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STABILITY ANALYSIS OF OLDROYD-B FLUID PIPE FLOW

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Abstract. *Several relevant flows in practice involve viscoelastic fluids, and determining whether these flows occur in laminar or turbulent states is important. Although the hydrodynamics of viscoelastic fluids is significantly influenced by the balance between inertial and elastic forces in the flow, the impact of elasticity on the stability of inertial flows has not yet been fully clarified. In this study, we investigate the stability analysis for the incompressible three-dimensional Hagen-Poiseuille flow in a pipe. We consider a viscoelastic fluid modeled by the linear Oldroyd-B constitutive equation. Linear Stability Theory was used to analyze the stability of these viscoelastic flows against non-stationary disturbances. In the Linear Stability Theory approach, the equations are linearized to incorporate a viscoelastic fluid. Several numerical simulations were carried out, varying the dimensionless parameters of the viscoelastic fluid flow. This study aims to determine the neutral stability curves and investigate how the presence of elasticity alters the laminar-turbulent transition process within this specific tube flow configuration.*

Keywords: *Laminar-Turbulent Transition, Oldroyd-B Fluid, Pipe Flow, Linear Stability Theory*

1. INTRODUCTION

Computational fluid dynamics covers a wide range of industrial applications in sectors such as chemical, food, aeronautical, and oil industries, facing significant challenges in dealing with non-Newtonian fluid flows (Souza *et al.*, 2016). The fluids, which continuously deform when subjected to shear stress, can be classified into categories such as Newtonian and non-Newtonian, the latter being characterized by the absence of a linear relationship between shear stress and strain rate.

Pipe flow stands as a central area of fluid dynamics, boasting a rich history and ongoing research efforts. The enigma lies in its transition to turbulence under real-world conditions. This has spurred the development of the Reynolds number, a paramount parameter for comprehending flow behavior. Despite traditional linear stability theory suggesting that Hagen-Poiseuille flow remains stable Drazin and Reid (1981), Meseguer and Trefethen (2003), the prevalence of turbulence necessitates further in-depth investigations Schmid and Henningson (1994).

According to Schmid and Henningson (2001) laminar flow in Newtonian fluid pipes is known to be linearly stable for any Reynolds number. Pioneering work by Garg *et al.* (2018) revealed the first instance of linear instability in Oldroyd-B viscoelastic fluids. This finding motivates the application of linear stability analysis to Oldroyd-B flow within a pipe. Such analysis provides us with information on how flow disturbances behave, including their potential for growth or decay along the pipe's axis.

This work investigates the hydrodynamic stability through the linear stability theory (LST) for three-dimensional, incompressible Hagen-Poiseuille flow of an Oldroyd-B model viscoelastic fluid. This analysis is particularly relevant for the optimization of the design of pipes and fluid transport systems that involve the use of this type of fluid. In addition, the stability study also helps to better understand the viscoelastic behavior of the Oldroyd-B fluid, contributing to the development of new models and analysis techniques.

2. MATHEMATICAL FORMULATION

The flow is assumed to be unsteady, non-Newtonian, three-dimensional and incompressible. The conservation of mass (continuity) and conservation of momentum equations governing the flow, in the dimensionless form, are given by

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{uu}) = -\nabla p + \frac{1-\beta}{Re}\nabla^2 \mathbf{u} + \nabla \cdot \frac{1}{Wi}P(\mathbf{C}), \tag{2}$$

where $\mathbf{u}$ denotes the velocity field, t is the time, p is the pressure and $\mathbf{C}$ is the non-Newtonian conformation tensor, given by

$$\mathbf{C} = \begin{bmatrix} C^{rr} & C^{r\theta} & C^{rz} \\ C^{\theta r} & C^{\theta\theta} & C^{\theta z} \\ C^{zr} & C^{z\theta} & C^{zz} \end{bmatrix},$$

where $P(\mathbf{C}) = (\mathbf{I} - \mathbf{C})$, in particular, for the Oldroyd-B model.

