $A^c = 2.0 \times 10^{-4}\ \mathrm{m}^2$, and β is the percentage factor that will cause variation of the uncertain parameter to occur, with β ranging from 0 to 2%. The areas of the cross sections of bars 5, 7 and 8 are deterministic, with $A_5 = A_7 = A_8 = 1.0 \times 10^{-4}\ \mathrm{m}^2$. Young's module is $E = 200 \times 10^9\ \mathrm{N/m^2}$ and the specific mass is $\rho = 7.8 \times 10^3\ \mathrm{kg/m^3}$.

By means of Fig. 3 it is possible to observe comparisons between the eigenvalues obtained through the proposed method and the eigenvalue inclusion principle, the lower and upper bounds calculated through the proposed method are more tight. Table 3 shows comparisons between the proposed method, the eigenvalue inclusion principle and the Monte Carlo method for $\beta = 2\%$. It can be noticed that the results obtained through the proposed method converge with those

Table 2: Interval eigenvalues for the three element stepped beam.

	Proposed method		Monte Carlo method		The parameter vertex solution theorem	
	Lower bound	Upper bound	Lower Bound	Upper bound	Lower bound	Upper bound
λ_1	3.620026E+05	3.682185E+05	3.615941E+05	3.685832E+05	3.612794E+05	3.689557E+05
λ_2	7.271976E+06	7.400345E+06	7.268218E+06	7.398023E+06	7.257447E+06	7.415161E+06
λ_3	4.780474E+07	4.864078E+07	4.779937E+07	4.862997E+07	4.770923E+07	4.873816E+07
λ_4	2.558403E+08	2.603177E+08	2.557887E+08	2.602688E+08	2.553292E+08	2.608389E+08
λ_5	8.842340E+08	8.998542E+08	8.839731E+08	8.995773E+08	8.824674E+08	9.016557E+08
λ_6	2.716636E+09	2.763361E+09	2.713185E+09	2.764558E+09	2.711208E+09	2.768894E+09

