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Numerical Stability of GFEM Evaluation for Free Vibration Analysis in Trussed Structures

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ABSTRACT

Since 1960s, Finite Element Methods (FEM) has been studied with the purpose of solving problems in structural analysis. However, there are some limitations of the method when these problems involve discontinuities for example. In order to overcome these difficulties, the Generalized Finite Element Method (GFEM) was developed. In the specific case of dynamic analysis of structures, when applied numerical methods for the solution, the free vibration problems, fall back a problem of generalized eigenvalues and eigenvectors. Several numerical methods have been developed in recent decades to solve these problems, among which the GFEM stand out for the high convergence rates presented. However, it is observed that the solution of the GFEM is very sensitive to the precision used in numerical algorithms. Analytically, restricted matrices of stiffness (K) and mass (M) are positive defined, so the eigenvalue problem is stable and should not present sensitivity issues. In this context, the objective of this work is investigate possible causes of the observed numerical instability and to establish parameters that allow to evaluate the numerical stability of different version of the GFEM for analysis of free vibration unidimensional bar elements. The numerical results obtained indicate that the condition number of the mass matrix can be used to evaluate the numerical stability of the GFEM in free vibration analysis.

Keywords: Stability; Generalized Eigenvalues and Eigenvectors Problems; Dynamic Analysis, Generalized Finite Element Method

1. INTRODUCTION

The analysis of mathematical models requires the use of numerical methods, which include the Finite Element Method (FEM) and its variations. These methods allow us to find the solutions of differential equation systems. The Generalized Finite Element Method (GFEM), a variation of FEM, was developed by [1]. It is based on the concept of unit partition from which the base of local approximation subspace consists of functions, not necessarily polynomial, that reflect a priori available information on the solution of the governing differential equation [2].

Since the GFEM has a flexibility in the choice of approximation functions, trigonometric functions has been applied in the GFEM dynamic analysis of trussed structures. Trigonometric shape functions were also applied in Enriched Methods for example in the works of [3], [4], among others.

As FEM and GFEM are approximated methods, approximation errors are found in the construction of mass and stiffness matrix. The work of [5], even presenting good results in the free vibration analysis of structures by GFEM, reports problems of solution sensitivity. It was observed that precision changes in the computational calculations for the determination of the matrices of mass and stiffness by numerical integration and solution of the problem of eigenvalues affect the accuracy and convergence of the method.

[6] present without justification or mathematical proof, the use of the condition number of the mass matrix as a measure of this sensitivity when comparing the Finite Element Method p-Fourier and the p-Hierarchical FEM.

By definition, we have that the condition number is the quotient between the largest and the smallest eigenvalues [7], so it can be said that it depends directly on the eigenvalues of the problem, and can serve as a measure of the conditioning of a matrix.

As already mentioned the objective of this work is to perform an analysis of the numerical stability of the generalized eigenvalues problem, generated by the equation $Ax = \lambda Bx$, of the GFEM. It is assumed that if a matrix is Hermitian, and receives a perturbation, then there will be an increase in its eigenvalues, in other words, the perturbation caused in the matrix influences the condition number [8]. Therefore, the analysis of the approximations of mass and stiffness matrices is an important topic to be studied in order to discover until where the perturbations generated by the approximations of the method of construction of these matrices influence the performance and precision of the methods for the dynamic analysis of structures.

2. Problem of Generalized Eigenvalues with Hermitian Matrices

A generalized eigenvalue problem is defined by the equation:

$$P(\lambda)x = (A - \lambda B)x = 0 \Longleftrightarrow Ax = \lambda Bx \tag{1}$$

If the matrix $B = I$, identity matrix, the generalized problem is called *standard eigenvalue problem*, or *standard problem*.

For this work the following definitions are necessary:

- An matrix A, is called hermitian, if $A = (\overline{A})^T$, where $\overline{A}$ is the conjugate matrix of A.
- Be λ_i's matrix eigenvalues of A, so:
  - If $\lambda_i > 0\ \forall i$, so A is called *positive definite*;
  - If $\lambda_i \geq 0\ \forall i$, so A is *positive semidefinite*;
  - If $\lambda_i < 0\ \forall i$, so A is *negative definite*;
  - If $\lambda_i \leq 0\ \forall i$, so A is *negative semidefinite*;

– The matrix A is *indefinite* when there are positive and negative eigenvalues.

The concept of conditioning can have several definitions however the one used in this work is defined through the sensitivity analysis of the solution of the eigenvalue problem to small variations in the input data.

