MECSOL
Joinville 2017

April 26-28, 2017
Joinville, Santa Catarina, Brazil

NON SATURATING KINEMATIC HARDENING LAW COUPLED WITH DAMAGE EVOLUTION FOR FATIGUE

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ABSTRACT

The present contribution proposes the use of an advanced plasticity model coupled to an improved continuum damage model in order to estimate low cycle fatigue life under proportional and non-proportional loading conditions. The plasticity model is based on a non-linear and non-saturating power law for the kinematic hardening modeling, proposed by Desmorat (2010) and Prisacari et al (2016). This model describes the non symmetry of the hysteresis loops and captures the effect of partial mean stress relaxation. For a better description of multiaxial fatigue – than by the original Continuum Damage Mechanics (CDM) approach of Lemaitre type – a modified microcracks closure effect is derived and introduced in the damage evolution law Malcher & Mamiya (2012). Here we represent the microcraks closure effect by a "hydrostatic sensitivity to damage", keeping the relation between the deviatoric stress and damage unaltered. The improvement consists in substituting the damage strength, constant material parameter S in initial Lemaitre law, by a function, dependent on both stress triaxiality and normalized third invariant. An implicit numerical integration scheme is suggested and implemented. Numerical results are compared with experimental data available in literature and parameters governing partial mean stress relaxation, evolution of damage/stress/strain variable and fields as well as fatigue life, are analyzed.

Keywords: Damage, Mean stress, Kinematic Hardening, Stress triaxiality, Third invariant.

1 INTRODUCTION

In times of so severe economic competiveness, the request for designs even more efficient, light and cheap, without gave up safety, has encouraged engineers and researchers to propose innovative methods capable to reduce the conservatism and pragmatism in their approaches currently applied in mechanical projects. On different types of machines and structures variable loads are acting. A precise description of the material response to such efforts is crucial to performance increase, not only the economic point of view, but the safety and reliability also. Many industrial sectors are affected by unexpected failure of mechanical components, especially the aerospace whose catastrophic fractures results in loss of human lives and serious financial prejudice.

Mechanical components for industrial applications are often submitted to fatigue complex loadings. Models common used to handle random fatigue usually require the loading simplification, made periodic or periodic by blocks, with the difficulty then to appreciate the errors made. The difficulties become most important when the loadings are multiaxial, non-proportional or anisothermal. More moderns approaches consist in define a variable capable of represent the material degradation accumulated over the time. In these context, the Continuum Damage Mechanics (CDM) arise as powerful tool to model complex fatigue, at least up to crack initiation ([1]; [2]; [3]; [4]; [5]). The kinetic (evolution) damage laws simply need to be time integrated to calculate the damage increment per cycle if the loading is periodic or over the whole loading if not. Recent studies has been demonstrated the importance of the third invariant effect as an important elasto-plastic parameter that needs to be accounted damage evolution law, in [6] a model which incorporate the effect of the third invariant of the stress tensor and the triaxiality was proposed.

The response of damage approaches is directly dependent to a good description of the plasticity phenomena. Nowadays, efforts have been made in plasticity to better describe the mean stress relaxation and the ratcheting under cyclic mutilaxial loads. The mean stress effect in HCF, well known and often well represented in uniaxial by Goodman and Soderberg linear rules, becomes quiet complex in multiaxial solicitations, [7] as Sines and Crossland criterions cannot describe the full triaxiality range nor properly handle non-proportional loadings. Even in uniaxial cases, standard models for backstress like A-F, or based in variations of it, predict a complete mean stress relaxation over first cycle. Recent approaches proposing the plasticity behavior is governed by kinematic non-saturating and non-linear kinematic hardening law [8] and [9]. Such models are demonstrated to be more adaptable and gave a good description for the mean stress relaxation

Considering the framework above described, the aim of this work is to propose a damage formulation, coupled to an elasto-plasticity model, capable to better describe the mean stress relaxation. More specifically, to proposes a modified crack closure effect associated to the improved damage evolution law by [6] and couple it to the kinematic non-saturating hardening law [8] with partial mean stress relaxation and [9]. An implicit numerical integration scheme is suggested and some preliminary results are present to evaluate the methodology response.

