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GFEM STABILIZATION TECHNIQUES APPLIED TO DYNAMIC ANALYSIS OF NON-UNIFORM SECCTION BARS

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ABSTRACT

The Finite Element Method (FEM), although widely used as an approximate solution method, has some limitations when applied in dynamic analysis. As the loads excite the high frequency and modes, the method may lose precision and accuracy. To improve the representation of these high-frequency modes, we can use the Generalized Finite Element Method (GFEM) to enrich the approach space with appropriate functions according to the problem under study. However, there are still some aspects that limit the GFEM applicability in problems of dynamics of structures, as numerical instability associated with the process of enrichment. Due to numerical instability, the GFEM may lose precision and even result in numerically singular matrices. In this context, this paper presents the application of two proposals to minimize the problem of sensitivity of the GFEM: an adaptation of the Stable Generalized Finite Element Method for dynamic analysis and a stabilization strategy based on preconditioning of enrichment. Examples of one-dimensional modal and transient analysis are presented. Numerical results obtained are discussed analyzing the effects of the adoption of preconditioning techniques on the approximation and the stability of GFEM in dynamic analysis.

Keywords: GFEM; Dynamic Analysis; Numerical Stability

1. INTRODUCTION

Dynamic analysis of a structure can be considered as an extension of static analysis. The difference is summarized in adding to the analysis the variation in time, and thus the effect of the resulting inertial actions. In this context, computational modeling is a relevant tool in the area of engineering projects, since in a dynamic analysis it is often not possible to test all hypothesis and models. In practice Numerical Methods, of which the Finite Element Method (FEM) is the most widespread, are of great importance for the definition and analysis of complex structures.

Although being a robust approximation method, FEM solves very costly problems with solutions that contain singularities, discontinuities or specific characteristics. Thus, the improvement of Finite Element models applied to the structural dynamics is part of a vast field of dynamic analysis of structures, constituting an area of current research and industrial applications.

Using particular information to improve the approximation characteristics is the core of the enriched methods, such as the Generalized Finite Element Method (GFEM), where enriching functions are used to improve the approximation space. In this context, the GFEM has applications in several areas, such as fracture mechanics, flow of biphasic fluids, electromagnetism, heat transfer with high gradients and, as approached in this work, in dynamics of structures. Despite the excellent properties of the GFEM, there are aspects that still limit its practical applicability and its efficiency.

One of these limiting factors is the numerical instability associated with the present enrichment process even in well-defined contour value problems. Several studies have been directed to the treatment of this problem in recent years, mainly in the context of fracture mechanics. However, there is a shortage of more in-depth studies discussing the stability of the GFEM in dynamic analysis and, therefore, this work focuses on this question.

2. METHODOLOGY

2.1 Bar Model Formulation

Consider a bar or straight axis of length l and variable cross-sectional area A, according to Fig. 1. For simplicity, it is considered that the cross section does not vary abruptly and maintains a constant shape, only by varying its area as a function of the horizontal relative position x. The constituent material is considered of linear behavior with modulus of elasticity E being able to vary as a function of x and has specific mass [1, 2].

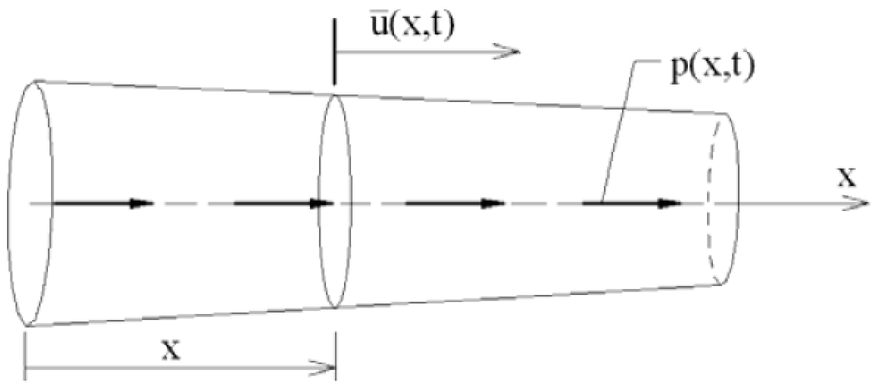

Figure 1: Bar Model.

The vibration of the bar is a time dependent problem, and the equation of motion governing this problem is a partial differential equation. The problem is to find the axial displacement u satisfying:

$$\rho A \frac{\partial^2 \bar{u}}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial \bar{u}}{\partial x}\right) = p(x,t) \tag{1}$$

where $A = A(x)$ is the cross section area, $E = E(x)$ is the Young modulus, ρ is the specific mass, p is the externally applied axial force per unit length and t is the time. The solution $\bar{u} = \bar{u}(x,t)$ must satisfy the boundary and initial conditions defined in the problem.

According to Carey and Oden [3], in order to obtain the variational form of a time dependent problem, time t must be considered as a real parameter and develop a family of variational problems in t. This consists in selecting test functions w = w(x), independent of t, and applying the weighted-residual method. If the Finite Element Method is used to represent the spatial behavior of the solution, one obtains a system of ordinary differential equations in terms of the tine dependent degrees of freedom. This approach is called the semi discrete formulation of the problem.

For all admissible functions $w \in H^1(0,L)$ bilinear forms $B : H^1 \times H^1 \implies \Re$ and $F : H^1 \times H^1 \implies \Re$ may be obtained from

$$B(u,w) = \int_0^L EA \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} dx \tag{2}$$

$$F(u,w) = \int_0^L \rho A u w dx \tag{3}$$

Bilinear forms B and F generate stiffness K and mass M matrices, respectivelly, assuming interpolation functions $\phi(x)$ to describe u and w, as shown below.

$$K_{ij} = \int_0^L EA \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} dx \tag{4}$$

$$M_{ij} = \int_0^L \rho A \phi_i \phi_j dx \tag{5}$$

2.2 Bar with cross section area variation

This paper studies a bar with polynomial cross area variation modeled as:

$$A(x) = (a + bx)^4 \tag{6}$$

where a and b are polynomial coefficients.

According to Kumar and Sujith [4], the analytical solution of the free vibration problem of a bar with polynomial variation of cross-sectional area in the form of equation 6 may be obtained using Bessel functions.