The dimensionless parameter $Re = \rho U_\infty L/\eta_0$ is the Reynolds number, where L and U_∞ are the length and velocity scales, respectively, ρ is specific mass of the fluid, and η_0 is the dynamic viscosity of the fluid. Moreover, $Wi = \lambda U_\infty/L$ is the Weissenberg number, which represents the ratio between the elastic forces and the viscous forces for a viscoelastic fluid, where λ is the relaxation-time of the fluid. The amount of Newtonian solvent is controlled by the dimensionless solvent viscosity coefficient $\beta = \eta_s/\eta_0$, where $\eta_0 = \eta_s + \eta_p$ denotes the total shear viscosity, where η_s and η_p the Newtonian solvent and polymeric viscosities, respectively. In this paper, we worked with viscoelastic fluid flow governed by the non-linear Oldroyd-B constitutive equation Petrucci Orefice *et al.* (2013), which can be given by:

$$\frac{\partial \mathbf{C}}{\partial t} + \nabla \cdot (\mathbf{uC}) = (\nabla \mathbf{u})\mathbf{C} \ + \mathbf{C}(\nabla \mathbf{u})^T + \frac{1}{Wi}P(\mathbf{C}). \tag{3}$$

3. LINEAR STABILITY THEORY

The linear stability theory analyzes the behaviour of a flow to disturbances of infinitesimal amplitude. The linear stability analysis conducted in this study assumed that the base flow remains unchanged in the z-direction. In other words,

$$u = 0, \quad v = 0, \quad w = W(r), \quad p = P(z), \quad \mathbf{C} = \hat{\mathbf{C}}(r).$$

Given that the instantaneous flow can be decomposed into a base flow and a disturbance flow, the dependent variables can be decomposed as follows Souza *et al.* (2016)

$$\mathbf{u} = \mathbf{U} + \tilde{\mathbf{u}}, \quad p = P + \tilde{p}, \quad \mathbf{C} = \hat{\mathbf{C}} + \tilde{\mathbf{C}}.$$

The disturbances can be written generally as given by

$$\tilde{\mathbf{u}}(r,\theta,z,t) = \overline{\mathbf{u}}(r)e^{i(m\theta+\alpha z-\omega t)}, \quad \tilde{p}(r,\theta,z,t) = \overline{p}(r)e^{i(m\theta+\alpha z-\omega t)}, \quad \tilde{\mathbf{C}}(r,\theta,z,t) \quad = \overline{\mathbf{C}}(r)e^{i(m\theta+\alpha z-\omega t)}.$$

Note that the value of $i = \sqrt{-1}$, $\overline{u} = (\overline{u}, \overline{v}, \overline{w})$ and $m \in \mathbb{R}$. In this case, we can observe that the disturbances propagate as waves with frequency ω, wavelength $\lambda_c = 2\pi/\alpha$, and wave velocity $c = \omega/\alpha$, where α is the wave number in the z direction and the amplitudes of the disturbances are represented by $\overline{u}$, $\overline{v}$, $\overline{w}$, $\overline{p}$, and $\overline{\mathbf{C}}$ (Souza *et al.*, 2016). Employing the separation of variables method with normal modes, the system comprised of conservation equations and non-Newtonian tensor equations can be transformed into a system of ordinary differential equations as given below

$$\left(\frac{1}{r} + \frac{d}{dr}\right)\overline{u}(r) + i\frac{m}{r}\overline{v}(r) + i\alpha\overline{w}(r) = 0, \tag{4}$$

$$\begin{aligned} -i\omega\overline{u} + i\alpha W\overline{u} + \frac{d\overline{p}}{dr} = \frac{\beta}{Re}\left(-\frac{\overline{u}}{r^2} + \frac{1}{r}\frac{d\overline{u}}{dr} + \frac{d^2\overline{u}}{dr^2} - \frac{m^2}{r^2}\overline{u} - \alpha^2\overline{u} + 2i\frac{m}{r^2}\overline{v}\right) + \\ + \frac{1-\beta}{ReWi}\left[\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)\overline{C}^{rr} + i\frac{m}{r}\overline{C}^{r\theta} + i\alpha\overline{C}^{rz} - \frac{\overline{C}^{\theta\theta}}{r}\right], \end{aligned} \tag{5}$$

$$-i\omega\overline{v} + i\frac{m}{r}\overline{p} + i\alpha W\overline{v} = \frac{\beta}{Re}\left(-\frac{\overline{v}}{r^2} + \frac{1}{r}\frac{d\overline{v}}{dr} + \frac{d^2\overline{v}}{dr^2} - \frac{m^2}{r^2}\overline{v} - \alpha^2\overline{v} + 2i\frac{m}{r^2}\overline{u}\right) + \\ + \frac{1-\beta}{ReWi}\left[\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)\overline{C}^{r\theta} + i\frac{m}{r}\overline{C}^{\theta\theta} + i\alpha\overline{C}^{\theta z}\right], \tag{6}$$