2 DAMAGE: TRIAXIALITY AND THIRD INVARIANT DEPENDENCY

The CDM can be understood as a continuum representation of the initiation, growth, and coalescence of microdefects and micro-cracks present at a scale smaller than the scale of the representative volume element of classical continuum mechanics. The isotropic case consists in a scalar representation variable, D, [1],[10],[11]. One of the most well-known CDM model for ductile materials is the Lemaitre's [12], [13]. There is a large range of factors in order to characterize different state of stress and indeed to quantify its influence in the ductile fracture [14]. Among them, the hydrostatic stress (σ_H), the stress triaxiality (T_X) and the Lode Angle(Θ), determined in terms of the invariants of the second order stress tensor, plays an important role:

$$\sigma_H = \frac{1}{3} I_1 = \frac{1}{3} tr\ (\boldsymbol{\sigma}) \tag{1}$$

$$T_X = \frac{J_1}{\sqrt{3J_2}} = \frac{\sigma_H}{\sigma_{eq}} \tag{2}$$

$$\Theta = \frac{1}{3} \arccos \left[\frac{27}{2} \frac{J_3}{(\sqrt{3J_2})^3} \right] = \frac{1}{3} \arccos \left[\frac{27}{2} \frac{\det \boldsymbol{\sigma}^D}{\left(\sigma_{eq}\right)^3} \right] \tag{3}$$

Where $(*)^D$denotes the deviatoric part of a second tensor; $J_2 = \frac{1}{2} \boldsymbol{\sigma^D} : \boldsymbol{\sigma^D}$ and $J_3 = \det \boldsymbol{\sigma}^D$are the second and third invariants of the deviatoric stress tensor. Here the subscript $(*)_{eq}$ denotes the von Mises equivalent norm of any second order tensor $\boldsymbol{A}$, as $(\boldsymbol{A})_{eq} = \sqrt{\frac{3}{2} \boldsymbol{A}^D : \boldsymbol{A}^D}$. From the Lode angle ($\theta$) definition, the so-called normalized third invariant of the deviatoric stress tensor is obtained and presented below:

$$\Xi = \frac{\frac{27}{2} \det \boldsymbol{S}}{{\sigma_{eq}}^3} \ ; \Xi \in [-1,1] \tag{4}$$

At [6] it was suggested the introduction of the effect of the stress state on the development of a new constitutive formulation, by the stress triaxiality and third invariant, for determining the correct displacement at fracture onset, regarding wide range of stress triaxiality. For that, the Lemaitre's original model was used as reference model, regarding isotropic hardening and damage.

$$S(T_X, \Xi\ ) = \frac{S_{0.33}}{3|T_X| + \frac{S_{0.33}}{S_{0.0}} (1 - \Xi^2)} \tag{5}$$

The equation (5) is the proposed function and represents the coupling behavior of the denominator of damage within the regions of the generalized stress triaxiality and the Lode angle. $S_{0.33}$ and $S_{0.0}$ are two material constants to be calibrated from standard tension/torsion tests. Under pure shear stress condition the values of normalized stress traxiality and the normalized third invariant are both null. Hence, the denominator of damage function recovers the value of $S_{0,0}$. In cases where the stress triaxiality is elevated, the denominator of damage will be equal to or less than $S_{0.33}$.

3 KINEMATIC HARDENING AND A MODEL FOR MEAN STRESS RELAXATION IN PLASTICITY

. A number of constitutive models have been proposed to describe the elastoplastic behaviour under cyclic loading conditions ([15]; [16]; [17], [18], [19], [8]). Some alloys are reported as a particular behavior, like the TiAl6V and INCO718DA [20],[9]. They observed cyclic loops are more "sharp", in the sense that the exit out of the elastic domains is done with a high tangent modulus and a tendency to non- saturated curve for large strain amplitudes. [9], it was show the material softens cyclically and that there is a slight tension-compression asymmetry. Such materials are better represented by a non-saturating law as that developed by [8].

$$\dot{\boldsymbol{X}} = \frac{2}{3}C\dot{\boldsymbol{\varepsilon}}^p - \Gamma X_{eq}^{M-2}\boldsymbol{X}\langle \dot{X}_{eq}\rangle;\ \ where\ \dot{X}_{eq} = \frac{d}{dt}\sqrt{\frac{3}{2}\boldsymbol{X}:\boldsymbol{X}} \tag{6}$$

Where C is the modulus of Kinematic Hardening, Γ is a material parameter of the springback term, M is an exponent to control the curvature of the stress strain curve and $X_{eq}$is the von Mises norm of $\boldsymbol{X}$. The symbol $\langle\ \ \rangle$ represents the *Macauley bracket*, that means for any scalar: $\langle\alpha\rangle = \begin{cases}\alpha\ if\ \ \alpha \geq 0 \\ 0\ if\ \alpha < 0\end{cases}$. This model is based on a power-law representing the springback term instead the usual exponential behavior of the AF. Another important feature, the backstress is now governed by von Mises $\dot{X}_{eq}$, leading to a non-vanishing rate $\dot{\boldsymbol{X}}$.