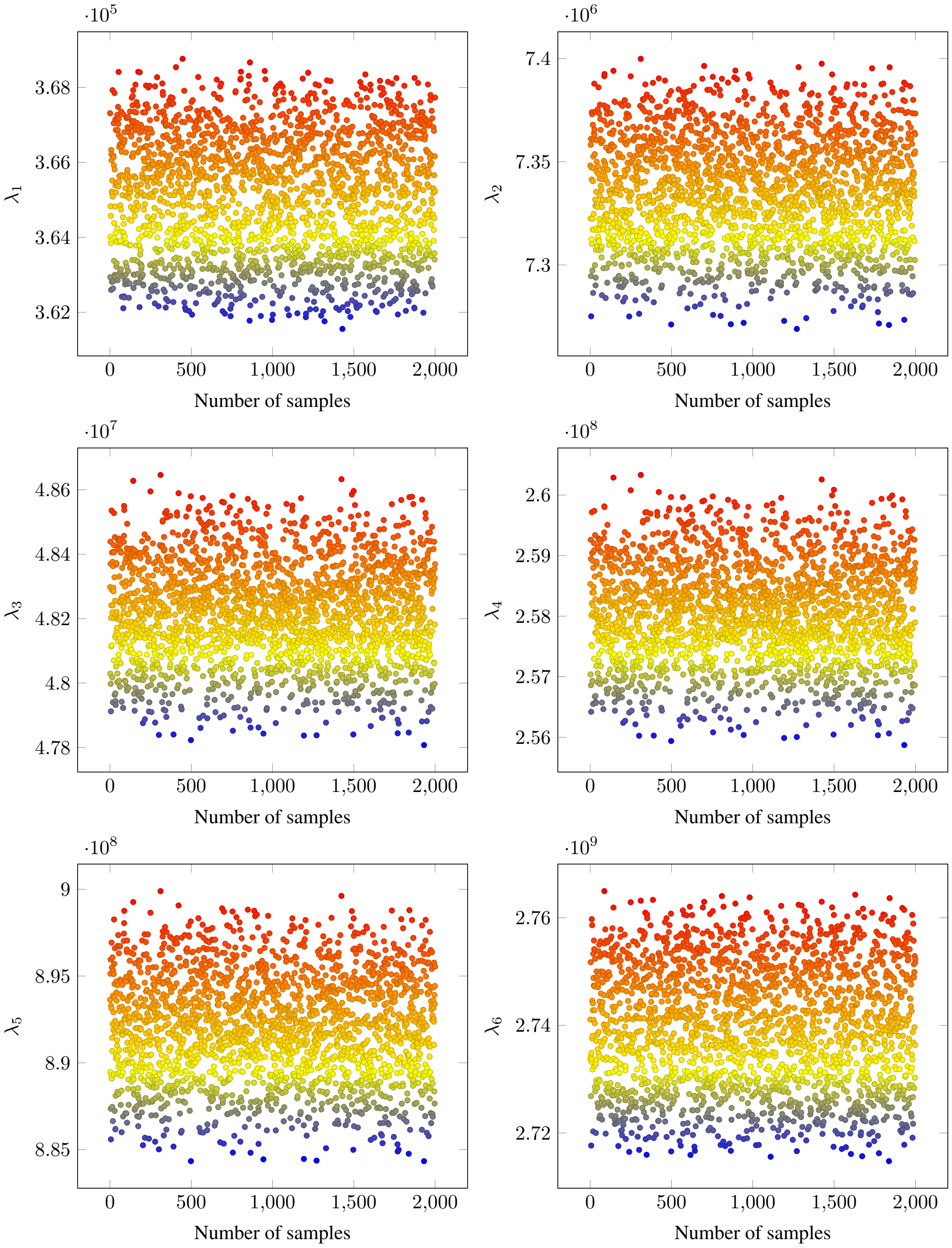

Figure 2: Simulations Monte Carlo method for three element stepped beam.

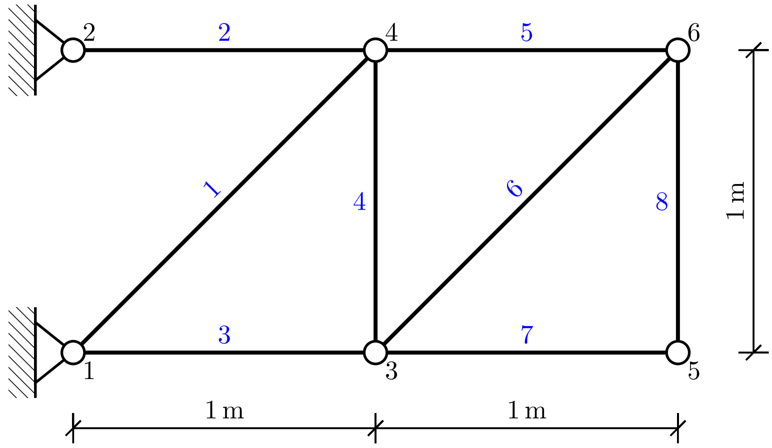

Figure 3: The eight bars truss.

Table 3: The first four interval eigenvalues for eight bars truss for $\beta = 2\%$.

	Proposed method		Monte Carlo method		Eigenvalue inclusion principle	
	Lower bound	Upper bound	Lower Bound	Upper bound	Lower bound	Upper bound
λ_1	7.842575E+05	7.957449E+05	7.842597E+05	7.957429E+05	7.668948E+05	8.137204E+05
λ_2	6.563958E+06	6.573943E+06	6.563960E+06	6.573942E+06	6.396250E+06	6.746699E+06
λ_3	8.891796E+06	8.976180E+06	8.891812E+06	8.976165E+06	8.687945E+06	9.184929E+06
λ_4	2.461287E+07	2.489588E+07	2.461293E+07	2.489583E+07	2.411954E+07	2.539569E+07

obtained by the Monte Carlo method at least up to the fourth decimal place. The graphical results obtained through simulations using the Monte Carlo method for $\beta = 2\%$ are shown by means of Fig. 5 for 2,000 samples.