$$-i\omega\overline{w} + W'\overline{u} + i\alpha W\overline{w} + i\alpha\overline{p} = \frac{\beta}{Re}\left(\frac{d^2\overline{w}}{dr^2} + \frac{1}{r}\frac{d\overline{w}}{dr} - \frac{m^2}{r^2}\overline{w} - \alpha^2\overline{w}\right) + \\ + \frac{1-\beta}{ReWi}\left[\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)\overline{C}^{rz} + i\frac{m}{r}\overline{C}^{\theta z} + i\alpha\overline{C}^{zz}\right], \tag{7}$$

$$-i\omega\overline{C}^{rr} = 2\left(\frac{d}{dr} + iWiW'\alpha\right)\tilde{u} - \left(iW\alpha + \frac{1}{Wi}\right)\overline{C}^{rr}, \tag{8}$$

$$-i\omega\overline{C}^{r\theta} = i\frac{1}{r}m\overline{u} + \left(\frac{d}{dr} - \frac{1}{r} + iWiW'\alpha\right)\overline{v} - \left(iW\alpha + \frac{1}{Wi}\right)\overline{C}^{r\theta}, \tag{9}$$

$$-i\omega\overline{C}^{rz} = \left\{WiW'\frac{d}{dr} - Wi\frac{d^2W}{dr^2} + i\left[1 + 2Wi^2W'^2\right]\alpha\right\}\overline{u} + \\ + \left(\frac{d}{dr} + iWiW\alpha\right)\overline{w} + W'\overline{C}^{rr} - \left(iW\alpha + \frac{1}{Wi}\right)\overline{C}^{rz}, \tag{10}$$

$$-i\omega\overline{C}^{\theta\theta} = \frac{2}{r}\tilde{u} + i\frac{2}{r}m\overline{u} - \left(iW\alpha + \frac{1}{Wi}\right)\overline{C}^{\theta\theta}, \tag{11}$$

$$-i\omega\overline{C}^{\theta z} = \left\{WiW'\frac{d}{dr} - \frac{Wi}{r}W' + i\left[1 + 2(Wi)^2(W')^2\right]\alpha\right\}\overline{v} + i\frac{1}{r}m\overline{w} + W'\overline{C}^{r\theta}, \tag{12}$$

$$-i\omega\tilde{C}^{zz} = -4(Wi)^2W'W''\overline{u} + 2\left\{WiW'\frac{d}{dr} + i\left[1 + 2(Wi)^2(W')^2\right]\alpha\right\}\overline{w} + \\ + 2W'\overline{C}^{rz} - \left(iW\alpha + \frac{1}{Wi}\right)\overline{C}^{zz}. \tag{13}$$

3.1 Numerical Method

The stability analysis is performed by obtaining the solution of the system of equations (4) – (13). The system of equations is written in matrix form, and the stability analysis problem becomes an eigenvalue/eigenvector problem. In the present work, the temporal analysis (ω is complex and α is real) of the disturbances was performed where the growth rate ω_i is analyzed. Rewriting the system of equations in matrix form as

$$L\mathbf{v} = \omega F\mathbf{v}, \tag{14}$$

for the eigenvector **v**,

$$\mathbf{v} = [\overline{u} \quad \overline{v} \quad \overline{w} \quad \overline{p} \quad \overline{C}^{rr} \quad \overline{C}^{r\theta} \quad \overline{C}^{rz} \quad \overline{C}^{\theta\theta} \quad \overline{C}^{\theta z} \quad \overline{C}^{zz}]^T. \tag{15}$$

It is possible to solve the stability analysis problem by finding the eigenvalue ω, using a method for calculating the eigenvalue. When the wave number elements are real numbers and the frequency $\omega = \omega_r + i\omega_i$ is a complex number, it is called temporal analysis. The real part ω_r represents the frequency of the perturbation, while the imaginary part ω_i represents the amplification rate. The classification of instabilities, on the temporal analyses is presented in Tab. 1.

In this work, the temporal analysis is performed considering the system composed by the conservation Eqs. (4) – (13).

Table 1. Instabilities classification.