For the isothermal cases $\boldsymbol{X}:\dot{\boldsymbol{X}}$ has the same sign as $\boldsymbol{X}:\dot{\boldsymbol{\varepsilon}}^p$, so after some algebraic work the equation can be rewritten as:

$$\dot{\boldsymbol{X}} = \frac{2}{3}C\dot{\boldsymbol{\varepsilon}}^p - \frac{C\Gamma X_{eq}^{M-3}}{1-\Gamma X_{eq}^{M-2}}\langle\boldsymbol{X}:\dot{\boldsymbol{\varepsilon}}^p\rangle\boldsymbol{X} \tag{7}$$

3.1 Memory Effect

The concept of memory effect in plasticity was firstly introduced by [21]. It was proposed the introduction of an index surface to represent an equivalent limit of plastic strain described of two new internal variables capable to recorder the maximum value reached for plastic strain. The mathematical description of the index function is given by:

$$\mathrm{F} = \sqrt{2/3(\boldsymbol{\varepsilon}^p - \boldsymbol{\xi}):(\boldsymbol{\varepsilon}^p - \boldsymbol{\xi})} - q \tag{8}$$

Such a surface define an hypersphere, similarly to the elasticity yield surface, by a scalar isotropic variable q, which determine the radius of the surface, and a tensorial kinematic variable $\boldsymbol{\xi}$, which gives the coordinate of the its center. Analogously to the rules of plastic flow, it is possible to establish the consistency rule for the evolution the index memory function. ($F = 0\ and\ \dot{F} = 0\ while\ \dot{\boldsymbol{\varepsilon}}^p > 0$). Thus, the evolution laws of the two variables q and $\boldsymbol{\xi}$ are defined as:

$$\dot{q} = b\mathcal{H}(\mathrm{F})\langle\boldsymbol{N}:\boldsymbol{N}^\star\rangle\,\dot{\lambda}\ ;\quad \dot{\boldsymbol{\xi}} = \sqrt{\frac{3}{2}}\,(1-b)\mathcal{H}(\mathrm{F})\langle\boldsymbol{N}:\boldsymbol{N}^\star\rangle\boldsymbol{N}^\star\,\dot{\lambda} \tag{9}$$

Where the term $\mathcal{H}(\mathrm{F})$ is the Heaviside function applied to F, b is a scalar weight parameter, $\boldsymbol{N}$ and $\boldsymbol{N}^{\star}$are the normals to the index function and the yield surface, respectively:

$$\boldsymbol{N}^{\star} = \frac{(\boldsymbol{\varepsilon}^{\boldsymbol{p}} - \boldsymbol{\xi})}{\|\boldsymbol{\varepsilon}^{\boldsymbol{p}} - \boldsymbol{\xi}\|} \qquad \boldsymbol{N} = \frac{(\boldsymbol{\sigma} - \boldsymbol{X})}{\|\boldsymbol{\sigma} - \boldsymbol{X}\|} \tag{10}$$

The parameter b works as a weight function between the isotropic and the kinematic parts of the memory surface. High values of b imply in an evolution mainly isotropic of the hysphere. For small values of b the evolution is mainly kinematic. A negative normal product indicates an unloading, which does not change $\boldsymbol{\xi}$ and q.

3.2 Mean Stress relaxation (Dependency Between Γ and Memory effect)

The analysis of the evolution of mean stress behavior in relation to the strain amplitude of imposed strain tests reveals a progressive mean stress relaxation. Generally such behavior has the general form presented in figure 1, where three main domains can be distinguished. At first domain, the strain amplitude is small and the stress amplitude dos not exceed the yield stress. The material remains always in elastic regime and there is no relaxation at all.

