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A COMPARISON OF TWO NUMERICAL ALGORITHMS FOR COMPUTING OPTIMAL LOW-THRUST TRAJECTORIES

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Abstract. *In this paper, two different numerical algorithms for computing optimal low-thrust limited power trajectories in an inverse-square force field are discussed. Both algorithms are based on indirect approach of solving optimal control problems. In this approach the two-point boundary value problem resulting from the necessary conditions expressed by the Pontryagin Maximum Principle is solved by means of a neighboring extremals algorithm based on state transition matrix.*

Keywords*: Low-thrust, limited power, Pontryagin Maximum Principle, neighboring extremals algorithm, infinitesimal canonical transformation.*

1. INTRODUCTION

The analysis presented in this work has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions in the last thirty years. Low-thrust electric propulsion systems are characterized by high specific impulse and low-thrust capability and have their greatest benefits for high-energy planetary missions (Marec, 1979; Racca, 2003). In the last fifty years, several researchers have obtained numerical and analytical solutions for several maneuvers involving specific initial and final orbits and specific thrust profiles (Edelbaum, 1965; Marec and Vinh, 1977; Haissig et al, 1993; Kiforenko, 2005).

In this paper, the transfer problem to be analyzed is concerned to the transfers between coplanar circular orbits with large radius ratio and moderate time of flight. Two different numerical algorithms for solving this problem are discussed. In the first algorithm, the optimization problem is formulated as a Mayer problem of optimal control with the radial distance and the components, radial and circumferential, of the velocity vector as state variables. In the second approach, the optimization problem is formulated as a Mayer problem of optimal control with a set of non-singular elements as state variables. Second order terms in eccentricity are then considered in the development of the state equations. In both cases, the fuel consumption is described by an auxiliary state variable that is a monotonic decreasing function of the mass of the space vehicle. The minimization of the final value of this consumption variable is equivalent to the maximization of the final mass of the vehicle or the minimization of the fuel quantity spent in the maneuver. In the first algorithm, the solution of the two-point boundary value problem of going from an initial circular orbit at time to a final circular orbit at a prescribed final time is obtained by applying straightforwardly the neighboring extremals algorithm. But, in the second algorithm, the solution of such boundary value problem is obtained in two stages. In the first one, is solved a two-point boundary value problem defined by an average canonical system describing the secular behavior of the optimal trajectories. The maximum Hamiltonian governing the average canonical system is derived by applying Hori method. So, an infinitesimal canonical transformation is built, and, the short periodic terms can be included in the solution by computing the Poisson brackets of the generating function. This technique provides an approximation, at a first order in a small parameter closely related to the magnitude of the optimal thrust acceleration, to the time behavior of the state variables considering the complete maximum Hamiltonian. In other words, it provides an approximation to the numerical integration of the complete canonical system. However, when the short periodic terms are included in the solution of the two-point boundary value problem obtained in the first stage, small deviations from the prescribed final conditions arise. Accordingly, the initial values of the adjoint variables computed in the first stage must be adjusted. This last step is performed by a simple Newton-Raphson algorithm with the partial derivatives of terminal constraints computed by a method of centered differences in which the state variables are obtained by the approximate solution using the infinitesimal canonical transformation.

Numerical results show the great agreement between the algorithms. Although the second algorithm has been developed including second order terms in eccentricity in the state equations, time behavior of the eccentricity along the optimal trajectory shows excellent agreement to the exact results provided by the first algorithm.

2. FIRST APPROACH OF THE OPTIMIZATION PROBLEM

In this section the general planar low-thrust limited power transfer problem is formulated as a Mayer problem of optimal control theory with the radial distance and the components of the velocity vector as state variables.

Consider the motion of a space vehicle $\mathcal{M}$, powered by a limited-power engine in an inverse-square force field. At time t, the state of a space vehicle $\mathcal{M}$ is defined by the radial distance r from the center of attraction, the radial and circumferential components of the velocity, v_r and v_s, and the variable J defined as (Marec, 1979).

$$J = \frac{1}{2}\int_{t_0}^{t} \Gamma^2 dt \tag{1}$$

where Γ is the magnitude of the thrust acceleration vector $\mathbf{\Gamma}$, used as a control variable. The consumption variable J is a monotonic decreasing function of the mass m of the space vehicle,

$$J = P_{\max}\left(\frac{1}{m} - \frac{1}{m_0}\right)$$

where $P_{\max}$ is the maximum power and m_0 is the initial mass. The minimization of the final value of the fuel consumption, J_f, is equivalent to the maximization of m_f.