In the work of Kumar and Sujith [4] it was verified that the lower natural frequencies are more affected by the variation of the cross section, and the highest ones are very close to the frequencies of the equivalent uniform bar. Arndt [5] in his work verified that the adaptive GFEM did not reach the same level of precision obtained in the problems of uniform bar. However, the error in the solution of the six frequencies is often smaller than that obtained by the h-version of the linear FEM with a greater number of degrees of freedom, especially at higher frequencies. It was also observed that the results of the analysis with the adaptive GFEM are better for the second frequency, always using a smaller number of degrees of freedom.

2.2.1 Trigonometric Enrichment

For free vibration problem was proposed by [5] a set of enrichment functions to the problem of dynamic analysis with GFEM. This group of functions consists of building a couple of clouds, a sine and a cosine, subordinated to the cover of enriched node. These clouds are written in the domain of element as two pairs of sine and cosine functions. The basic domain is considered to $\xi \in [0,1]$.

Sine cloud:

$$\begin{aligned} \gamma_{1j} &= sin(\beta_j L_e \xi) \\ \gamma_{2j} &= sin(\beta_j L_e (\xi - 1)) \end{aligned} \tag{7}$$

Cosine cloud:

$$\begin{aligned} \varphi_{1j} &= cos(\beta_j L_e \xi) - 1 \\ \varphi_{2j} &= cos(\beta_j L_e (\xi - 1)) - 1 \end{aligned} \tag{8}$$

Where L_e is the length of the element and $\beta_j = j\pi$ is a hierarchical enrichment parameter proposed by [5] to j function levels.

2.3 Transient bar problem

The trigonometric enrichment of GFEM applied to one-dimensional transient analysis was discussed earlier by [6] and [7] covering several examples. This work will continue the discussion by presenting the application of p refinement with stabilization proposals set out in three enlightening examples. Stabilization alternatives in transient analysis, such as HHT used by [7], were avoided in order to keep focus on the analysis of the interactions of the enrichment process and the stabilization proposals.

For the following examples, transient analysis arises in the application of the Newmark Method, as described in [8], using mass and stiffness matrices generated by the application of GFEM with different approaches. Parameters were setted such as $\sqrt{\frac{E}{\rho}} = c = 1$, neglecting damping, and adopting a uniform mesh of 20 finite elements. For the time discretization, the 20 seconds analysis interval was divided in 2000 steps of 10^{-2} seconds.

The model considered for the three examples consists of a bar with a clamped end and the other free end where the load is applied. Trigonometric enrichment was adopted using $\beta_1 = \frac{3\pi}{4}$ due to its performance in modal analysis as presented by [9].

2.4 Stabilization Strategies

2.4.1 SGFEM-based Stabilization

Stable Generalized Finite Element Method (SGFEM) was firstly proposed to address numeric conditioning issues of GFEM [10]. This methods consists in the application of a subtle modification of enrichment function prior to its inclusion in GFEM approximation space, see [11, 12].

In the SGFEM, the enrichment function are modified as described in Eq. 9, as presented by [10]:

$$\widetilde{\varphi}_i(x) = \varphi_i(x) - I_\omega(\varphi_i(x)) \tag{9}$$

where,

- $\widetilde{\varphi}_i$: i-th stabilized enrichment function
- φ_i: i-th enrichment function
- $I_\omega(\varphi_i(x))$: linear interpolant of the i-th enrichment function subordinated to support ω

The proposed stabilization of the first level of enrichment is shown in Fig.2.

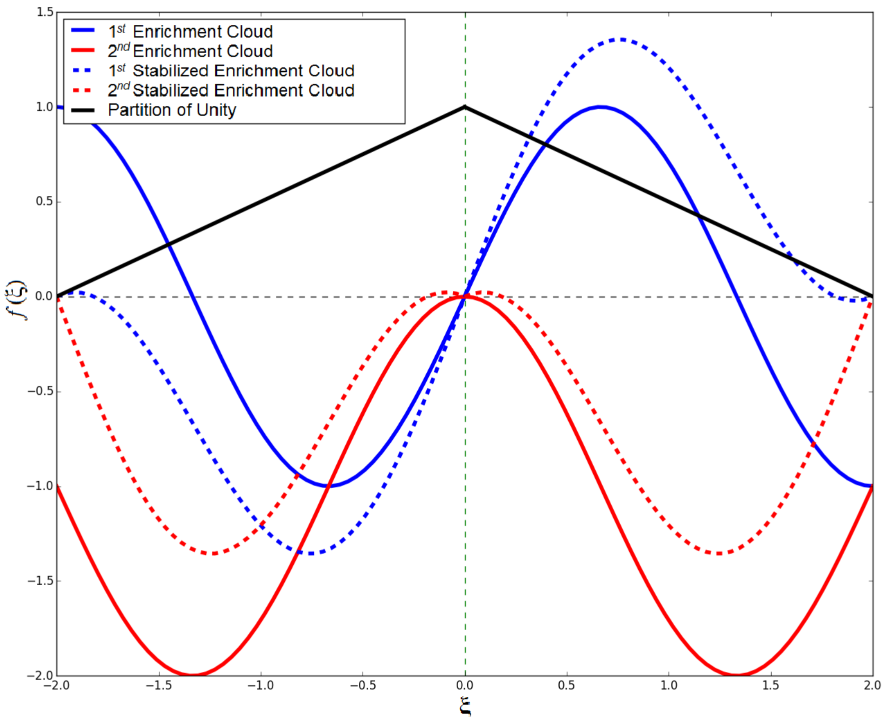

Figure 2: Stabilization process for the first level of enrichment functions.

2.4.2 Heuristic Modification Stabilization

Note that the variation in the enrichment parameter β_1 implies different characteristics of approaching, as already pointed out by [5] and [6]. However, the gain in accuracy for certain frequencies does not appear to be associated with numerical stability gain. In fact, apparently there is a certain trade-off between accuracy and numerical stability, regarding the choice of the parameter β_1.

A second point of interest is the observation of β_j family of parameters. Parameters described by β_j are intrinsically related with enrichment functions and, consequently, with relations between the enriched functions present in the approximation space base. Thus, evolution of β_j influences the numerical characteristics of the approximation, such as stability. Therefore, we sought a change in the enrichment function group to stabilize its continuous application, avoiding the construction of approach spaces that tend to linear dependence.

The proposed modification is basically the change parameter formation rule of β_j enrichment parameter, resulting in a subtle modification of each level of enrichment.