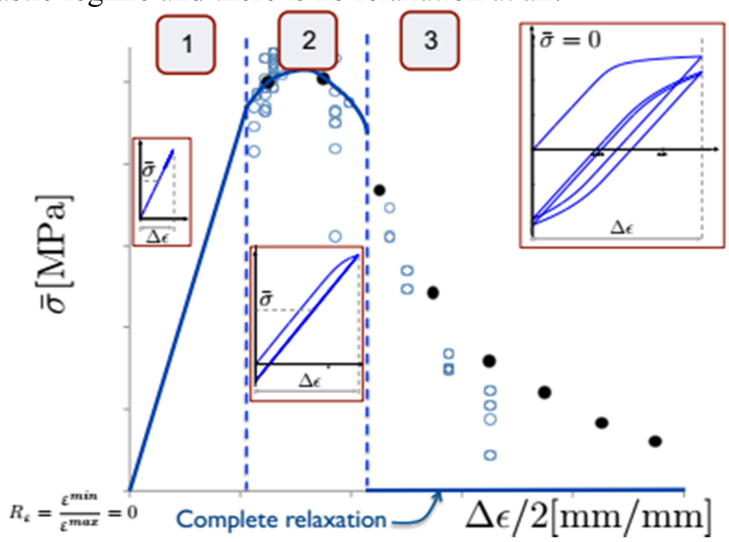

Figure 1 - Mean stress relaxation in relation to increase of instrain amplitude. (adapted from [9])

In 2, the strain amplitude achieves sufficient levels to induce the plastic flow, but, after the unloading, the adaptation (elastic stabilization) occurs due to hardening and the material returns to an elastic response after the first cycle. In the third zone, the strains amplitude reaches values capable to generate plastic flow at during both, loading and unloading, as consequence the mean stress progressively relax. Standard models in plasticity, like the A-F lead to a complete mean stress relaxation in a few cycles.

In [9] the behavior of the Inco718DA alloy was studied under strain controlled tests with strain ratios $\left(R_\varepsilon = \frac{\varepsilon_{min}}{\varepsilon_{max}}\right)$, $R_\varepsilon = -1$ and $R_\varepsilon = 0$ using variable amplitude, and a monotonic test with progressive unloading. In order to achieve sharp cycles at a saturated isotropic cyclic plasticity domain, the kinematic hardening was considered governed by the eq.(6).

It was demonstrated the existence of a dependency between Γ and the maximum amplitude of the plastic strain experienced by the material. For the strains ratio $R_\varepsilon = -1$, the computed Γ with respect to the maximum plastic strain presented a distribution very close to linear. They decided to introduce into the memory surface of the plastic strain an approach similar to the one used by [21]. The initially constant $\Gamma$of the non-saturating kinematic hardening rule, eq.(7), was replaced by $\Gamma(q, \boldsymbol{\xi})$ where q is the memory effect variable corresponding to b = 1. This approach allows adjusting the non-saturating cyclic loops, with their corresponding curvatures Γ at each level of strain amplitude.

In the monotonic case was noticed that the distribution of the parameter $\Gamma(q)$ is no longer linear, but affine. To correctly represent the monotonic behavior using this kinematic hardening rule, we need to introduce an offset term (Γ_0) in the equation. Thus, the equation for computing Γ in the monotonic case is:

$$\Gamma_{Mono} = \langle \Gamma_0' q + \Gamma_0 \rangle \tag{11}$$

In order to find a solution to the complete mean stress relaxation in cyclic plasticity zone, the cyclic loops were analyzed in detail. They noted differences in the evolution of the kinematic hardening on the two parts of the loops (ascending and descending). The kinematic hardening on the descending part X_{Down}, and the absolute value of the kinematic hardening on the ascending part $|X_{Up}|$, were noted with different values. This dual behavior was fundamental to their model.