The optimization problem can be formulated as a Mayer problem of optimal control as follows (Marec, 1979): It is proposed to transfer the space vehicle $\mathcal{M}$ from the initial state $\left(r_0, v_{r_0}, v_{s_0}, 0\right)$ at time t_0 to the final state $\left(r_f, v_{r_f}, v_{s_f}, J_f\right)$ at time t_f, such that the final consumption variable J_f is a minimum. The duration of the transfer $t_f - t_0$ is specified.

In the two-dimensional formulation, the state equations are given by

$$\frac{dr}{dt} = v_r$$

$$\frac{dv_r}{dt} = \frac{v_s^2}{r} - \frac{\mu}{r^2} + R$$

$$\frac{dv_s}{dt} = -\frac{v_r v_s}{r} + S$$

$$\frac{dJ}{dt} = \frac{1}{2}\left(R^2 + S^2\right) \tag{2}$$

where μ is the gravitational parameter, R and S are the radial and circumferential components of the thrust acceleration vector, respectively. The performance index is then defined by

$$IP = J(t_f)\,. \tag{3}$$

For limited power system, it is assumed that there are no constraints on the thrust acceleration vector (Marec, 1979).

According to (Da Silva Fernandes et al, 2016), the optimal trajectories are governed by the maximum Hamiltonian H^*,

$$H^* = v_r p_r + \left(\frac{v_s^2}{r} - \frac{\mu}{r^2}\right) p_{v_r} - \frac{v_r v_s}{r} p_{v_s} + \frac{1}{2}\left(p_{v_r}^2 + p_{v_s}^2\right), \tag{4}$$

where p_r, p_{v_r} and p_{v_s} are adjoint variables to r, v_r and v_s, respectively.

The optimal thrust acceleration is given by

$$R^* = p_{v_r} \qquad S^* = p_{v_s} \,. \tag{5}$$

From the maximum Hamiltonian H^*, one finds that the two-point boundary value concerning transfers between coplanar circular orbits is defined by:

$$\frac{dr}{dt} = v_r \qquad \frac{dp_r}{dt} = \left(\frac{v_s^2}{r^2} - 2\frac{\mu}{r^3}\right) p_{v_r} - \frac{v_r v_s}{r^2} p_{v_s}$$

$$\frac{dv_r}{dt} = \frac{v_s^2}{r} - \frac{\mu}{r^2} + p_{v_r} \qquad \frac{dp_{v_r}}{dt} = \frac{v_s}{r} p_{v_s} - p_r$$

$$\frac{dv_s}{dt} = -\frac{v_r v_s}{r} + p_{v_s} \qquad \frac{dp_{v_s}}{dt} = -2\frac{v_s}{r} p_{v_r} + \frac{v_r}{r} p_{v_s} \,, \tag{6}$$

with the boundary conditions

$$r(0) = 1 \qquad v_r(0) = 0 \qquad v_s(0) = 1 \,, \tag{7}$$

and,

$$r(t_f) = r_f \qquad v_r(t_f) = 0 \qquad v_s(t_f) = \sqrt{\frac{\mu}{r_f}} \,. \tag{8}$$

Equations (7) and (8) are given in canonical units and they define the initial circular orbit O_0 and the final circular orbit O_f. The conditions $v_r(t_f)$, $v_s(t_f)$ and $r(t_f)$ denote the state variables at the prescribed final time t_f, and, 0, $\sqrt{\mu/r_f}$ and r_f are the prescribed values defining the final circular orbit. Similar definition applies at the initial time $t_0 = 0$. The fuel consumption variable J is obtained by quadrature, after solving the boundary value problem. The boundary conditions defined by Eqns. (7) and (8) can be extended to elliptical orbits by using well-known equations of elliptic motion connecting orbital elements with position and velocity vectors.

In order to solve the two-point boundary value problem described above, a neighboring extremals method, based on the state transition matrix (Da Silva Fernandes et al, 2016) is applied with the continuation technique described in (Roberts et al, 1968) to assure the convergence.