It is said that the problem is *well conditioned* if small disturbances in the input data result in small variations in the output data. Otherwise, when small perturbations in the input data cause major disturbances in the output data, we have a *ill conditioned* problem. This definition in some areas of Computational Mechanics is also known as stability of the method, so a problem is stable when it is well conditioned, and otherwise unstable.

It is known that a generalized eigenvalue problem, as defined in (1), can be well conditioned or ill conditioned. And when the A matrix is Hermitian and this is a standard problem, then this is a well conditioned problem [9]. But when it is a generalized problem, it is usually ill conditioned, especially when the matrices are not hermitian [10].

Dealing with hermitian and definite positive matrices has some advantages, one of which is the existence of algorithms that make it easy to find the eigenvalues of the matrices, in order to have a measure of conditioning. Another advantage is that a generalized eigenvalue problem can be transformed into a standard problem. If the matrices A and B are hermitian and B is positive definite, then B can be decomposed as the product of non-singular matrices, $B = LL^*$, through the Cholesky Decomposition, for example. So the problem becomes:

$$(L^{-1}A(L^*)^{-1})\bar{x} = \lambda \bar{x} \tag{2}$$

Since the domain spaces are invariant [11], the properties are preserved after the transformation, that is, the eigenvalues are the same, and if $\bar{x}$ is eigenvector at (2), then $x = (L^*)^{-1}\bar{x}$ satisfies the equation (1). Therefore, when a generalized eigenvalue problem has the matrices A and B hermitian and B is positive definite, then this is a well conditioned problem because it can be reduced to a standard problem with the same eigenvalues and in turn is well conditioned

3. Dynamic Analysis

The analysis of free vibrations in non-damped structures became on the problem [12]:

$$K\phi = \omega^2 M\phi \tag{3}$$

where K is stiffness matrix, M the mass matrix, ω the natural frequency and ϕ the natural vibration mode vector.

The matrices K and M when derived from Galerkin's weak form concerning the dynamic equilibrium of the system for bar vibrations are given in the form:

$$K = [k_{ij}] = EA \int_\Omega \frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi_j}{\partial x} d\Omega \tag{4}$$

$$M = [m_{ij}] = \rho A \int_\Omega \Phi_i \Phi_j d\Omega \tag{5}$$

Where Φ's are interpolation functions, A is the cross section area, E is the elasticity modulus, ρ is the density, and Ω is the global domain of the problem. The choice of interpolation functions depends on the approximate method to be employed.

3.1 Enriched Methods

For this work, it is considered a uniform straight bar element composed of two nodes and one degree of freedom per node. Making use of conventional FEM shape functions, by adding other enrichment functions, the approximate solution in the domain can be defined as:

$$u = u_{FEM} + u_{ENRICH} = \mathbf{N}^T \mathbf{q} + \Phi^T \mathbf{c} \tag{6}$$

where

$$\begin{aligned} \mathbf{N}^T &= [\ \psi_1 \quad \psi_2\ ] \\ \mathbf{q}^T &= [\ u_1 \quad u_2\ ] \\ \Phi^T &= [\ F_1 \quad F_2 \quad \dots \quad F_r \quad \dots \quad F_n\ ] \\ \mathbf{c}^T &= [\ c_1 \quad c_2 \quad \dots \quad c_n\ ] \end{aligned} \tag{7}$$

where **q** corresponds to the nodal degrees of freedom vector of the conventional FEM element, **N** the vector containing the linear shape functions of the FEM, Φ the vector of the enrichment functions, and **c** the vector of field degrees of freedom.

3.1.1 p-Hierarchical FEM - Lobatto Polynomials

The shape functions based in Lobatto polynomials are obtained from the integration of the Legendre polynomials and can be found in more detail in [13]. In the case of a domain $\Omega = [-1, 1]$, the polynomials of Lobatto have the following form:

$$\begin{aligned} l_0(x) &= \frac{1-x}{2}; \\ l_1(x) &= \frac{1+x}{2}; \\ l_k(x) &= \frac{1}{\|L_{k-1}\|_2} \int_{-1}^{x} L_{k-1}(\varepsilon) d\varepsilon, \quad 2 \leq k \end{aligned} \tag{8}$$

where $\|L_{k-1}\|_2 = \int_{-1}^{1} L_{k-1}^2(x) dx = \sqrt{\frac{2}{(2k-1)}}$. By the properties of the Legendre polynomials we have:

$$\begin{aligned} L_0(x) &= 1; \\ L_1(x) &= x; \\ L_k(x) &= \frac{2k-1}{k} x L_{k-1}(x) - \frac{k-1}{k} L_{k-2}(x) \quad k \geq 2 \end{aligned} \tag{9}$$

3.2 Generalized Finite Element Method (GFEM)

The approximation of the solution proposed by the GFEM in the domain of the master element can be written as a combination of the components[14]:

$$u = u_{FEM} + u_{ENRICH} \tag{10}$$

where u_{FEM} is the FEM component based on the nodal degrees of freedom and u_{ENRICH} is the enrichment component generated by the unit partition and based on field degrees of freedom, as in other enriched methods.