Recalling that parameter β_j is calculated by $\beta_j = j\alpha\pi$, it was proposed a modification, creating new stabilized parameters $\bar{\beta_n}$ given by:

$$\bar{\beta_j} = \left[2\left(j-1\right) + \frac{\beta_1}{\pi}\right]\pi \qquad j \geq 1 \tag{10}$$

It's interesting to note that $\bar{\beta_1} = \beta_1$, since:

$$\bar{\beta_1} = \left[2\left(1-1\right) + \frac{\beta_1}{\pi}\right]\pi = \left[0 + \frac{\beta_1}{\pi}\right]\pi = \beta_1 \tag{11}$$

This implies that there is no difference between approximation taken by this approach and the standard trigonometric first level of enrichment.

3. NUMERICAL RESULTS

In this section are presented modal and transient analysis of a bar with polynomial cross area section variation in the form $A(x) = (1+x)^4$, as described.

3.1 Prior Modal Analysis

In order to validate the numerical results, we present the comparison of the numerical values obtained through the exposed methods in relation to the analytical solution presented by [4]. For the firsty six frequencies of a double-clamped bar with area variation as choosen, the solutions obtained were:

Analytical Solution	FEM - 100 elements	GFEM - 20 elements 1 enrichment layer	GFEM - 20 elements 2 enrichment layer
3.286007	3.286175	3.286008	3.286007
6.360678	6.361800	6.360690	6.360678
9.477196	9.480820	9.477233	9.477196
12.605890	12.614341	12.605973	12.605890
15.739656	15.756038	15.739809	15.739656
18.876001	18.904194	18.876251	18.876001

As one may observe in the table presented, GFEM solution achieve better approximation using less elements and only one enrichment layer. Applying the second enrichment layer improve even more the accuracy, as shown in the last column of the table. However, the approximation of higher frequencies may require more enrichment levels, as discussed in [9]. This enrichment process may result in numerical instabilities arising from ill-conditioned matrices. The Fig. 3 presents the evolution of the condition number of the mass matrix related to a 20-element uniform mesh successively enriched.

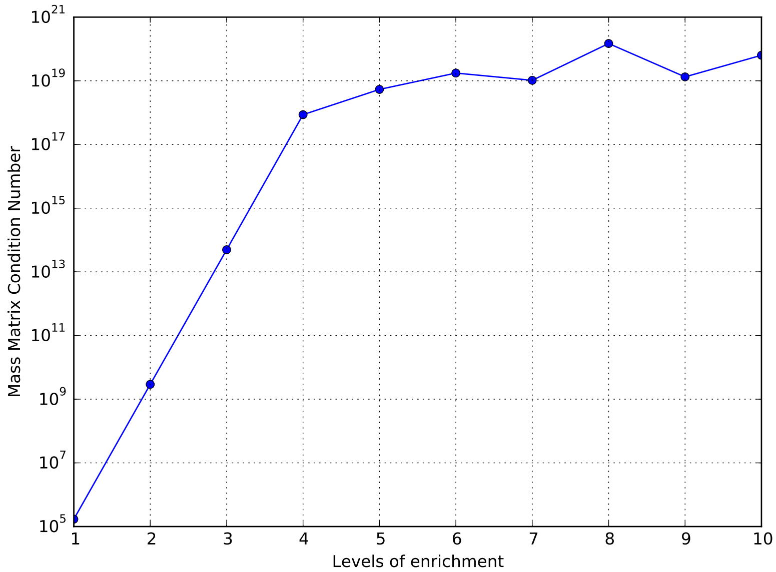

Figure 3: Evolution of Mass Matrix Condition Number (Numerically Calculated).

Intending to address this issue, stability approaches are applied in the transients analysis context and results are presented in the following section.

3.2 Transient Analysis

Firstly is compared an approximation performed using 20-element uniform mesh with: no enrichment (FEM); standard one-level trigonometric enrichment (GFEM); and Stable one-level trigonometric enrichment (SGFEM). The results are presented in the Figs. 4 and 5. The external load is applied with a value of $1N$ in the time range from 0 to 10^{-2} seconds featuring a impulse loading.

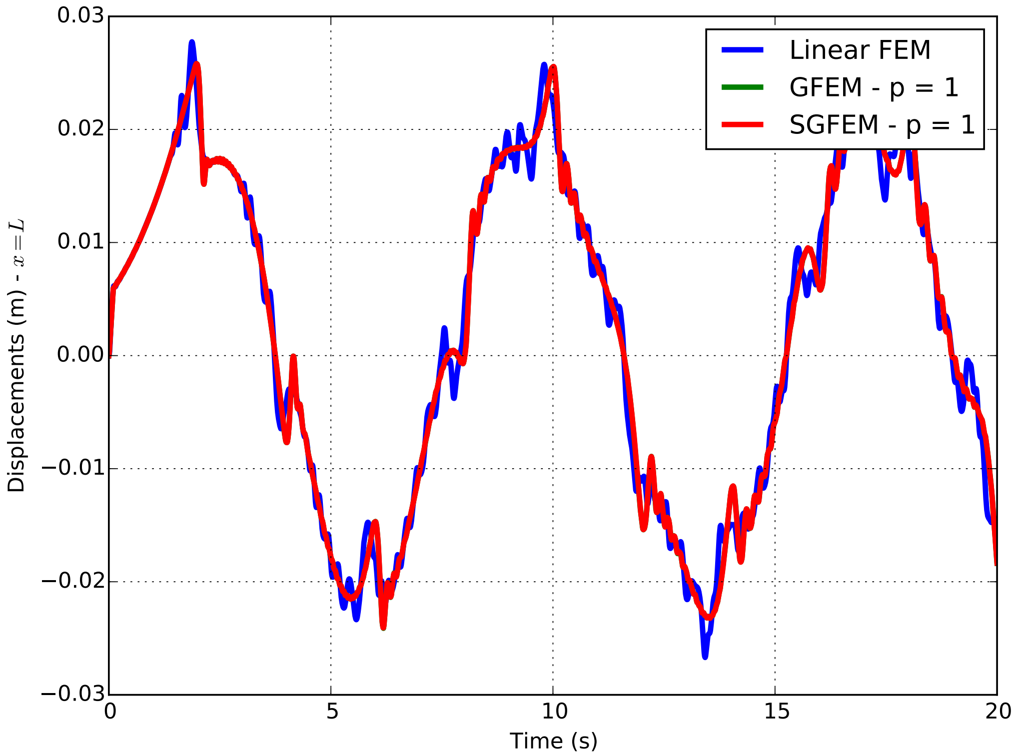

Figure 4: Displacements over time.