Neighboring extremals algorithms are based on the solution of a linearized two-point boundary value problem that involves the derivatives of the right-hand side of the Eqn. (6) with respect to the state and adjoint variables. These equations can be put in the following form

$$\frac{d\delta\mathbf{x}}{dt} = A\delta\mathbf{x} + B\delta\mathbf{p}$$

$$\frac{d\delta\mathbf{p}}{dt} = C\delta\mathbf{x} - A^T\delta\mathbf{p}$$

with $\delta\mathbf{x}(t) = \mathbf{x}^{n+1}(t) - \mathbf{x}^n(t)$ and $\delta\mathbf{p}(t) = \mathbf{p}^{n+1}(t) - \mathbf{p}^n(t)$, where n denotes the iterate, and A, B and C are matrices given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -a & 0 & \frac{2v_s}{r} \\ \frac{v_r v_s}{r^2} & -\frac{v_s}{r} & -\frac{v_r}{r} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} d & -\frac{v_s}{r^2} p_{v_s} & b \\ -\frac{v_s}{r^2} p_{v_s} & 0 & \frac{1}{r} p_{v_s} \\ b & \frac{1}{r} p_{v_s} & -\frac{2}{r} p_{v_r} \end{bmatrix}$$

where

$$a = \frac{v_s^2}{r^2} - 2\frac{\mu}{r^3} \qquad b = \frac{2v_s}{r^2} p_{v_r} - \frac{v_r}{r^2} p_{v_s}$$

$$c = p_{v_r}^2 + p_{v_s}^2 \qquad d = \left(-\frac{2v_s^2}{r^3} + \frac{6\mu}{r^4} \right) p_u + \frac{2v_r v_s}{r^3} p_{v_s}$$

In above equations, $\boldsymbol{x}$ represents the state vector and $\boldsymbol{p}$ represents the adjoint vector.

3. SECOND APPROACH OF THE OPTIMIZATION PROBLEM

In this section, the second approach of the optimization problem is briefly described. This approach is based on an infinitesimal canonical transformation built by means of Hori method (1966). The state variables are defined by a suitable set of non-singular orbital elements as described in what follows.

Consider the motion of a space vehicle M, powered by a limited-power engine in an inverse-square force field. Since the terminal orbits are coplanar, the entire motion of the space vehicle is confined in a plane. So, at time t, the state of the space vehicle can be defined by the following set of non-singular orbital elements:

$$a \qquad h = e\cos\omega \qquad k = e\sin\omega \qquad l = M + \omega \tag{9}$$

and, the fuel consumption defined by variable J. Herein, a is the semi-major axis, e is the eccentricity, ω is the argument of pericenter and M is the mean anomaly.

The optimization problem can be formulated as a Mayer problem of optimal control as follows: It is proposed to transfer the space vehicle $\mathcal{M}$ from initial state $(a_0, h_0, k_0, l_0, 0)$ at time $t_0 = 0$ to the final state $(a_f, h_f, k_f, l_f, J_f)$ at time t_f, such that the final consumption variable J_f is a minimum. The duration of the transfer $t_f - t_0$ is specified, and, the final position of the vehicle is free, since only simple transfers (no rendezvous) are considered in this study.

The state equations, expressed up to the second order in eccentricity, are given by

$$\begin{aligned} \frac{da}{dt} = \frac{1}{na}\Big\{ & 2a\left[-k\cos l + h\sin l - 2hk\cos 2l + \left(h^2 - k^2\right)\sin 2l \right] R \\ & + 2a\left[1 - \frac{1}{2}\left(h^2 + k^2\right) + h\cos l + k\sin l + \left(h^2 - k^2\right)\cos 2l + 2hk\sin 2l \right] S \Big\} \end{aligned} \tag{10}$$

$$\frac{dh}{dt}=\frac{1}{na}\left\{\left[-k-\frac{1}{4}hk\cos l-\left(\frac{11}{8}h^2+\frac{13}{8}k^2-1\right)\sin l-k\cos 2l+h\sin 2l-\frac{9}{4}hk\cos 3l+\frac{9}{8}\left(h^2-k^2\right)\sin 3l\right]R\right.$$
$$\left.+\left[-\frac{3}{2}h-\left(\frac{5}{2}h^2+\frac{3}{2}k^2-2\right)\cos l-hk\sin l+\frac{3}{2}h\cos l+\frac{3}{2}k\sin 2l+\frac{3}{2}\left(h^2-k^2\right)\cos 3l+3hk\sin 3l\right]S\right\} \quad (11)$$

$$\frac{dk}{dt}=\frac{1}{na}\left\{\left[h+\left(\frac{13}{8}h^2+\frac{11}{8}k^2-1\right)\cos l+\frac{1}{4}hk\sin l-h\cos 2l-k\sin 2l-\frac{9}{8}\left(h^2-k^2\right)\cos 3l-\frac{9}{4}hk\sin 3l\right]R\right.$$
$$\left.+\left[-\frac{3}{2}k-hk\cos l-\left(\frac{3}{2}h^2+\frac{5}{2}k^2-2\right)\sin l-\frac{3}{2}\mathrm{k}\cos 2l+\frac{3}{2}\mathrm{h}\sin 2l-3hk\cos 3l+\frac{3}{2}\left(h^2-k^2\right)\sin 3l\right]S\right\} \quad (12)$$