For this work the functions proposed by [5] and [15] were applied in the problem of free vibration and presented good results.

The trigonometric enrichment reproduces some fundamental vibration modes in the structure, allowing a great accuracy of the obtained results. Thus the trigonometric approximation of GFEM is given by:

$$u(\xi) = \sum_{i=1}^{2} \eta_i(\xi) + \eta_1(\xi) \left[\sum_{j=1}^{n_l} \left(f_{1j}^S(\xi) + f_{1j}^C(\xi) \right) \right] + \eta_2(\xi) \left[\sum_{j=1}^{n_l} \left(f_{2j}^S(\xi) + f_{2j}^C(\xi) \right) \right] \tag{11}$$

In this work we use the enrichment functions, based on [5] and [16], given by:

$$\begin{cases} f_{1j}^S = sin\left(\beta_j(1+\xi)\right) \\ f_{1j}^C = \cos\left(\beta_j(1+\xi)\right) - 1 \\ f_{2j}^S = sin\left(\beta_j(\xi-1)\right) \\ f_{2j}^C = \cos\left(\beta_j(\xi-1)\right) - 1 \end{cases} \tag{12}$$

for $j = 1, 2, ..., n_l$, being n_l the number of levels of enrichment.

And the unit partition functions used were:

$$\begin{cases} \eta_1 = \dfrac{(1-\xi)}{2} \\ \eta_2 = \dfrac{(1+\xi)}{2} \end{cases} \tag{13}$$

For this work it was considered $\xi \in (-1, 1)$ e $\beta_j = j\frac{3}{4}\pi, \forall j = 1, 2...n_l$.

3.2.1 Heuristic Modification Stabilization

Note that the variation in the enrichment parameter $\beta_1 = \frac{3}{4}\pi$ implies different characteristics of approaching, as already pointed out by [5] and [17]. However, the gain in accuracy for certain frequencies does not appear to be associated with numerical stability gain. In fact, apparently there is a certain trade-off between accuracy and numerical stability, regarding the choice of the parameter β_1.

A second point of interest is the observation of β_j family of parameters. Parameters described by β_j are intrinsically related with enrichment functions and, consequently, with relations between the enriched functions present in the approximation space base. Thus, evolution of β_j influences the numerical characteristics of the approximation, such as stability. Therefore, we sought a change in the enrichment function group to stabilize its continuous application, avoiding the construction of approach spaces that tend to linear dependence.

The proposed modification is basically the change parameter formation rule β_j enrichment parameter, resulting in a subtle modification of each level of enrichment.

Recalling that parameter β_j is calculated by $\beta_j = j\alpha\pi$, it was proposed a modification, creating new stabilized parameters $\bar{\beta_n}$ given by:

$$\bar{\beta_j} = \left[2(j-1) + \frac{\beta_1}{\pi} \right] \pi \qquad j \geq 1 \tag{14}$$

It's interesting to note that $\bar{\beta_1} = \beta_1$, since:

$$\bar{\beta_1} = \left[2(1-1) + \frac{\beta_1}{\pi} \right] \pi = \left[0 + \frac{\beta_1}{\pi} \right] \pi = \beta_1 \tag{15}$$

This implies that there is no difference between approximation taken by this approach and the standard trigonometric enrichment first level of enrichment.

In constructing the method, as shown by [5], it is known that matrices K and M are hermitian matrices. As M is positive definite [12], the decomposition can be applied in M, transforming the generalized eigenvalue problem into a standard problem well conditioned . And if $\bar{x}$ is eigenvector of Eq.(2), then $x = (L^*)^{-1}$ satisfies Eq.(3), so by the above definitions it follows that the eigenvalues are real and positive, that is, $\omega^2 \in \mathbb{R}$.

As the mass matrix is by definition positive definite in the problems of free vibration there would be no reason to measure the conditioning of the system, since it is known that the problem is well conditioned and the eigenvalues are real and positive. However, according to the precision used in calculating the mass and stiffness matrices, it is observed that the problem still numerically ill conditioned, so it is necessary to verify the numerical sensitivity of the mass matrix, since the well conditioning of the problem depends on its being numerically definite positive.