A partition of the distribution $\Gamma(q, \boldsymbol{\xi})$ in a mean value $\bar{\Gamma}$ and amplitude $\Gamma_a \geq 0$ is considered. During the ascending loading ($\dot{\boldsymbol{\varepsilon}}^p > 0$) the parameter $\Gamma = \Gamma_{up} = \bar{\Gamma} - \Gamma_a$, otherwise $\Gamma = \Gamma_{down} = \bar{\Gamma} + \Gamma_a$. The fitting equations in terms of $(q, \boldsymbol{\xi})$ are given by:

$$\bar{\Gamma} = A_q + B\xi_{eq} - c - \xi_{eq}/q \quad ; \quad \Gamma_a = (\Gamma_a)_M \left[1 - \left\langle \frac{\xi_M - \xi_{eq}}{\xi_M - \xi_l} \right\rangle^{a_l} - \left\langle \frac{\xi_{eq} - \xi_M}{\xi_r - \xi_M} \right\rangle^{a_r} \right] \tag{12}$$

Therefore, to apply that model, during a simulation there will be three distinct situations, which translate in three sets of values for the parameter Γ. The first case is the transition phase, when either the maximum or the minimum value of the strain is evolving. Here, the parameter Γ is computed as the eq. 11 .When the maximum or minimum value of the strain isn't surpassed during loading the cyclic phase is activated. Thus, when the triaxiality of the strain rate is positive, the ascending part will be used Γ_{Up}, when it's negative, the descending part Γ_{Down} is considered.

4 PROPOSED METHODOLOGY

4.1 Isotropic Damage with Modified Microcrack Closure Effect

Since damage evolution is characterized as a growth of the defects and micro-cracks, under a compressive state, such voids will partially close and its effect on the stiffness reduction is strongly minimized. The classical solution for isotropic case was proposed by [22], see also [23] and [24]. The effective stress concept is fundamental to write the coupling with plasticity without unilateral conditions. However, it cannot be used coupled with the microdefects closure effect, [25]. The yield surface may become non convex [23]. In order to avoid that problem, here is introduced an adaptation of this formulation by the modification the elastic part of the free energy density. It is proposed to include the closure effect conditions just into hydrostatic part of the state potential performed by a power law:

$$\psi_{\boldsymbol{e}}^{*}(\boldsymbol{\sigma}, D) = \frac{1}{4G}\frac{\boldsymbol{\sigma}^{D}:\boldsymbol{\sigma}^{D}}{(1-D)} + \frac{1}{18K}\left[\frac{\langle tr(\boldsymbol{\sigma})\rangle^{2}}{(1-D)^{\eta}} - \frac{\langle -tr(\boldsymbol{\sigma})\rangle^{2}}{(1-D)^{h}}\right] \tag{13}$$

Where η and h are material parameter to damage sensibility to hydrostatic stress. Such modification on the original model lies in the fact of mechanism of plasticity is controlled by the slips produced by shear stress, represented by deviatoric of the stress tensor $\boldsymbol{\sigma}^{D}$, which is the almost same regardless the of their sign. Hence, the open or closure of microcracks has much more influence over the hydrostatic part of the stress tensor. The proposed power law coupling between damage and the closure effect has the aim of better describe the energetics dissipation aspects of the damage, especially when it goes to the unity, in both cases, traction and compression when .

The elasticity law for this potential

$$\varepsilon^{e} = \frac{\rho\partial\psi_{e}^{*}}{\partial\boldsymbol{\sigma}} = \frac{1}{2G}\frac{\boldsymbol{\sigma}^{D}}{(1-D)} + \frac{1}{9K}\left[\frac{\langle tr(\boldsymbol{\sigma})\rangle}{(1-D)^{\eta}} - \frac{\langle -tr(\boldsymbol{\sigma})\rangle}{(1-D)^{h}}\right]\boldsymbol{I} \tag{14}$$

An advantage is the possibility of direct use of the effective stress into standard plasticity formulations, simplifying the coupling between damage and plasticity. The effective stress is defined as:

$$\widetilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}^{\boldsymbol{D}}}{(1-D)} + \frac{1}{3}\left[\frac{\langle tr(\boldsymbol{\sigma})\rangle}{(1-D)^{\eta}} - \frac{\langle -tr(\boldsymbol{\sigma})\rangle}{(1-D)^{h}}\right]\boldsymbol{I} \tag{15}$$

The energy density release rate Y is given by

$$Y = \frac{\partial\psi_{e}^{*}}{\partial D} = \frac{1}{4G}\frac{\boldsymbol{\sigma}^{D}:\boldsymbol{\sigma}^{D}}{(1-D)^{2}} + \frac{1}{18K}\left[\eta\frac{\langle tr(\boldsymbol{\sigma})\rangle^{2}}{(1-D)^{\eta+1}} + h\frac{\langle -tr(\boldsymbol{\sigma})\rangle^{2}}{(1-D)^{h+1}}\right] \tag{16}$$