$$\frac{dl}{dt}=n+\frac{1}{na}\left\{\left[-2-\frac{1}{2}\left(h^2+k^2\right)+\frac{3}{2}h\cos l+\frac{3}{2}k\sin l+\frac{1}{2}\left(h^2-k^2\right)\cos 2l+hk\sin 2l\right]R\right.$$
$$\left.+\left[-k\cos l+h\sin l+\frac{3}{2}hk\cos 2l+\frac{3}{4}\left(h^2-k^2\right)\sin 2l\right]S\right\} \quad (13)$$

$$\frac{dJ}{dt}=\frac{1}{2}\left(R^2+S^2\right) \quad (14)$$

where $n=\sqrt{\dfrac{\mu}{a^3}}$ is the mean motion, μ is the gravitational parameter, R and S are the radial and circumferential components of the thrust acceleration vector, respectively.

The performance index is defined, as before, by

$$IP=J\left(t_f\right) \quad (15)$$

Recall that there are no constraints on the thrust acceleration vector.

Following the Pontryagin Maximum Principle (Pontryagin et al, 1962), the adjoint variables p_a, p_h, p_k, p_l and p_J are introduced and the Hamiltonian function $H\left(a,h,k,i,\text{-},p_a,p_h,p_k,p_l,p_J,R,S\right)$ is formed:

$$H=p_a f_1+p_h f_2+p_k f_3+p_l f_4+p_J f_5 \quad (16)$$

where f_i, $i=1,\ 2,\ 3,\ 4,\ 5$ denotes the right-hand side of Eqns. (10) – (14), respectively.

The control variables R, S must be selected from the admissible controls such that the Hamiltonian function reaches its maximum along the optimal trajectory. Thus,

$$R^*=-\frac{1}{nap_J}\left\{2a\left[-k\cos l+h\sin l-2hk\cos 2\mathrm{l}+\left(h^2-k^2\right)\sin 2l\right]p_a+\left[-k-\frac{1}{4}hk\cos l-\left(\frac{11}{8}h^2+\frac{13}{8}k^2\right.\right.\right.$$
$$\left.\left.-1\right)\sin l-k\cos 2l+h\sin 2l-\frac{9}{4}hk\cos 3l+\frac{9}{8}\left(h^2-k^2\right)\sin 3l\right]p_h+\left[h+\left(\frac{13}{8}h^2+\frac{11}{8}k^2-1\right)\cos l\right.$$
$$\left.+\frac{1}{4}hk\sin l-h\cos 2l-k\sin 2l-\frac{9}{8}\left(h^2-k^2\right)\cos 3l-\frac{9}{4}hk\sin 3l\right]p_k+\left[-2-\frac{1}{2}\left(h^2+k^2\right)+\frac{3}{2}h\cos l\right.$$
$$\left.\left.+\frac{3}{2}k\sin l+\frac{1}{2}\left(h^2-k^2\right)\cos 2l+hk\sin 2l\right]p_l\right\} \quad (17)$$

$$S^* = -\frac{1}{nap_J}\left\{2a\left[1-\frac{1}{2}\left(h^2+k^2\right)+h\cos l+k\sin l+\left(h^2-k^2\right)\cos 2l+2hk\sin 2l\right]p_a+\left[-\frac{3}{2}h-\left(\frac{5}{2}h^2+\frac{3}{2}k^2\right.\right.\right.$$
$$\left.\left.-2\right)\cos l-hk\sin l+\frac{3}{2}h\cos 2l+\frac{3}{2}k\sin 2l+\frac{3}{2}\left(h^2-k^2\right)\cos 3l+3hk\sin 3l\right]p_h+\left[-\frac{3}{2}k-hk\cos l\right.$$
$$\left.-\left(\frac{3}{2}h^2+\frac{5}{2}k^2-2\right)\sin l-\frac{3}{2}k\cos 2l+\frac{3}{2}h\sin 2l-3hk\cos 3l+\frac{3}{2}\left(h^2-k^2\right)\sin 3l\right]p_k$$
$$\left.+\left[-k\cos l+h\sin l-\frac{3}{2}hk\cos 2l+\frac{3}{4}\left(h^2-k^2\right)\sin 2l\right]p_l\right\} \tag{18}$$

Note that p_J is a first integral and its value is obtained from the transversality condition, $p_J\left(t_f\right)=-1$. This result simplifies Eqns. (17) and (18).