4. Sensitivity Analysis

Taking into account the points raised previously, two analyzes were carried out .The first one varied the number of significant digits (precision) for the numerical approximation (numerical integration) of the mass and stiffness matrices, calculating the absolute error of the approximations related to exact values (analitical integration).

The second analysis considered the observation suggested by [6] to evaluate the number of condition of the mass matrix observing when the mass matrix becomes positive definite and what the order of the condition number of this matrix obtained numerically (numerical integration).

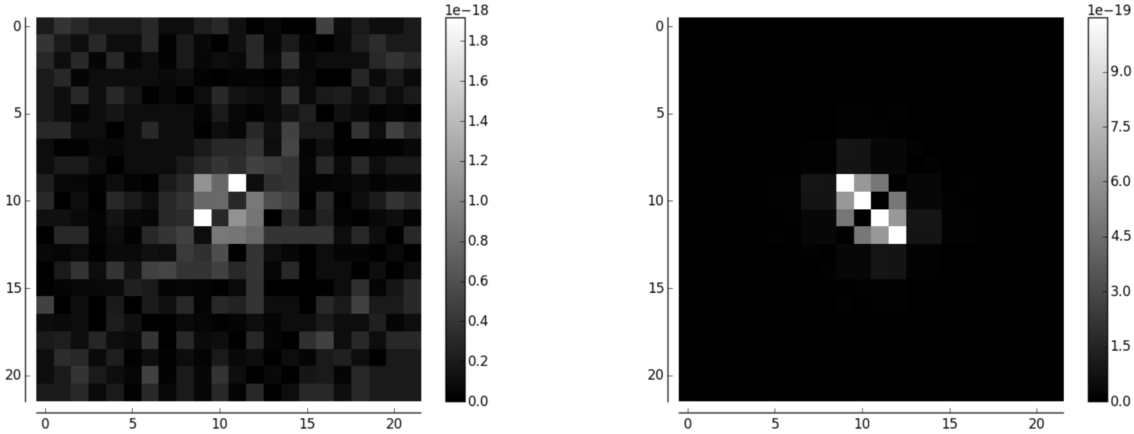
4.1 First analysis

As already mentioned, this first analysis aims to observe the absolute error between numerical and exact matrices. For this purpose, the matrices were generated in Maple software. Considering 5 levels of enrichment ($n_l = 5$) and varying the number of significant digits (19 to 22 significant digits for GFEM proposed by [5], 3 to 9 digits for the version of GFEM proposed by [15] and 3 to 10 digits for the p-Hierarchical FEM). Gauss quadrature was used to determine approximated matrices.

Figures 1 to 6 show, by way of illustration, the absolute error distribution in the mass and stiffness matrices for an accuracy of 3, 10, 19 and 22 significant digits.

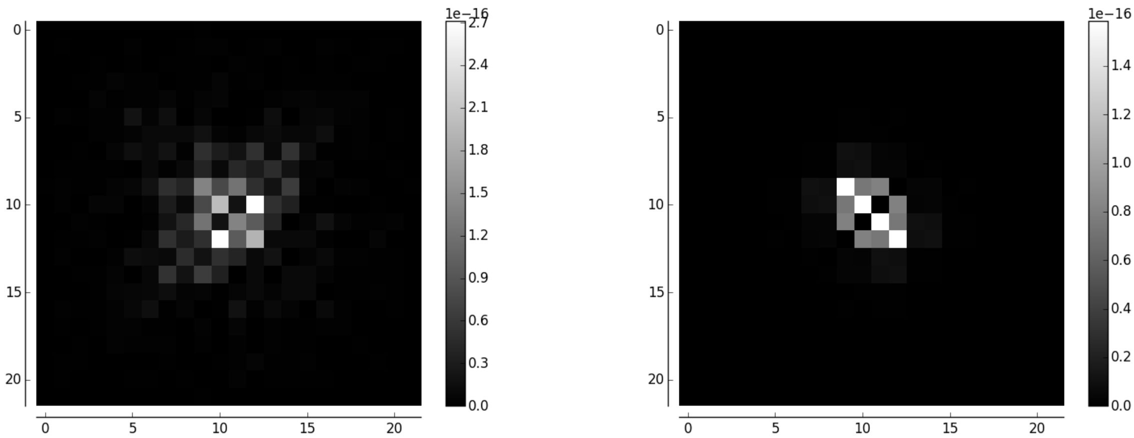
It is observed that the central part[1] of the matrices of mass and stiffness of the GFEM [5] and GFEM [15], is the one that contains the highest errors in the matrix, although they are quite small; already for p-Hierarchical FEM the errors are concentrated in the extremities of the matrix. In this region are the enrichment functions of the highest levels (in this case, level 5).