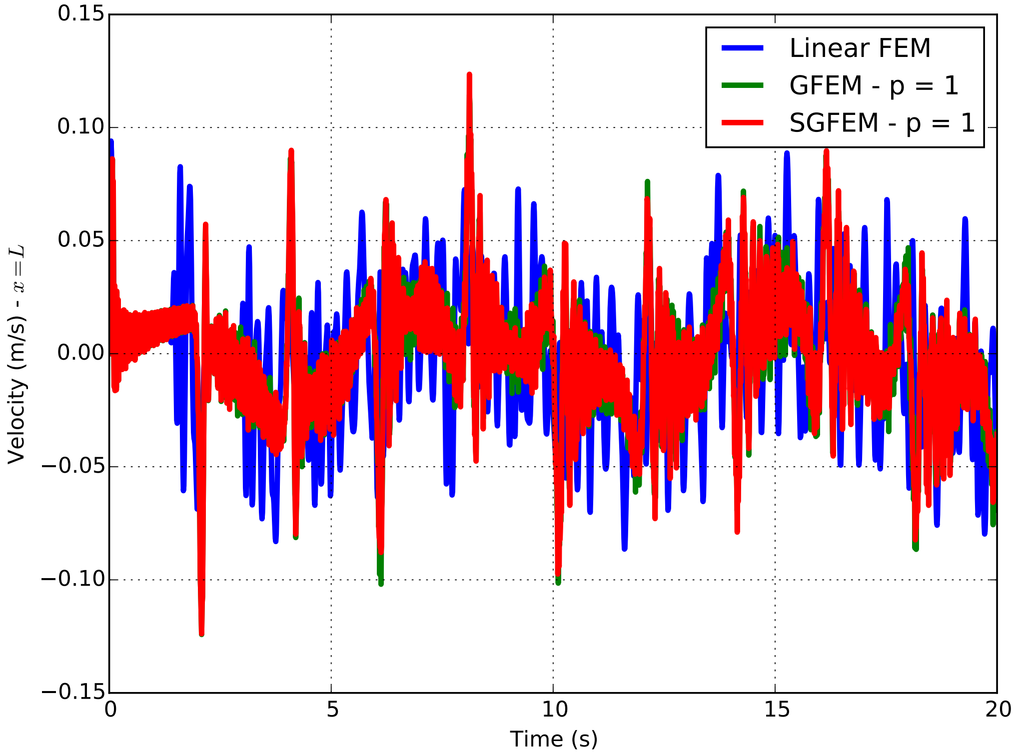

Figure 5: Velocities over time.

Raising enrichment level, the results obtained are presented in Figs. 6 and 7 for: GFEM; SGFEM; and Heuristically Modified GFEM.

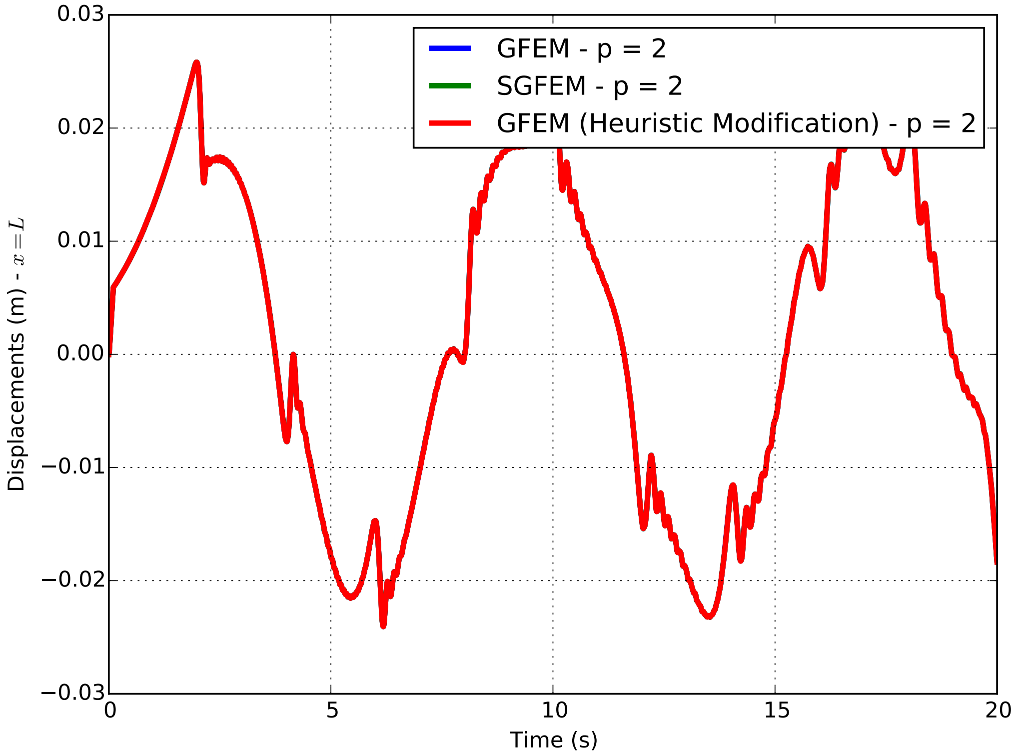

Figure 6: Displacements over time.