From Eqns. (16), (17) and (18) one finds that the expression of the Hamiltonian, taking into account only terms up to the second order in eccentricity, can be put in the form

$$H^* = H_0 + H_\Gamma^* \tag{19}$$

where

$$H_0 = np_M$$

denotes the undisturbed Hamiltonian and H_Γ^* is the part related to the optimal thrust acceleration, as described in (da Silva Fernandes et al, 2017).

By applying the perturbation technique based on Hori method (1966) and described in previous works (da Silva Fernandes, 2003; da Silva Fernandes and das Chagas Carvalho, 2008; da Silva Fernandes et al, 2016; das Chagas Carvalho et al, 2016), one finds the following formal solution for the canonical system governed by the maximum Hamiltonian H^*:

$$a = a' + \frac{1}{4n'^3a'^2}\left\{8\left[\left(2h'^2-2k'^2\right)\sin 2l'-4k'\cos l'-4h'k'\cos 2l'+4h'\sin l'\right]a'^2\,p'_a+2a'\left[\left(-9h'^2-7k'^2+8\right)\sin l'\right.\right.$$
$$\left.+\left(3h'^2-3k'^2\right)\sin 3l'+4h'\sin 2l'+2h'k'\cos l'-6h'k'\cos 3l'-4k'\cos 2l'\right]p'_h+2a'\left[\left(-3h'^2+3k'^2\right)\cos 3l'\right.$$
$$\left.\left.+\left(7h'^2+9k'^2-8\right)\cos l'-4h'\cos 2l'-2h'k'\sin l'-6h'k'\sin 3l'-4k'\sin 2l'\right]p'_k\right\} \tag{20}$$

$$h = h' + \frac{1}{4n'^3a'^2}\left\{2\left[\left(\frac{5}{4}h'^2-\frac{5}{2}k'^2\right)\sin 4l'+\left(-4h'^2-\frac{7}{2}k'^2+\frac{3}{2}\right)\sin 2l'-4k'\cos l'-4h'\sin l'+\frac{1}{2}h'k'\cos 2l'\right.\right.$$
$$\left.-\frac{5}{2}h'k'\cos 4l'+\frac{4}{3}h'\sin 3l'-\frac{4}{3}k'\cos 3l'\right]p'_h+2a'\left[\left(-9h'^2-7k'^2+8\right)\sin l'+\left(3h'^2-3k'^2\right)\sin 3l'+4h'\sin 2l'\right.$$
$$\left.+2h'k'\cos l'-6h'k'\cos 3l'-4k'\cos 2l'\right]p'_a+\left[\left(\frac{15}{2}h'^2+\frac{15}{2}k'^2-3\right)\cos 2l'+\left(\frac{5}{2}k'^2-\frac{5}{2}h'^2\right)\cos 4l'-\frac{8}{3}k'\sin 3l'\right.$$
$$\left.\left.+8h'\cos l'-5h'k'\sin 4l'-8k'\sin l'-\frac{8}{3}h'\cos 3l'\right]p'_k\right\} \tag{21}$$

$$k = k' + \frac{1}{4n'^3 a'^2}\Bigg\{2\Bigg[\left(\frac{7}{2}h'^2 + 4k'^2 - \frac{3}{2}\right)\sin 2l' + \left(-\frac{5}{4}h'^2 + \frac{5}{4}k'^2\right)\sin 4l' + \frac{5}{4}h'k'\cos 4l' + \frac{1}{2}h'k'\cos 2l' + 4h'\sin l'$$
$$+4k'\cos l' - \frac{4}{3}h'\sin 3l' + \frac{4}{3}k'\cos 3l'\Bigg]p'_k + 2a'\Big[\left(-3h'^2 + 3k'^2\right)\cos 3l' + \left(7h'^2 + 9k'^2 - 8\right)\cos l' - 4h'\cos 2l'$$
$$-2h'k'\sin l' - 6h'k'\sin 3l' - 4k'\sin 2l'\Big]p'_a + \Bigg[\left(\frac{15}{2}h'^2 + \frac{15}{2}k'^2 - 3\right)\cos 2l' + \left(\frac{5}{2}k'^2 - \frac{5}{2}h'^2\right)\cos 4l' - \frac{8}{3}k'\sin 3l'$$
$$+8h'\cos l' - 5h'k'\sin 4l' - 8k'\sin l' - \frac{8}{3}h'\cos 3l'\Bigg]p'_h\Bigg\} \tag{22}$$

where a', h', k', p'_a, p'_h and p'_k satisfies the canonical system of differential equations governed by the average Hamiltonian

$$F_1 = \frac{1}{2n^2a^2}\left\{4a^2p_a^2 + \frac{5}{2}\left[p_h^2 + p_k^2 - \left(hp_h + kp_k\right)^2\right] - 2\left(kp_h - hp_k\right)^2\right\}, \tag{23}$$

and , l' is given by numerical integration of the differential equation

$$\frac{dl'}{dt} = n' + \frac{7}{4}\frac{a'}{\mu}\left(k'p'_h - h'p'_k\right). \tag{24}$$