Type of analysis	Amplification Rate	Amplitude	Classification
	$\omega_i < 0$	decreases	stable
Temporal analysis	$\omega_i = 0$	constant	neutral
	$\omega_i > 0$	increase	unstable

3.2 Base Flow

The base flow is known as laminar or fully developed. To calculate the flow in the pipe, it is first assumed that all variables are dependent only on the axis r, except for the pressure whose gradient is constant in the z-direction. The domain in the r direction is understood to be between $[0, 1]$. According to Batchelor (1967), the velocity profile is given by:

$$U = V = 0, \quad W(r) = 1 - r^2. \tag{16}$$

In addition, the base flow of the Oldroyd-B constitutive model in a pipe according Petrucci Orefice *et al.* (2013) is given by

$$C^{rr} = C^{\theta\theta} = 1, \quad C^{r\theta} = C^{\theta z} = 0, \quad C^{rz} = WiW' \quad C^{zz} = 1 + 2Wi^2(W')^2, \tag{17}$$

where the symbol prime $'$ denotes the derivative with respect to r.

3.3 Boundary Conditions

The boundary conditions adopted for the Hagen-Poiseuille flow problem are given by

$$r = 1: \quad \tilde{u} = \tilde{v} = \tilde{w} = 0. \tag{18}$$

These boundary conditions correspond to the no-slip condition at the wall and the decay of disturbances away from the wall.

In addition, according to Georgievskii (2015), in order to avoid the formation of vortices at the axis of the pipe, the following conditions must be imposed

$$r = 0: \quad \frac{\partial \tilde{u}}{\partial \theta} = 0, \quad \tilde{v} = 0, \quad \frac{\partial \tilde{w}}{\partial \theta} = 0. \tag{19}$$

Moreover, in this work we impose that the flow is axisymmetric, that is, we take $m = 0$. Therefore, we have that

$$C^{r\theta} = C^{\theta z} = 0. \tag{20}$$

4. NUMERICAL RESULTS

Stability analysis results of a three-dimensional pipe flow for an Oldroyd-B viscoelastic fluid using Linear Stability Theory are presented. In this analysis, ω is considered complex and α is real, as shown in Table 1. In order to evaluate the neutral stability curves, different numerical simulations were performed by varying the dimensionless parameters for the Oldroyd-B fluid flow. The temporal analysis was performed by constructing stability curves for each set of parameters. The parameters that varied in this study were the Reynolds number (Re), the Weissenberg number (Wi), and the β constant.

For each simulation, it is possible to determine a neutral stability curve, where the values of $\omega_i = 0$. The neutral stability curves are presented with the Reynolds number on the x-axis and the perturbation frequency α on the y-axis. These diagrams exhibit a characteristic banana shape. Amplification rates greater than zero ($\omega_i < 0$), or the stable region, are outside the banana, while amplification rates less than zero ($\omega_i > 0$), or the unstable region, are inside the banana.

To investigate the influence of the solvent polymeric contribution (β) on the stability of the Hagen-Poiseuille flow of Oldroyd-B fluid, numerical simulations were performed considering the parameters $\beta = 0.3, 0.5,$ and 0.65, and $Wi = 30, 40, 50,$ and 65. Using the LST code developed for the Oldroyd-B model, the neutral stability curves were obtained for each case and are presented in Figs. 1, 2 and 3.

The neutral curves in Fig. 3 exhibit an increase in the region of instability compared to those in Figs. 1 and 2 for the same values of Wi, as the β parameter increases. This demonstrates the influence of β on the stability of the Hagen-Poiseuille flow of the Oldroyd-B fluid in pipe.