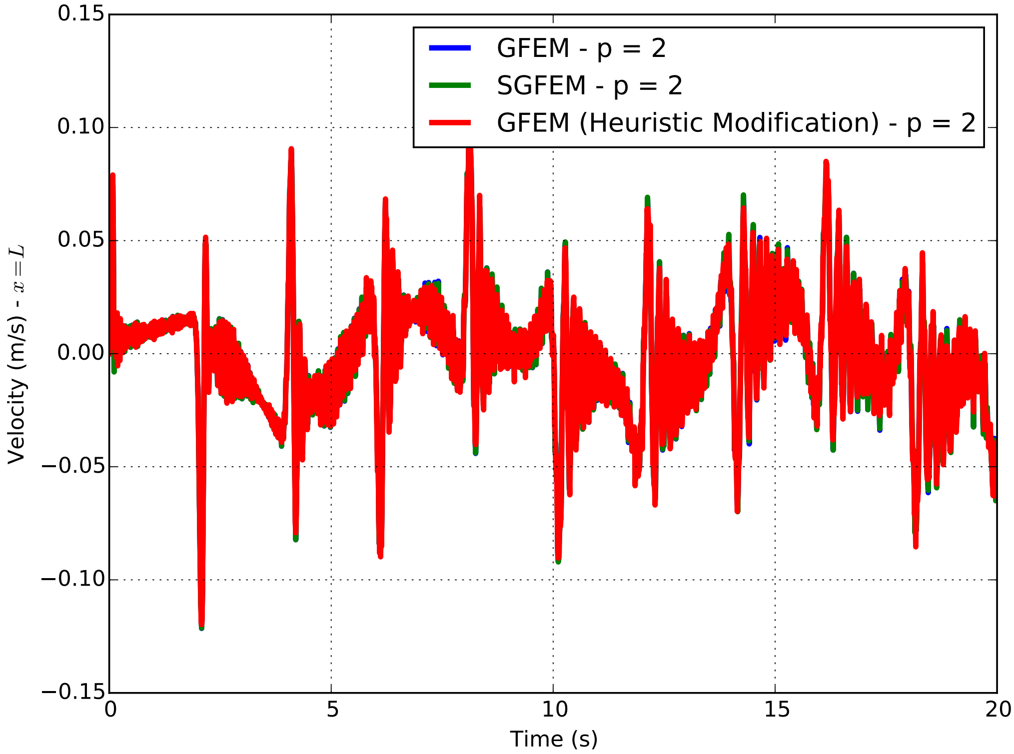

Figure 7: Velocities over time.

Continuing the process of successive enrichment, Figs. 8 and 9 present the respective results.

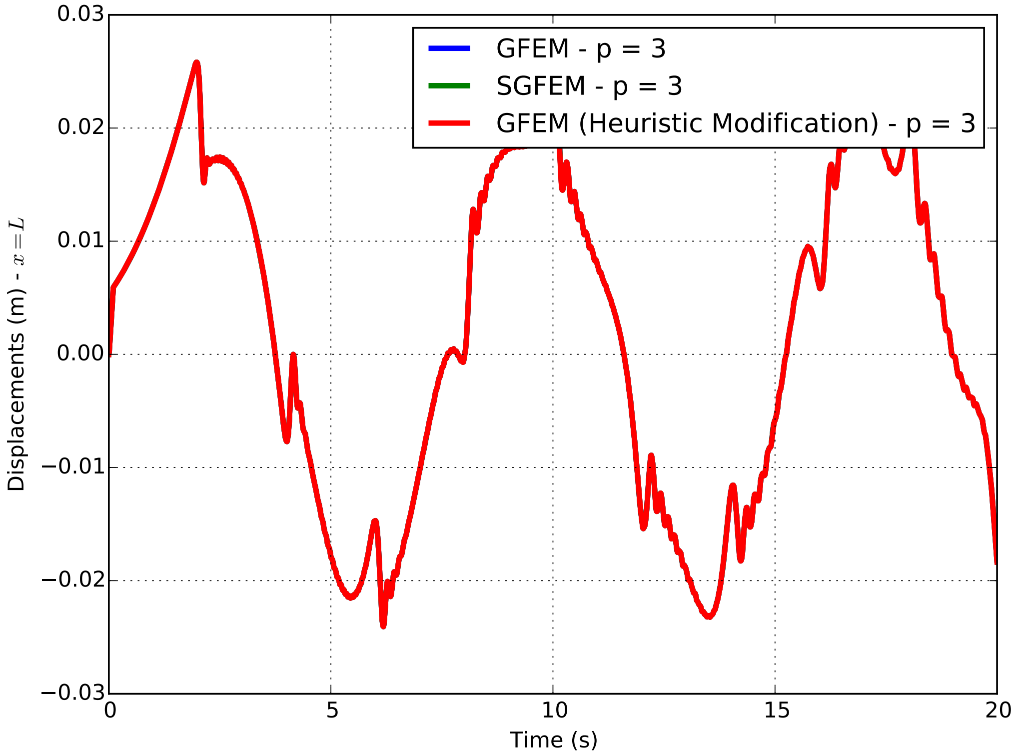

Figure 8: Displacements over time.

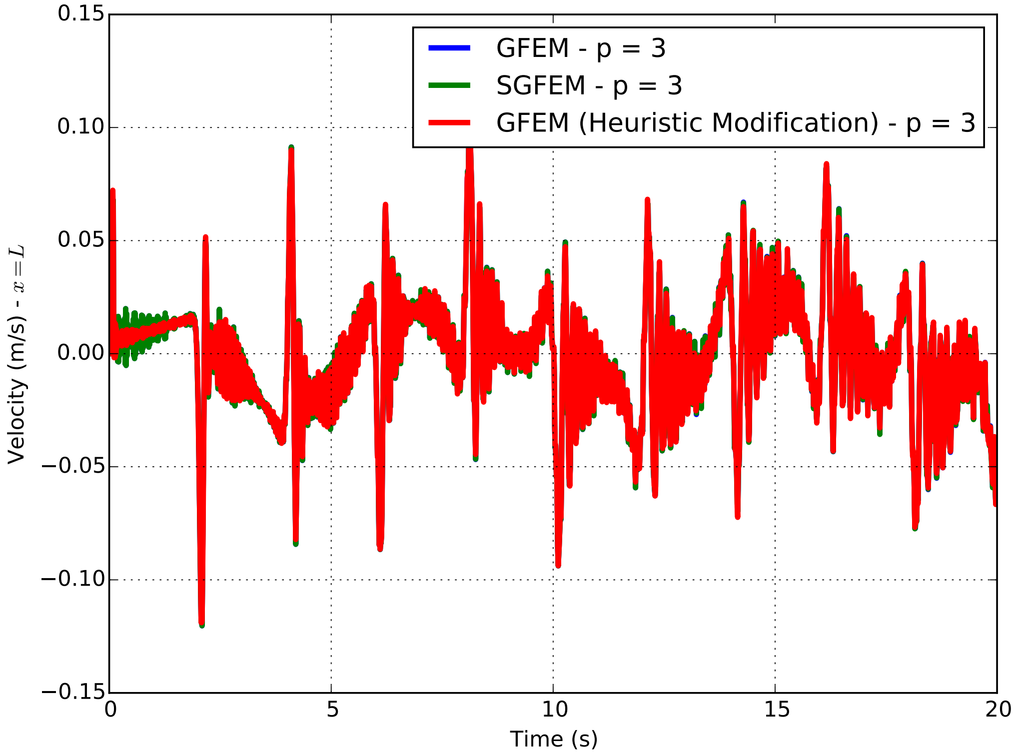

Figure 9: Velocities over time.