5. CONCLUSIONS

In this paper, through the examples analyzed, it was demonstrated that the proposed method, together with the Finite Element Method and the Interval Analysis are important tools for the quantification of parametric uncertainties in structures, being of great relevance to guarantee the structural reliability, avoiding failures. The comparisons made with the other two methods show that the proposed method obtains better numerical results, and the results for the eight bars truss are exact, because it is a reticulated structure. For this, a computer program was developed to solve the interval eigenvalue problem, which demonstrated a rapid solution of the problems, and provided reliable solutions, with the possibility of application in practical situations. The Monte Carlo method was used to compare the results, demonstrating the reliability of the results obtained with the proposed method.

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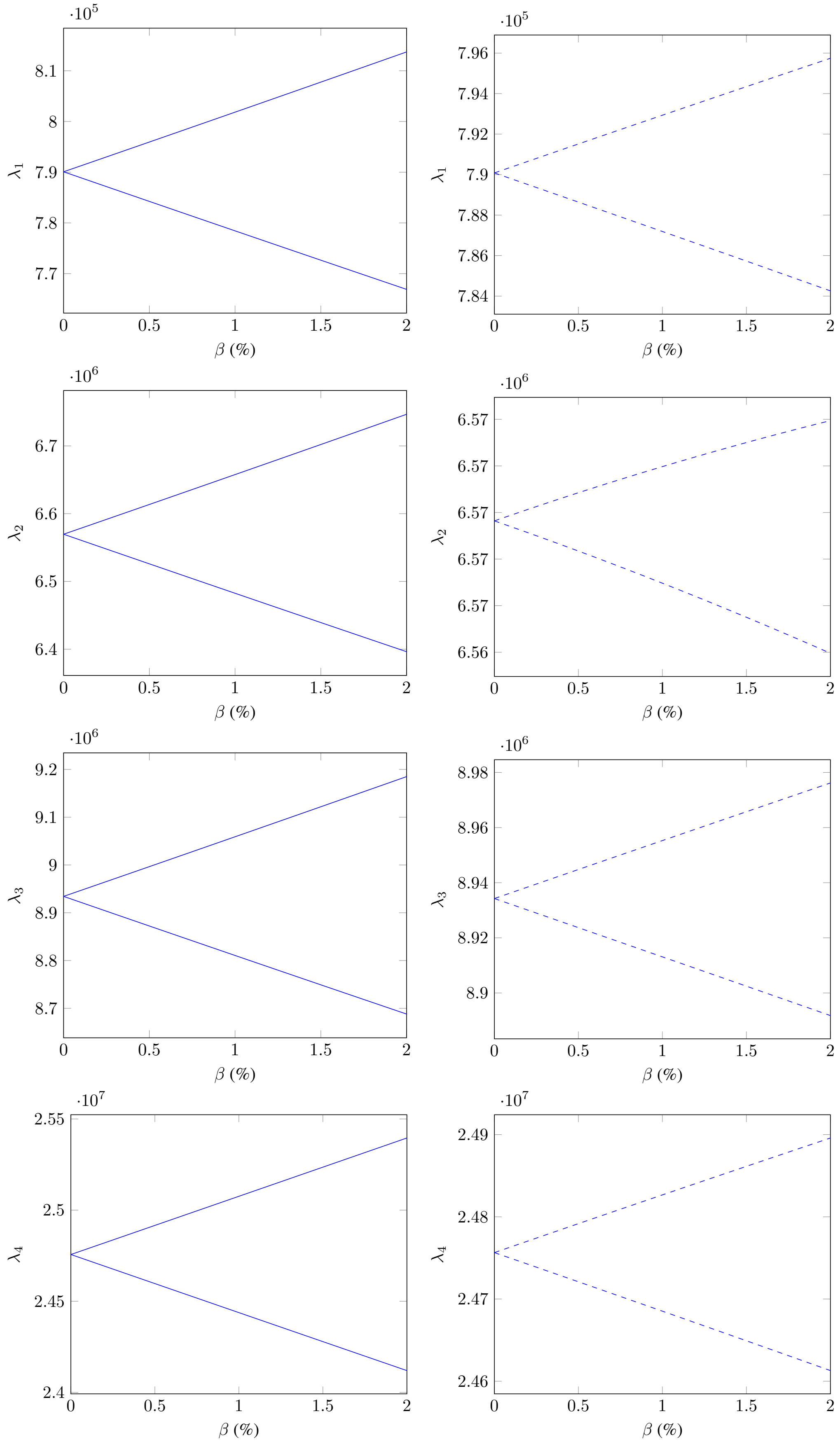