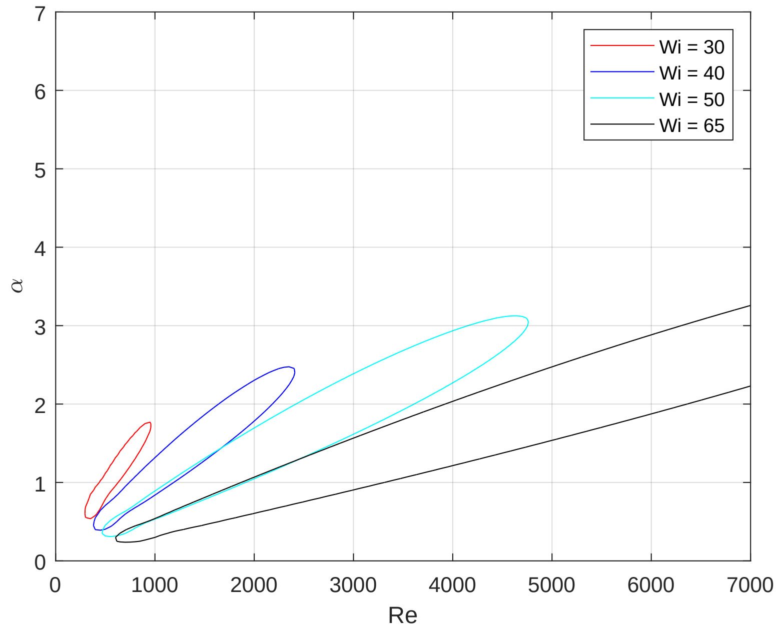

Figure 1. Neutral curves for parameter $\beta = 0.3$ and $Wi = 30, 40, 50$ and 65.

Nevertheless, it is observed that for the same values of Wi, as β increases, the neutral curves tend to get closer together, with the region of instability decreasing as the Reynolds number decreases.

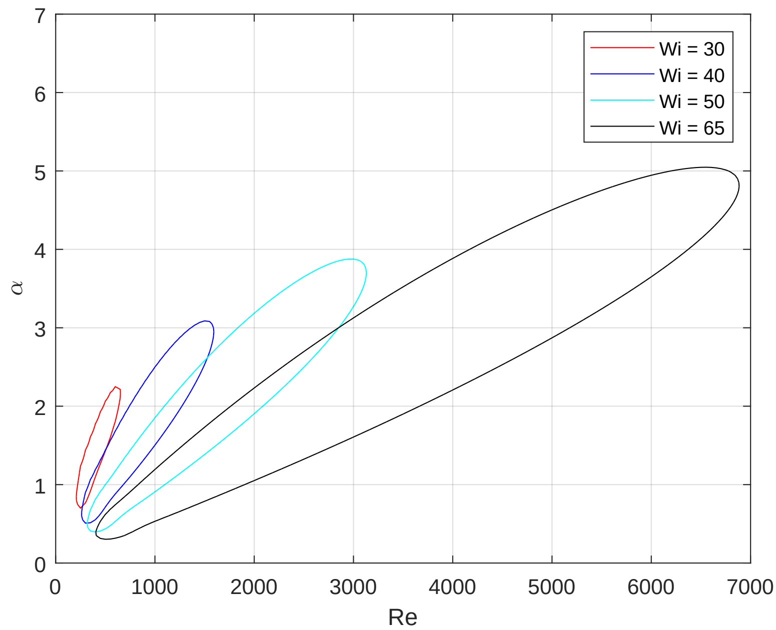

Figure 2. Neutral curves for parameter $\beta = 0.5$ and $Wi = 30, 40, 50$ and 65.

In particular, for the simulation of $\beta = 0.65$ and $Wi = 30, 40, 50$, and 65 the results shown in the Fig. 3 are consistent with those presented in the works of Chaudhary *et al.* (2021)and Wan *et al.* (2021). This demonstrates the robustness of the LST code developed for the analysis of Oldroyd-B fluid in a pipe.

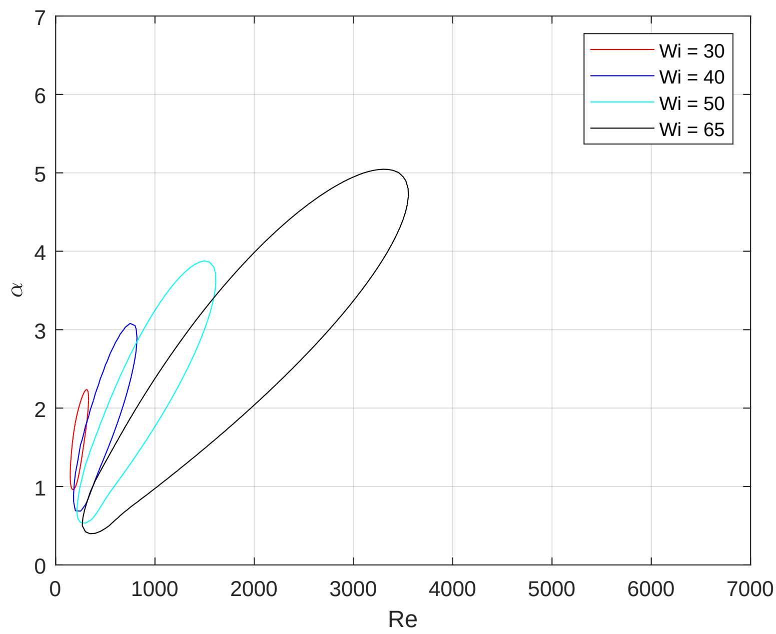

Figure 3. Neutral curves for parameter $\beta = 0.65$ and $Wi = 30, 40, 50$ and 65.