For 4 enrichment level results are shown in Figs. 10 and 11.

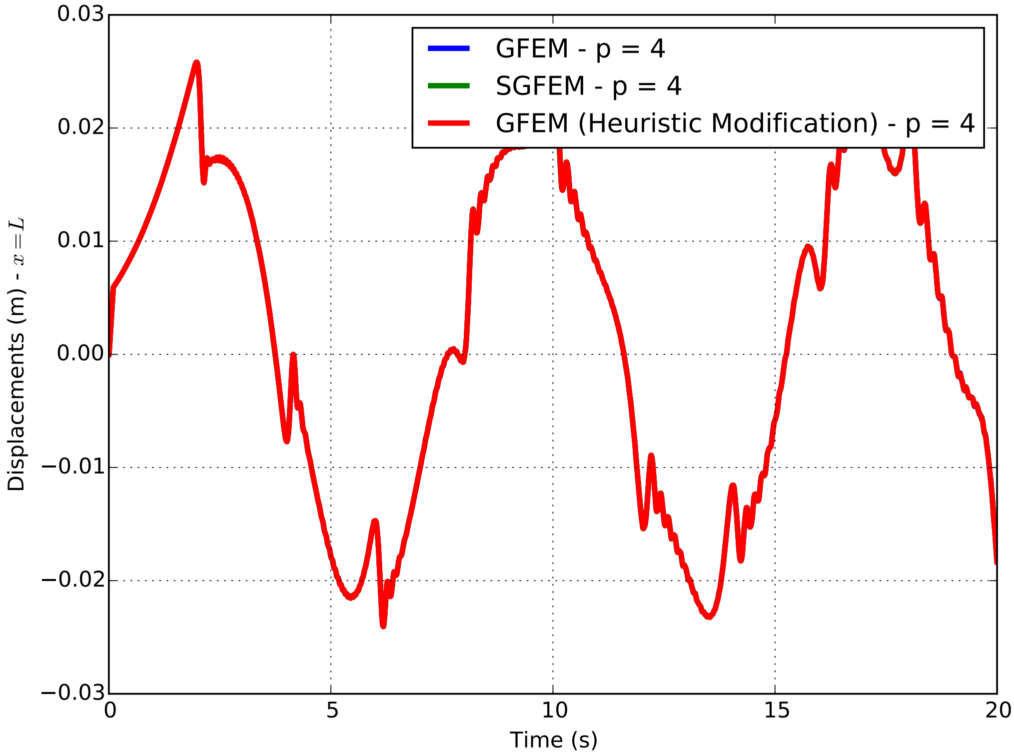

Figure 10: Displacements over time.

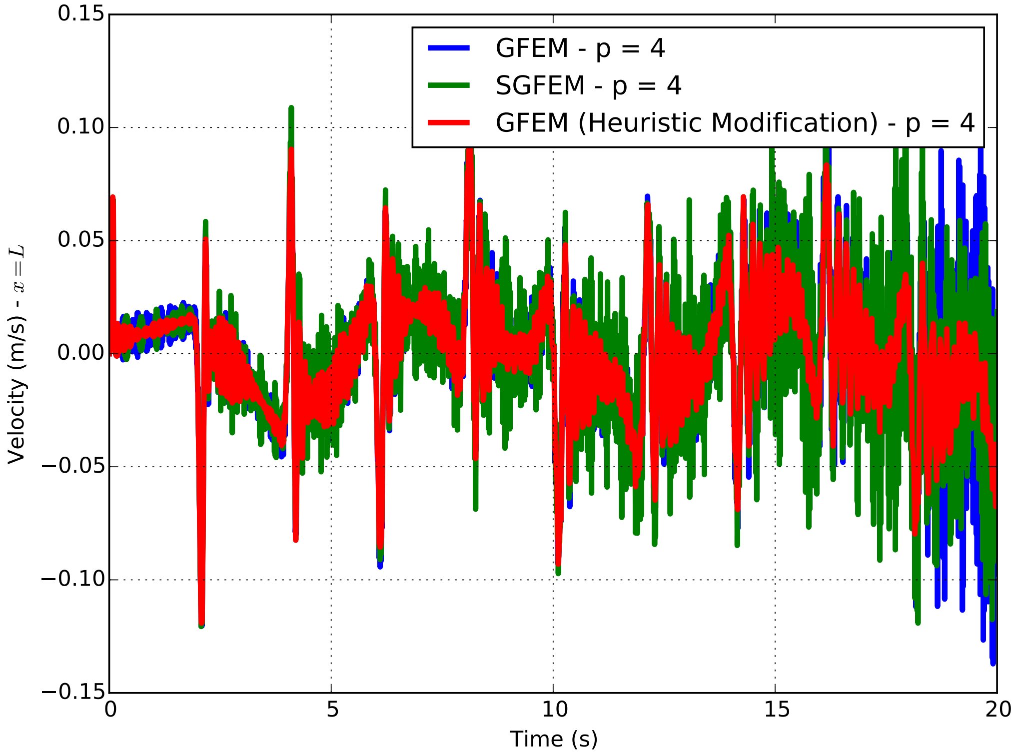

Figure 11: Velocities over time.

Results aiming to highlight the stabilization proposal are presentend in Fig. 12 and Fig. 13, taking up to p-refinement to high-order.