Figure 4: Bounds for the first four eigenvalues of the eight bars truss using eigenvalue inclusion principle (left) and proposed method (right).

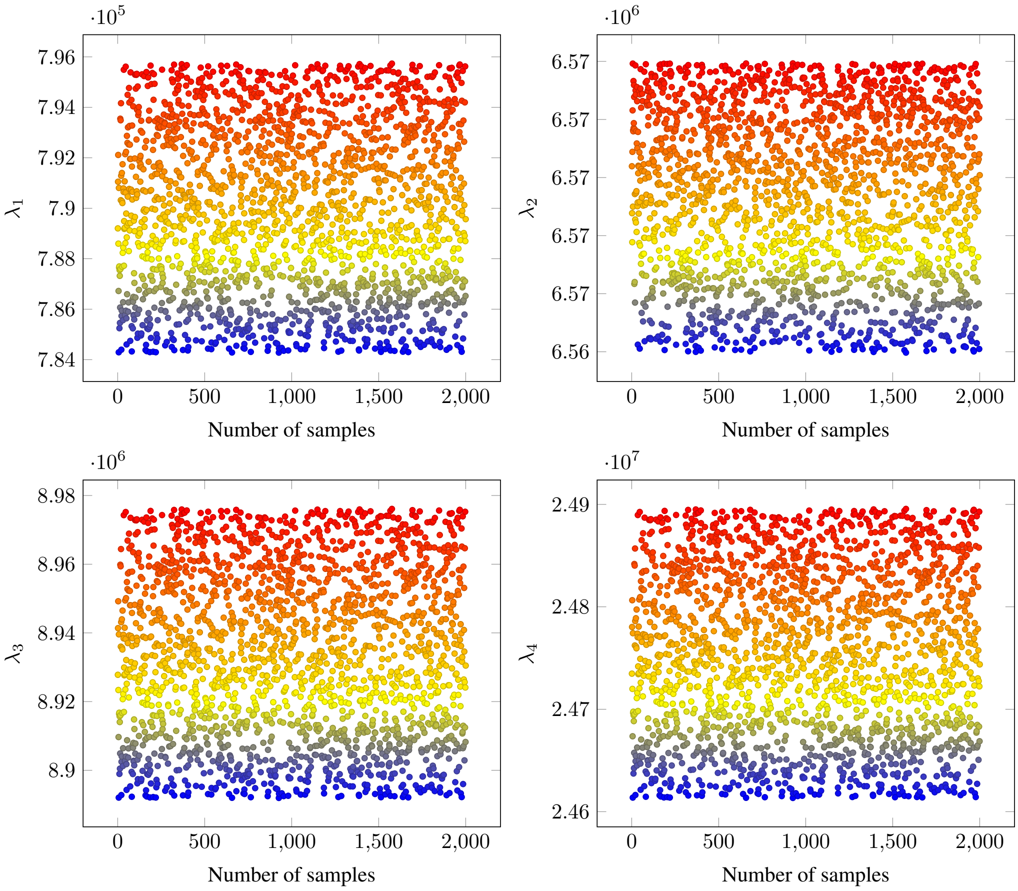

Figure 5: Simulations Monte Carlo method for the eight bars truss for $\beta = 2\%$.

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