4.2 Dissipation potential and Evolution laws

The strain equivalence principle proposed by [1] is considered. Thus, despite the replacement of $\boldsymbol{\sigma}$ by $\widetilde{\boldsymbol{\sigma}}$ into the plasticity formulation, the others variables in plasticity remain unchanged at the original formulation [26]. As the variables related to the memory effect just have the influence to correct the value of Γ into the backstress evolution law, they do not change the thermodynamic state of the material and there is no energy dissipated for their evolution. Assuming the normality rule and associative plasticity, the plastic flow rule and the evolution of the internal variables can be derived as:

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^{p} = \dfrac{\dot{\lambda}}{1-D}\dfrac{3}{2}\dfrac{\widetilde{\boldsymbol{\sigma}}^{\boldsymbol{D}} - \boldsymbol{X}}{(\widetilde{\boldsymbol{\sigma}}^{\boldsymbol{D}} - \boldsymbol{X})_{eq}} \\ \dot{D} = \dfrac{\dot{\lambda}}{1-D}\left(\dfrac{Y}{S(T_{X}, \Xi)}\right)^{s} \\ \dot{r} = \dot{\lambda} \\ \dot{\boldsymbol{\alpha}} = \dot{\lambda}\dfrac{3}{2}\dfrac{\widetilde{\boldsymbol{\sigma}}^{\boldsymbol{D}} - \boldsymbol{X}}{(\widetilde{\boldsymbol{\sigma}}^{\boldsymbol{D}} - \boldsymbol{X})_{eq}} - \dfrac{3}{2}\dfrac{\Gamma(\boldsymbol{\xi}, q)}{C}X_{eq}^{M-2}\boldsymbol{X}\langle\dot{X}_{eq}\rangle \end{cases} \tag{17}$$

For convenience at the numerical implementation, the equation for the back stress evolution will be considered as the form present in the eq. (7). The evolution laws of the memory effect internal variables are given by eq.(9). The equations for dependency of the Γ factor are that present ate the section 3.2

4.3 Mathematical Formulation

The full set of constitutive equations for isotropic damage coupled with the elasto-plasticity considering isotropic and non-linear hardening is resumed in next boxes.

Mathematical Model Isotropic Damage With Proposed Closure Effect

i. Additive Strain Decomposition

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$$

ii. State laws

$$\varepsilon^e = \mathbb{E}^{-1}:\left\{\frac{\boldsymbol{\sigma}^D}{(1-D)} + \frac{1}{3}\left[\frac{\langle tr(\boldsymbol{\sigma})\rangle}{(1-D)^\eta} - \frac{\langle -tr(\boldsymbol{\sigma})\rangle}{(1-D)^h}\right]\boldsymbol{I}\right\}$$

$$R = 0$$

$$\boldsymbol{X} = \frac{2}{3}C\boldsymbol{\alpha}$$

$$Y = \frac{1}{4G}\frac{\boldsymbol{\sigma}^D:\boldsymbol{\sigma}^D}{(1-D)^2} + \frac{1}{18K}\left[\eta\frac{\langle tr(\boldsymbol{\sigma})\rangle^2}{(1-D)^{\eta+1}} + h\frac{\langle -tr(\boldsymbol{\sigma})\rangle^2}{(1-D)^{h+1}}\right]$$

iii. Yield Function

$$f = \sqrt{3J_2(\tilde{\boldsymbol{\sigma}}^D - \boldsymbol{X})} - \sigma_{y0}$$

iv. Evolutions laws

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \dfrac{\dot{\lambda}}{1-D}\sqrt{\dfrac{3}{2}}\dfrac{\tilde{\boldsymbol{\sigma}}^{\boldsymbol{D}} - \boldsymbol{X}}{\|\tilde{\boldsymbol{\sigma}}^{\boldsymbol{D}} - \boldsymbol{X}\|} \\ \dot{r} = \dot{\lambda} = \dot{p}(1-D) \\ \dot{\boldsymbol{\alpha}} = \dot{\lambda}\,\dot{\boldsymbol{\varepsilon}}^p - \dfrac{C\Gamma X_{eq}^{M-3}}{1+\Gamma X_{eq}^{M-1}}\langle \boldsymbol{X}:\dot{\boldsymbol{\varepsilon}}^p\rangle\boldsymbol{\alpha} \\ \dot{D} = \dfrac{\dot{\lambda}}{1-D}\left(\dfrac{Y}{S(T_X,\Xi)}\right)^s \\ \dot{q} = b\mathcal{H}(\mathrm{F})\langle \boldsymbol{N}:\boldsymbol{N}^\star\rangle\,\dot{\lambda} \\ \dot{\boldsymbol{\xi}} = \sqrt{3/2}\,(1-b)\mathcal{H}(\mathrm{F})\langle \boldsymbol{N}:\boldsymbol{N}^\star\rangle\boldsymbol{N}^\star\,\dot{\lambda} \end{cases}$$