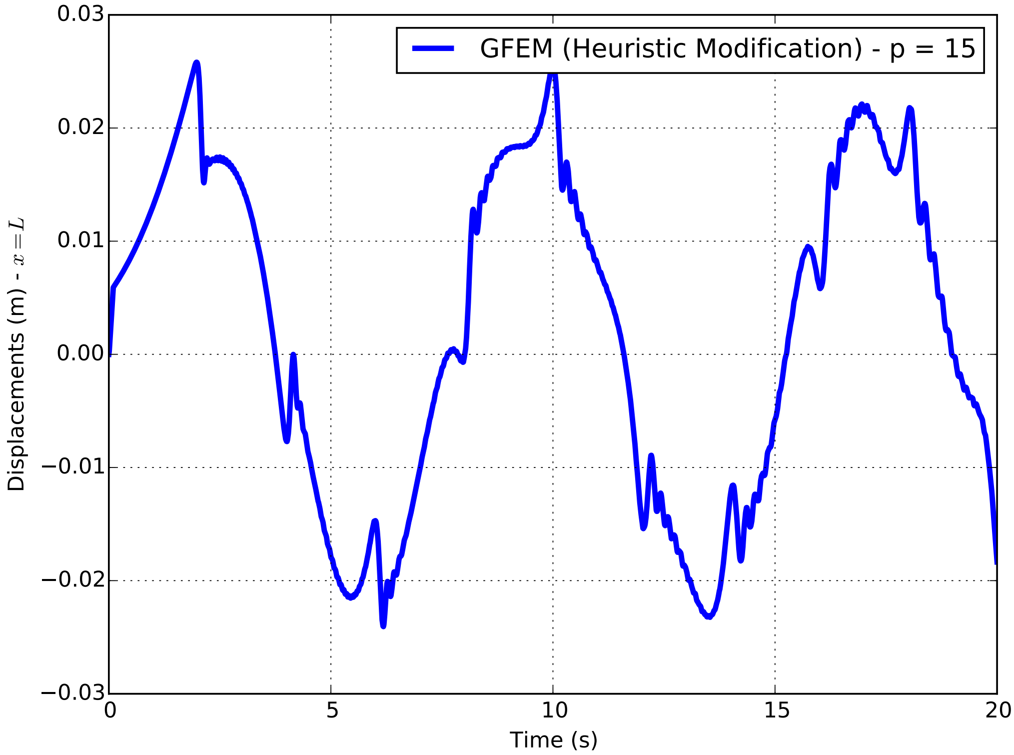

Figure 12: Displacements over time.

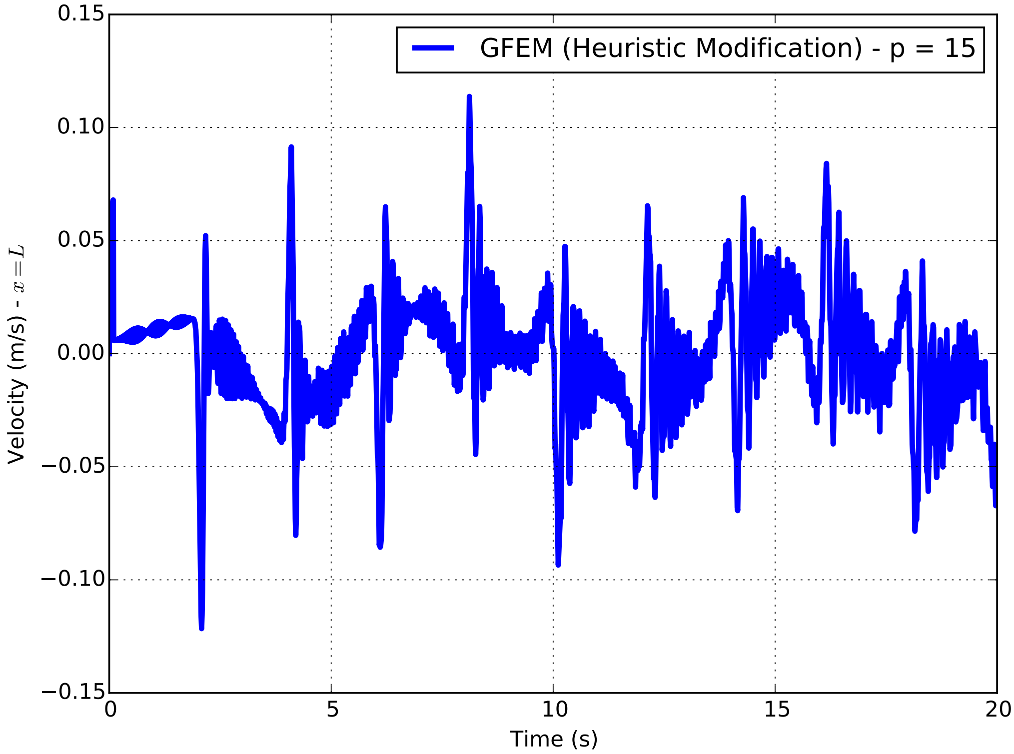

Figure 13: Velocities over time.

The impulse loading case was studied and responses were presented for displacements and velocities. Acceleration was omitted since its poorly approximated using continuous functions. Firstly, the approximation was made by one level of enrichment, concerning GFEM, SGFEM, and FEM. As shown in Fig.4 and Fig.5, displacements response has large oscillations for Linear FEM and a more accurate behavior for alternative enriched. On the other hand, the answer in terms of velocities present great disturbance for the three alternative approaches.

Results for displacements and velocities tend to present more accurate behavior as analytical refinement is performed, as shown in Figs. 6 and 7 for 2 levels and Figs. 8 and 9 for 3 levels of enrichment.

Applying 4 levels of enrichment, it was compared GFEM, SGFEM and the Heuristic Modification, as show in Fig.10 and Fig.11. The corresponding approximations for displacements are considerably close and more accurate than the previous results, and the Heuristic Modification resulted in a fairly accurate approximation for the first time steps. However, the velocities response for SGFEM presented a pretty deteriorated behaviour over time.

Testing the stability of the proposed Heuristic Modification, Fig.12 and Fig.13 show the results of application of 20 levels of enrichment. Apparently that high order p-refinement did not result in significant gains in accuracy for both displacements and velocities. However, it is worth noting that the application that many enrichment levels did not compromised the stability of the numerical approach, following the trend shown in modal analysis as presented by [9].

The Fig. 14 presents the evolution of the condition number of the mass matrix related to a 20-element uniform mesh successively enriched for different stabilization strategies, trying to tie the observations of numerical instabillity in the transient analysis with the condition number of mass matrices.