Note in the Fig. 4 that when Wi is fixed and β varies in the values of $\beta = 0.3, 0.5, 0.65$ and 0.9, the neutral curves begin to “flatten”. In other words, the unstable region decreases considerably as β increases.

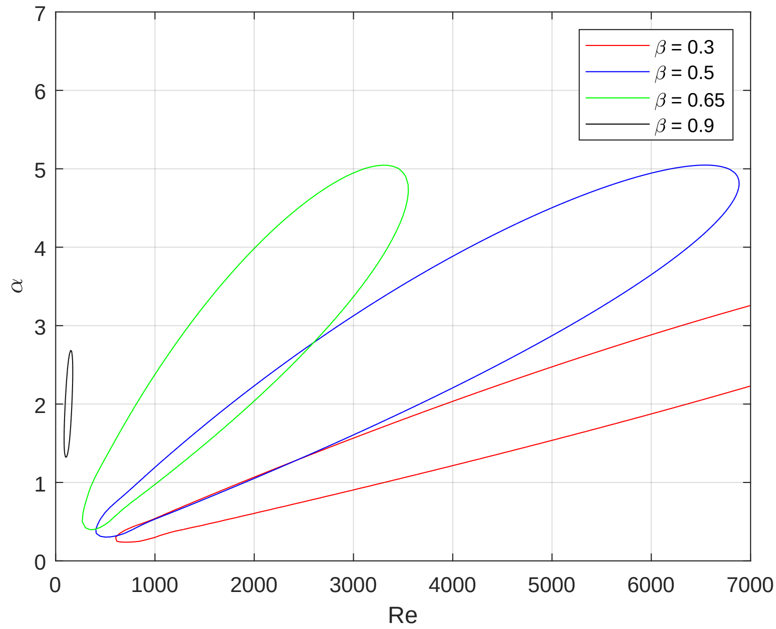

Figure 4. Neutral curves for parameter $Wi = 65$ and $\beta = 0.3, 0.5, 0.65$ and 0.9.

5. CONCLUSIONS

In this work, the equations for modeling incompressible, isothermal, three-dimensional flows for a non-Newtonian viscoelastic fluid were presented, using the constitutive equation of the Oldroyd-B model. The equations for modeling flows for a non-Newtonian viscoelastic fluid were presented in their dimensionless form. Moreover, only temporal analysis was used to investigate the stability of viscoelastic fluid flows using Linear Stability Theory, through neutral stability curves. These were presented for different dimensionless parameters of the Oldroyd-B model. The neutral stability curves were evaluated using two-dimensional perturbations for different values of the model's dimensionless parameters. In addition, the influence of the dimensionless parameters present in the Oldroyd-B model, such as β, Wi were studied, considering Reynolds numbers from $Re = 50$ to $Re = 7000$ and $\alpha = 0$ to $\alpha = 7$.

The effect of elastic forces, given by the fixed Weissenberg number (Wi), on the flow stability was simulated for four different values of Weissenberg, namely $Wi = 30, 40, 50$, and 65 for different values of β. The numerical results showed that as the value of Wi increases, there is a shift of the neutral curve to the left, indicating that the flow becomes unstable for smaller Reynolds number values.

The influence of the constant β, which controls the contribution of the Newtonian solvent polymer in the fluid, was also studied to analyze the stability of the Hagen-Poiseuille flow of Oldroyd-B fluid. Numerical simulations were performed considering seven different values of $\beta = 0.3, 0.5, 0.65$ and 0.9, for fixed Weissenberg number $Wi = 65$. The numerical results showed that the neutral stability curves of the Oldroyd-B model as the value of β increases, the value of α decreases. The numerical results showed that the neutral stability curves of the Oldroyd-B model were flattening and becoming smaller and smaller.

The numerical results obtained using the LST technique proved to be quite acceptable and suitable for the analysis of viscoelastic flow stability, considering the scarcity of literature on the viscoelastic Oldroyd-B fluid model in a pipe. This constitutes the main scientific contribution of this work, providing up-to-date results derived from a robust numerical tool for assessing the stability of three-dimensional flows involving Oldroyd-B fluid.

6. ACKNOWLEDGEMENTS

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