[1]The matrices of the GFEM [5] and GFEM [15], were not constructed in the traditional way, where the new levels of enrichment are placed in the end of the matrix, but in the central part.



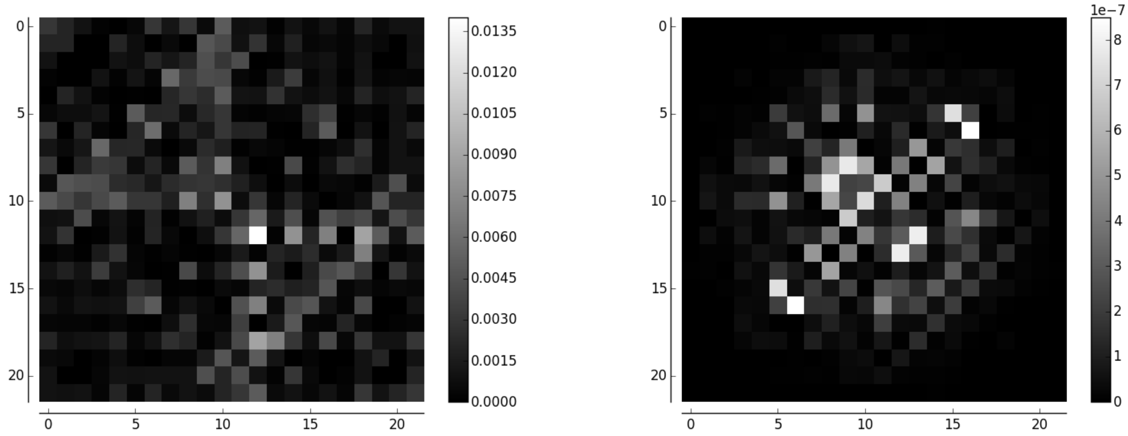
(a) 19 DIGITS (b) 22 DIGITS

Figure 1: GFEM [5]: MASS MATRIX ERROR



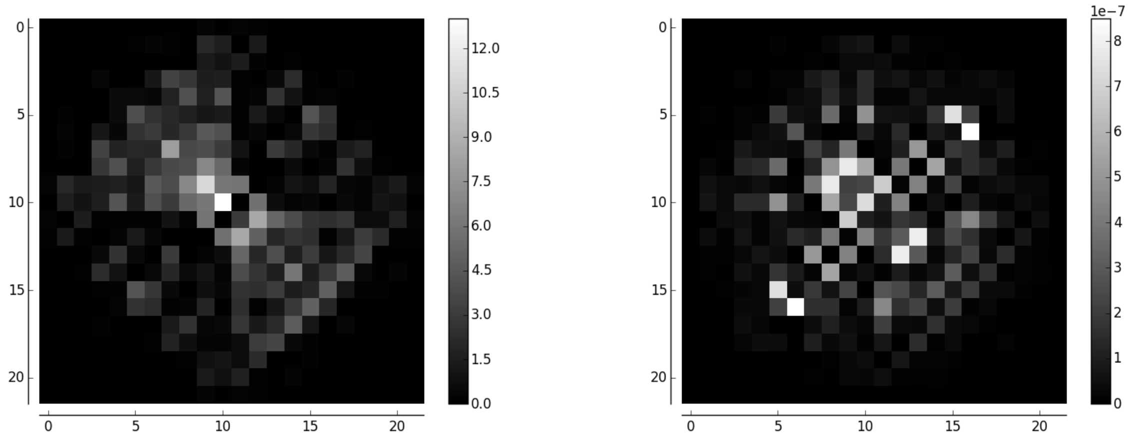
(a) 19 DIGITS (b) 22 DIGITS

Figure 2: GFEM [5]: STIFFNESS MATRIX ERROR



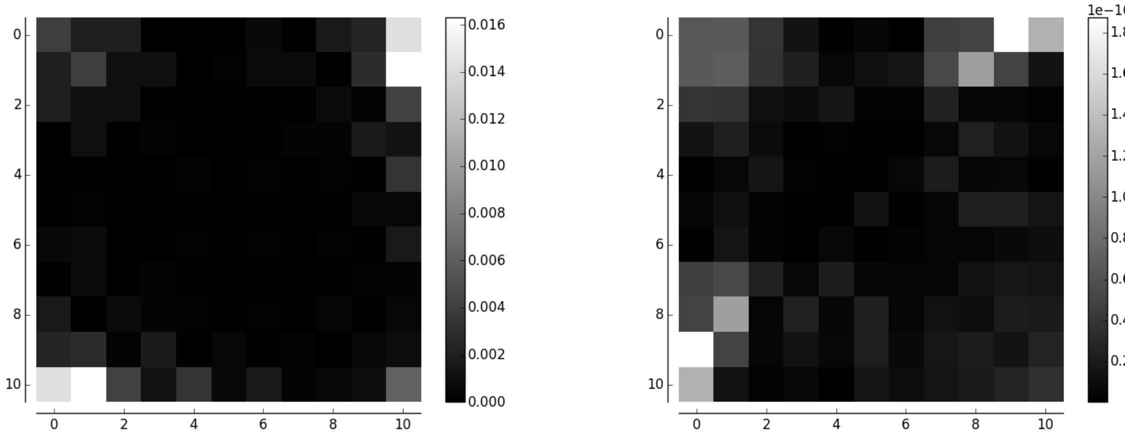
(a) 3 DIGITS (b) 10 DIGITS

Figure 3: GFEM [15]: MASS MATRIX ERROR



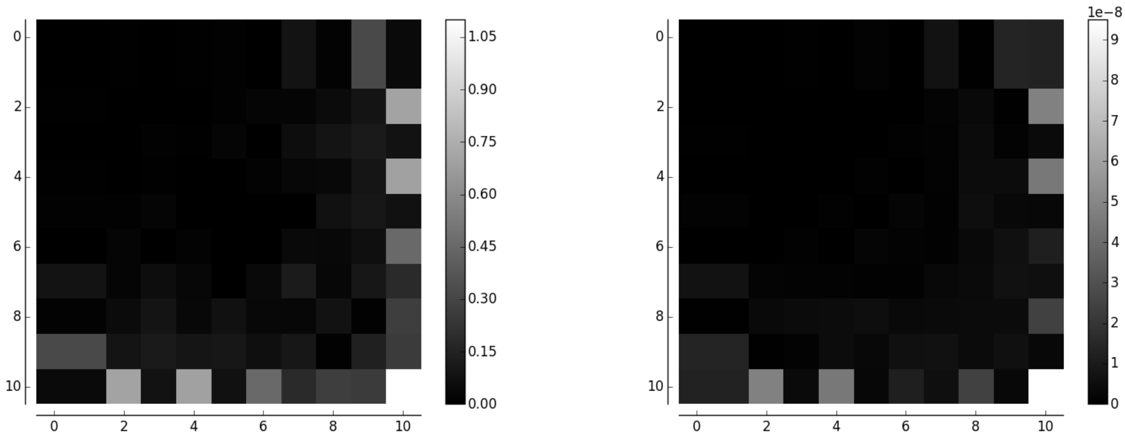
(a) 3 DIGITS (b) 10 DIGITS

Figure 4: GFEM [15]: STIFFNESS MATRIX ERROR



(a) 3 DIGITS (b) 10 DIGITS

Figure 5: P-HIERARCHICAL FEM: MASS MATRIX ERROR



(a) 3 DIGITS (b) 10 DIGITS

Figure 6: P-HIERARCHICAL FEM: STIFFNESS MATRIX ERROR

4.2 Second analysis

This second analysis was based on the argument that the generalized eigenvalue problem, $Ax = \lambda Bx$, can be reduced to a standard problem, which is well conditioned, when the matrix