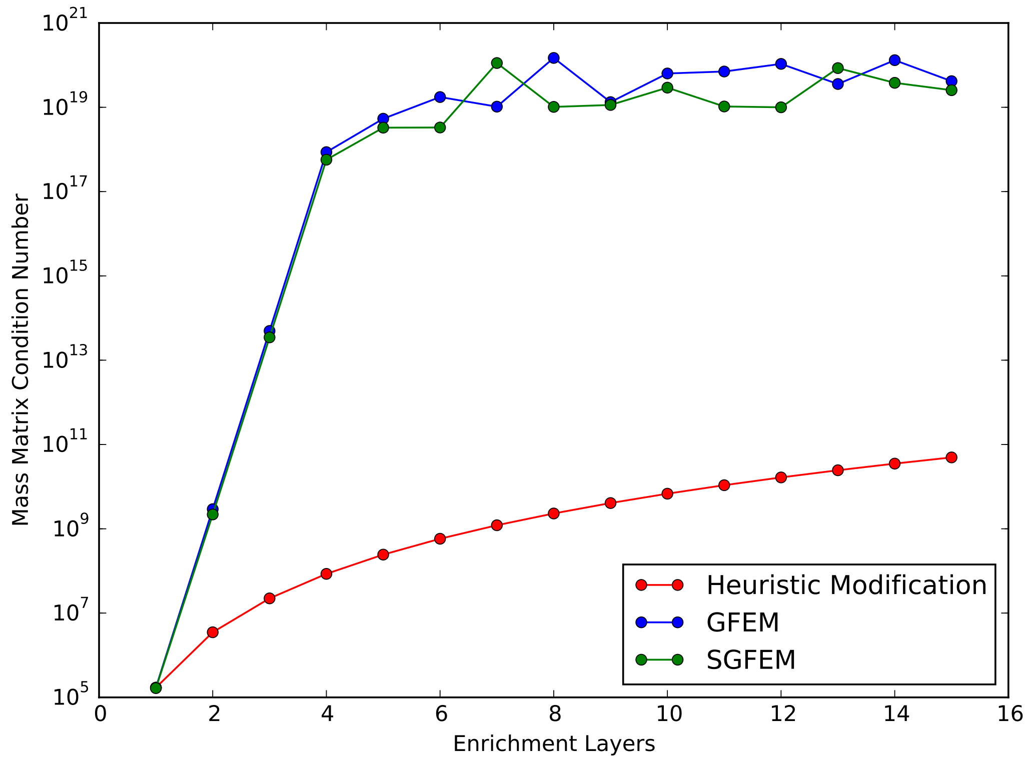

Figure 14: Evolution of Mass Matrix Condition Number (Numerically Calculated).

4. CONCLUDING REMARKS

This paper discussed issues relevant to the stability of the Generalized Method of Finite Elements applied to dynamic analysis. One-dimensional bar with non uniform cross area was presented contemplating modal and transient analysis. GFEM formulation of trigonometric enrichment was based on proposals of [5] and [6].

Seeking to address the stability issue, two stabilization alternatives. The first was based on an adaptation of the Stabilized Generalized Finite Element Method, initially proposed to problems falling within the system of equations for a resolution by [10]. The second proposal was the modification of the parameter β present in GFEM trigonometric functions enrichment proposed by [5] and [6].

For transient analysis it was used the Newmark method with the taking advantage of the mass and stiffness matrices generated by GFEM. Although the results presented inherent disturbances of Newmark method, it was possible to compare the proposals for approximation among each other, and the second proposal stabilization stood out in most examples. The possibility of making high-order return fines with consequent improvement in response without affecting the CFL stability condition (Courant-Friedrichs-Lewy, see [13]) was presented.

The results of this work point out that there are ways to overcome instability problems in GFEM applied to dynamic analysis, since simple proposals were able to positively impact the approaches. Additionally, it was possible to observe that the Heuristic Modification stood out in stabilizing the problem, despite the non-uniform cross section. Therefore, it's possible to suppose that stabilizantion strategies similar to those presented in this paper may be extended to other applications.

5. ACKNOWLEDGEMENTS

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REFERENCES

[1] Craig, R. R. (1981). Structural dynamics: an introduction to computer methods. John Wiley & Sons Inc.

[2] Arndt, M., Machado, R., and Scremin, A. (2010). "An adaptive generalized finite element method applied to free vibration analysis of straight bars and trusses." Journal of Sound and Vibration, 329(6):659–672.

[3] Carey, G. F. and Oden, J. T. (1984). Finite elements: computational aspects, volume 3. Prentice Hall.

[4] Sujith, R. (2010). "Exact solutions for the longitudinal vibration of non-uniform rods: Author's reply." Journal of Sound and Vibration, 329(19):4108.

[5] Arndt, M. (2009). O Método dos Elementos Finitos Generalizados Aplicado à Análise de Vibrações Livres de Estruturas Reticuladas. Tese de doutorado, Universidade Federal do Paraná.

[6] Torii, A. J. (2012). Análise dinâmica de estruturas com o Método dos Elementos Finitos Generalizado. Tese de doutorado, Universidade Federal do Paraná.

[7] Shang, Y. H. (2014). Análise dinâmica elastoplástica de problemas da mecânica de sólidos via métodos enriquecidos de elementos finitos. Tese de doutorado, Pontifícia Universidade Católica do Paraná.

[8] Bathe, K.-J. (1996). Finite element procedures. Prentice-Hall.

[9] Weinhardt, P. O., Rauen, M., Machado, R. D., and Arndt, M. (2015). "Aplicação do Método dos Elementos Finitos, Análise Isogeométrica e Métodos Enriquecidos à Análise Dinâmica 1D." Proceedings of CMN2015, 1(1).

[10] Babuška, I. and Banerjee, U. (2012). "Stable Generalized Finite Element Method (SGFEM)." Computer Methods in Applied Mechanics and Engineering, 201-204:91–111. doi: 10.1016/j.cma.2011.09.012.

[11] Gupta, V., Duarte, C. A., Babuška, I., and Banerjee, U. (2013). "A stable and optimally convergent generalized FEM (SGFEM) for linear elastic fracture mechanics." Computer Methods in Applied Mechanics and Engineering, 266:23–39.

[12] Li, H. (2014). Investigation of stability and accuracy of high order generalized finite element methods. Master's thesis, University of Illinois at Urbana-Champaign.

[13] de Moura, C. A. and Kubrusly, C. S. (2012). The Courant–Friedrichs–Lewy (CFL) Condition: 80 Years After Its Discovery. Springer Science & Business Media, New York.