Eqns. (20) – (22) can be put in the following compact form

$$a = a' + \delta a$$
$$h = h' + \delta h \tag{25}$$
$$k = k' + \delta k$$

where δa, δh and δk denotes the periodic terms in the right-hand side of Eqns. (20), (21) and (22).

Applying the initial conditions one finds that the time evolution of the state variables is written as

$$a(t) = a'(t) + \delta a(t) - \delta a(t_0)$$
$$h(t) = h'(t) + \delta h(t) - \delta h(t_0) \tag{26}$$
$$k(t) = k'(t) + \delta k(t) - \delta k(t_0)$$

where $\delta a(t)$, $\delta h(t)$ and $\delta k(t)$ are calculated at time t with a', h', k', p'_a, p'_h, p'_k obtained by numerical integration of the average canonical system from the initial conditions a_0, h_0, k_0, p_{a_0}, p_{h_0} and p_{k_0}; and, $\delta a(t_0)$, $\delta h(t_0)$, $\delta k(t_0)$ are calculated in terms of the initial conditions at $t = t_0$. Accordingly, the solution with periodic terms and the average solution satisfies the same initial conditions.

Equations (26) can be applied to get an approximation of the time evolution of the state variables. Note that if the two-point boundary value problem governed by the average canonical system is solved by means of a neighboring extremal algorithm, that is, p'_{a_0}, p'_{h_0} and p'_{k_0} are determined such that the final conditions are satisfied, then the periodic terms expressed by $\delta a(t) - \delta a(t_0)$, $\delta h(t) - \delta h(t_0)$ and $\delta k(t) - \delta k(t_0)$ provides small deviations from the final conditions, and p'_{a_0}, p'_{h_0}, p'_{k_0} must be adjusted in order to satisfy these conditions. This adjustment is made by a Newton-Raphson algorithm in the second stage of the algorithm as described in (da Silva Fernandes et al, 2017).

4. NUMERICAL RESULTS

In this section, the proposed algorithms for computing low-thrust limited power trajectories briefly described in the preceding sections are applied in a preliminary mission analysis which consists to transfer a space vehicle from a circular low Earth orbits (LEO) with 180 *km* of altitude to a final coplanar circular orbit with higher altitude, corresponding to an orbit of GPS constellation. The radius ratio $\rho = r_f / r_0$ is then equal to 4.0502, approximately.

Table 1 shows values of the consumption variable J for four values of transfer duration, $t_f - t_0 = T$. In this table, $J_{Numerical}$ denotes the consumption variable computed by solving the transfer problem by means of the algorithm 1 described in Section 2, and, $J_{Analytical-Numerical}$ denotes the consumption variable computed by means of the algorithm 2 described in the Section 3. The results are expressed in canonical units defined in terms of the radius and period of LEO: 1 D.U. = 6558.2 *km* and 1 T.U. = 841.2175 *s*. Accordingly, $r_f = 26657$ *km* and $T = 29.209$ *hs*; 35.050 *hs*; 40.892 *hs* and 46.734 *hs*.

Table 1. Consumption variable J

$\rho = r_f / r_0$	$t_f - t_0$	$J_{Analytical-Numerical}$	$J_{Numerical}$
4.0502	125.0	1.0124×10^{-3}	1.0301×10^{-3}
	150.0	8.4372×10^{-4}	8.5392×10^{-4}
	175.0	7.2319×10^{-4}	7.2978×10^{-4}
	200.0	6.3279×10^{-4}	6.3744×10^{-4}

It should be noted that for such transfer duration, the perturbations due to the second zonal harmonic of the geopotential can be significant; but, the purpose of this study is a preliminary mission analysis, as mentioned before, without considering other perturbations on the motion of the space vehicle.

In Figures 1 – 4, the time evolution of the semi-major axis and eccentricity are presented for both algorithms. It should be noted the contribution of the periodic terms in the time behavior of the orbital elements, mainly, of the eccentricity; as well as, the good agreement between the results obtained by the two algorithms.