B, that in the free vibration problem is the mass matrix, is definite positive. Here the numbers of significant digits (precision) and enrichment levels were varied, and the behavior of the condition number was observed. The following results show the accuracy from which the mass matrix M and stiffness matrix K stabilizes as positive-definite and the order of the corresponding condition number. Following the tables 1-6, Figures 8-13 graphically show these observations of the behavior of the order of the condition number versus the number of significant digits so that the matrix is positive definite.

For this data analysis, problem boundary conditions were included. A fixed-free bar was analyzed (figure 7).

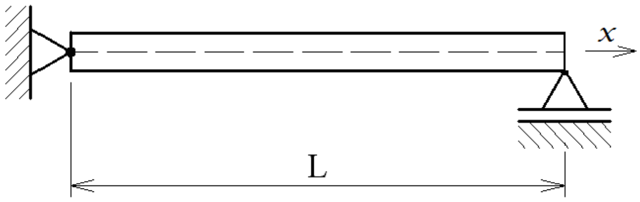


Figure 7: FIXED-FREE BAR

Table 1: GFEM [5] - MASS MATRIX

Number of Levels	1	2	3	4	5	6	7	8	9	10
Number of Significant Digits	5	8	10	17	21	24	28	34	35	40
Order of Condition Number	10^4	10^8	10^{12}	10^{15}	10^{21}	10^{25}	10^{29}	10^{33}	10^{38}	10^{42}