$$S(T_X,\Xi) = \frac{S_{0.33}}{3|T_X| + \frac{S_{0.33}}{S_{0.0}}(1-\Xi^2)}$$

$$T_X = \frac{\sigma_H}{\sigma_{eq}}$$

$$\Xi = \frac{27}{2}\frac{det\,\boldsymbol{\sigma}^D}{{\sigma_{eq}}^3}$$

$$\Gamma(\boldsymbol{\xi},q) = \Gamma_a \pm \bar{\Gamma}$$

$$\bar{\Gamma} = A_q + B\xi_{eq} - c - \xi_{eq}/q$$

$$\Gamma_a = (\Gamma_a)_M\left[1 - \left\langle\frac{\xi_M - \xi_{eq}}{\xi_M - \xi_l}\right\rangle^{a_l} - \left\langle\frac{\xi_{eq} - \xi_M}{\xi_r - \xi_M}\right\rangle^{a_r}\right]$$

v. Plastic multiplier

$$\dot{r} = \dot{\lambda} = \dot{p}\,(1-D)$$

vi. Load/unload criterion

$$\dot{\lambda} \geq 0 \;\;,\; f \leq 0\,,\, f\dot{\lambda} = 0$$

4.4 Numerical Implementation

The numerical integration of the constitutive equations Lemaitre's isotropic damage model was firstly proposed by [27]. By the non-linear nature of the internal variables evolution equations, is necessary to implement a numerical technique able to discretize the continuo time domain and determine the change from a known state variables, denoted by the subscript n, at a timet_n, to a new thermodynamic state variables $n+1$ at a time t_{n+1} by the established strain increment $\Delta\boldsymbol{\varepsilon}$.

Numerical Model Algorithm

1) Determine a trial elastic predictor state:

$$\boldsymbol{\varepsilon}_{n+1}^{e,trial} = \boldsymbol{\varepsilon}_n^e + \Delta\boldsymbol{\varepsilon} \quad \widetilde{\boldsymbol{\sigma}}_{n+1}^{trial} = \mathbb{E}: \boldsymbol{\varepsilon}_{n+1}^{e,trial} \quad D_{n+1}^{trial} = D_n$$

$$\boldsymbol{X}_{n+1}^{trial} = \boldsymbol{X}_n \qquad q_{n+1}^{trial} = q_n \qquad \boldsymbol{\xi}_{n+1}^{trial} = \boldsymbol{\xi}_n$$

2) Perform the decomposition of $\boldsymbol{\sigma}_{n+1}^{trial}$ and calculate:

$$\sigma_{eq}{}_{n+1}^{trial} = \sqrt{\frac{3}{2}\left(\widetilde{\boldsymbol{\sigma}}^{\boldsymbol{D}}{}_{n+1}^{trial} - \boldsymbol{X}_{n+1}^{trial}\right):\left(\widetilde{\boldsymbol{\sigma}}^{\boldsymbol{D}}{}_{n+1}^{trial} - \boldsymbol{X}_{n+1}^{trial}\right)}$$

3) Verify the plastic admissibility:

$$\phi_{n+1}^{trial} = \sigma_{eq}{}_{n+1}^{trial} - \sigma_{y0}$$

If $\phi_{n+1}^{trial} \leq 0$, elastic and $(*)_{n+1} = (*)_{n+1}^{trial}$;

Else, **plastic step**: Apply return algorithm:

4) Algorithm mapping: Solve the non-linar equantion system (Newton=Raphson), With the following internal variables: $\Delta\boldsymbol{\varepsilon}_{n+1}$, $\Delta\boldsymbol{\alpha}_{n+1}, \Delta r$, ΔD_{n+1} , $\Delta\boldsymbol{\sigma}_{n+1}, \Delta q_{n+1}$ and $\Delta\boldsymbol{\xi}_{n+1}$

$$\Delta\boldsymbol{\varepsilon}_{n+1}^e - \Delta\boldsymbol{\varepsilon}_{n+1}^{e,trial} + \Delta r\boldsymbol{N}_{n+1} = \boldsymbol{0}$$

$$\sigma_{eq}(\widetilde{\boldsymbol{\sigma}}_{n+1}^D, \boldsymbol{X}_{n+1}) - \sigma_{y0} = 0$$

$$\Delta\boldsymbol{\alpha}_{n+1} - \Delta r\boldsymbol{N}^X + \frac{C\Gamma(\boldsymbol{\xi}_{n+1}, q_{n+1}) X_{eq}{}_{n+1}^{M-3}}{X_{eq}{}_{n+1}^{M-1}}$$

$$\Delta D_{n+1} - \frac{\Delta r}{1 - D_{n+1}}\left(\frac{Y_{n+1}}{S(T_{X_{n+1}}, \xi_{n+1})}\right)^s = 0$$

$$\boldsymbol{\varepsilon}_{n+1}^e - \mathbb{E}^{-1}: \widetilde{\boldsymbol{\sigma}}_{n+1} = \boldsymbol{0}$$

$$\Delta q_{n+1} = 0$$

$$\Delta\boldsymbol{\xi}_{n+1} = \boldsymbol{0}$$

5) Actualize the internal and associated variables.

$$\boldsymbol{\varepsilon}_{n+1}^e = \boldsymbol{\varepsilon}_n^e + \Delta\varepsilon^e \qquad \boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p + \Delta r\boldsymbol{N}_{(n+1)} \qquad \boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_n + \Delta\boldsymbol{\sigma}$$

$$\boldsymbol{\alpha}_{\boldsymbol{n+1}} = \boldsymbol{\alpha}_n + \Delta\boldsymbol{\alpha} \qquad \boldsymbol{X}_{n+1} = \frac{2}{3}C\boldsymbol{\alpha}_{n+1} \qquad D_{n+1} = D_n + \Delta D$$

$$q_{\boldsymbol{n+1}} = q_n + \Delta q \qquad \boldsymbol{\xi}_{n+1} = \boldsymbol{\xi}_n + \Delta\boldsymbol{\xi}$$

6) End

The elasto-plastic constitutive model results initial value problem and the backward (or fully implicit) Euler method is applied. Stress update procedures, which are based on the so-called operator split concept, see [28] and [5]. A two-step algorithm is established in which the two steps are considered. The set of discretized evolution equations are called the return mapping equations:

Newton-Raphson Return mapping algorithm

i) Determination of the trial state:

ii) Solve the no-linear system of equations for the internal variables on the

$$\Delta\boldsymbol{\varepsilon}^{\boldsymbol{e}(0)} = \boldsymbol{\varepsilon}^{\boldsymbol{e}trial} \quad \Delta r^{(0)} = 0 \quad \Delta\boldsymbol{\alpha}^{(0)} = \boldsymbol{0} \quad \Delta D^{(0)} = 0$$
$$\tilde{\boldsymbol{\sigma}}_{n+1}^{trial} = \mathbb{E}{:}\,\boldsymbol{\varepsilon}_{n+1}^{e,trial} \quad \Delta q^{(0)} = 0 \quad \Delta\boldsymbol{\xi}^{(0)} = \boldsymbol{0}$$

iii) linearized form: $R_{n+1}^{k} - \left[\frac{\partial R}{\partial \boldsymbol{A}}\right]_{n+1}^{k} \cdot \delta A_{n+1}^{k}=0$

Update the Jacobian Matrix

$$\left[\frac{\partial R}{\partial \boldsymbol{A}}\right]_{n+1}^{k}$$

iv) calculate :

$$\Delta\boldsymbol{A}_{n+1}^{(k+1)} = \Delta\boldsymbol{A}_{n+1}^{(k)} + \boldsymbol{\delta A}_{n+1}^{(k_{+1})}$$

v) Verify the convergence:

$$error = \frac{Res_i^{\ k+1}}{|(Var)_{\mathrm{i}}|} \leq tolerance$$

vi) end.

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