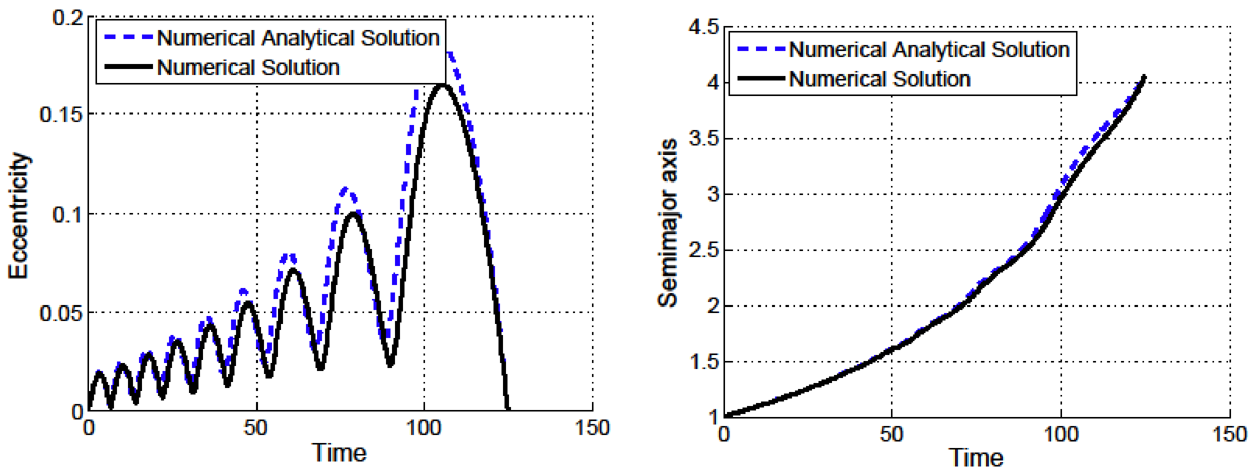

Figure 1. Numerical-analytical solution and numerical solution for $\rho = 4.0502$ D.U. and $t_f - t_0 = 125.0$ T.U.

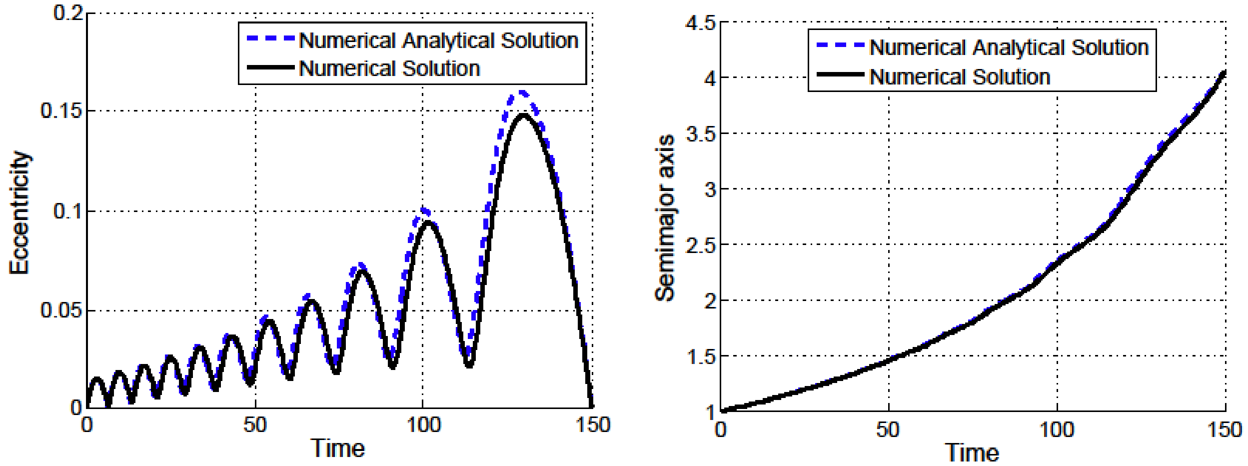

Figure 2. Numerical-analytical solution and numerical solution for $\rho = 4.0502$ D.U. and $t_f - t_0 = 150.0$ T.U.