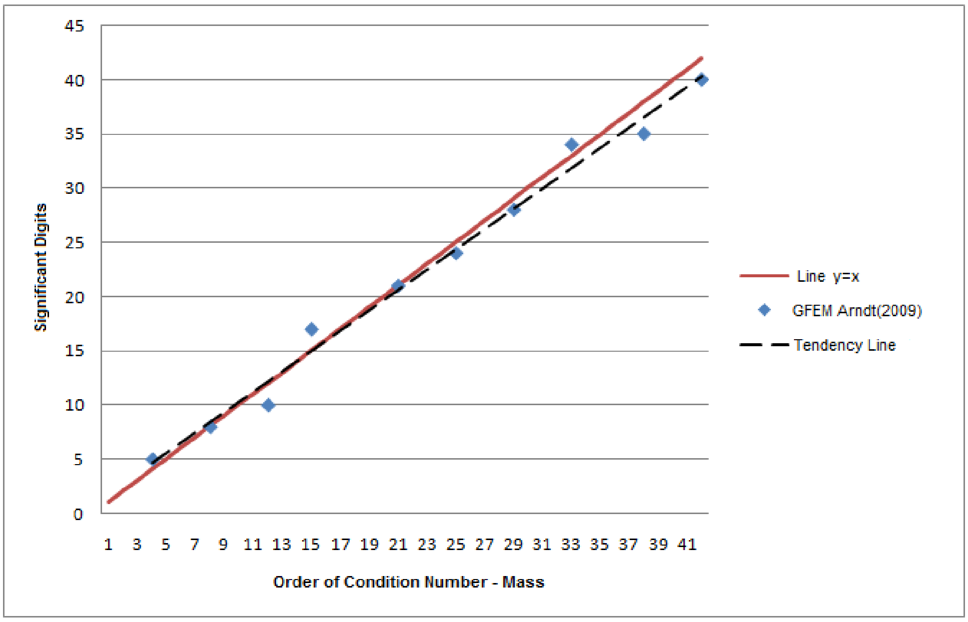


Figure 8: CONDITION NUMBER × SIGNIFICANT DIGITS - GFEM [5] - MASS MATRIX

Table 2: GFEM [5] - STIFFNESS MASS

Number of Levels	1	2	3	4	5	6	7	8	9	10
Number of Significant Digits	2	7	9	14	18	23	26	33	34	40
Order of Condition Number	10^2	10^6	10^{10}	10^{17}	10^{19}	10^{23}	10^{27}	10^{31}	10^{36}	10^{40}

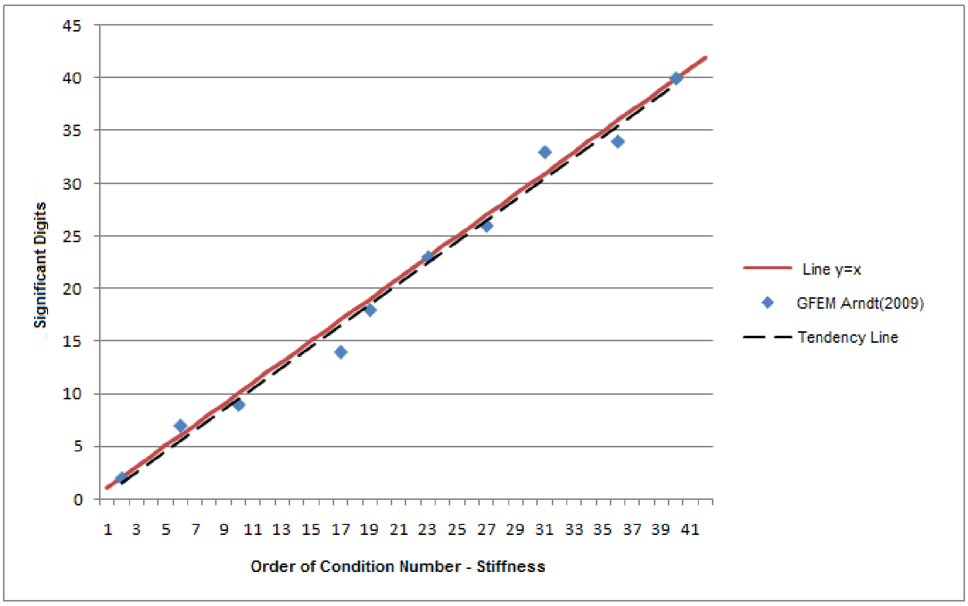


Figure 9: CONDITION NUMBER × SIGNIFICANT DIGITS - GFEM [5] - STIFFNESS MATRIX

Note that in Tables 1 and 2, the mass matrix always has a greater number of significant digits than the stiffness matrix.