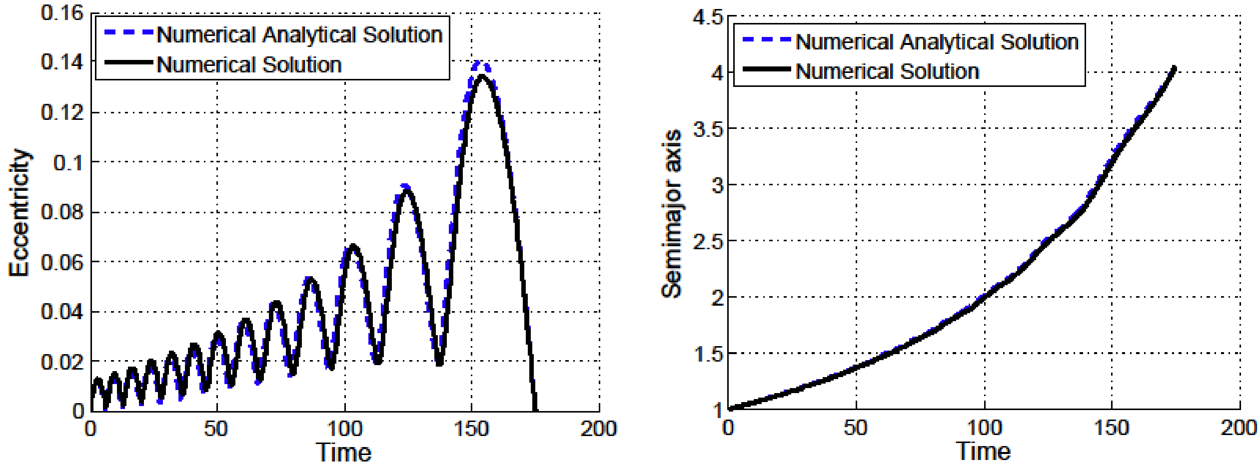

Figure 3. Numerical-analytical solution and numerical solution for $\rho = 4.0502$ D.U. and $t_f - t_0 = 175.0$ T.U.

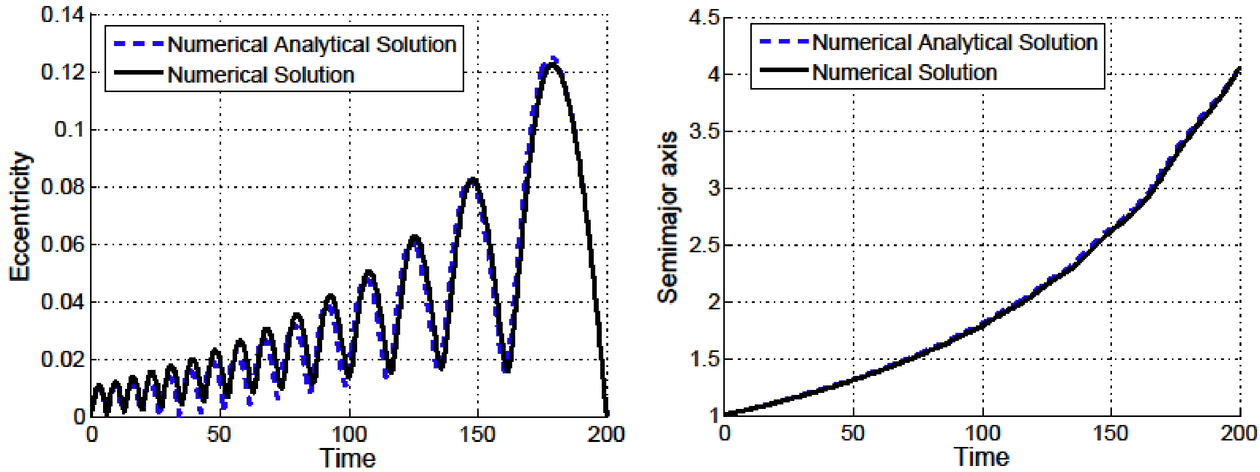

Figure 4. Numerical-analytical solution and numerical solution for $\rho = 4.0502$ D.U. and $t_f - t_0 = 200.0$ T.U.

5. CONCLUSION

In this work, two different numerical algorithms for computing optimal low-thrust limited-power trajectories concerning transfers between coplanar orbits in an inverse-square force field are discussed. In the first algorithm, the optimization problem is formulated as a Mayer problem of optimal control with the radial distance and the components, radial and circumferential, of the velocity vector as state variables. In the second approach, the optimization problem is formulated as a Mayer problem of optimal control, with a set of non-singular elements as state variables. The second approach is based on an infinitesimal canonical transformation built by applying Hori method. The proposed algorithms are applied in a preliminary mission analysis which consists to transfer a space vehicle from a circular low Earth orbits (LEO) to a final coplanar circular orbit with, higher altitude corresponding to an orbit of GPS constellation. Numerical results show the great agreement between the results provided by the proposed algorithms.

6. ACKNOWLEDGEMENTS

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