Table 3: GFEM [15] - MASS MATRIX

Number of Levels	1	2	3	4	5	6	7	8	9	10
Number of Significant Digits	5	4	4	4	4	8	8	8	8	8
Order of Condition Number	10^4	10^5	10^5	10^6	10^7	10^7	10^7	10^8	10^8	10^8

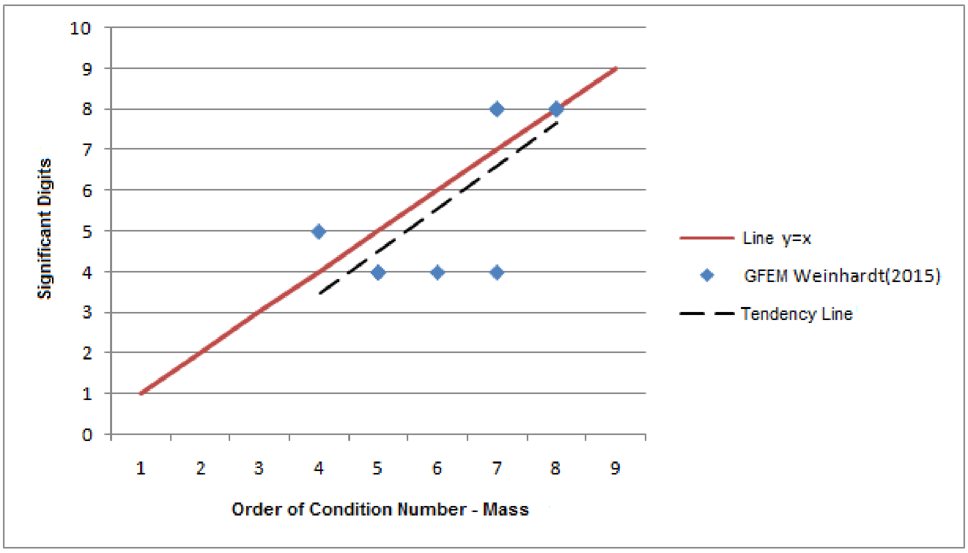


Figure 10: CONDITION NUMBER × SIGNIFICANT DIGITS - GFEM [15] - MASS MATRIX

Table 4: GFEM [15] - STIFFNESS MATRIX

Number of Levels	1	2	3	4	5	6	7	8	9	10
Number of Significant Digits	2	2	3	4	4	4	4	4	4	4
Order of Condition Number	10^2	10^4	10^4	10^5	10^5	10^6	10^6	10^6	10^6	10^7

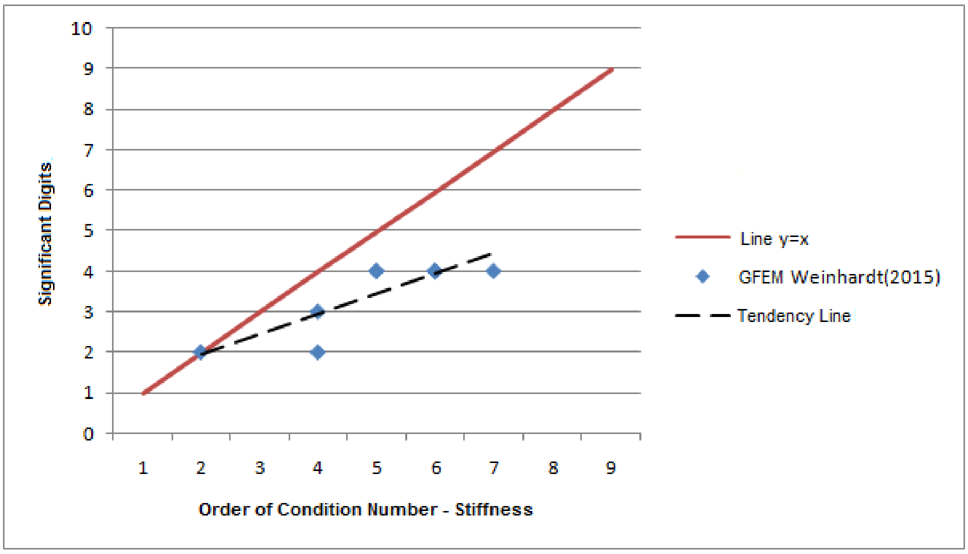


Figure 11: CONDITION NUMBER × SIGNIFICANT DIGITS - GFEM [15] - STIFFNESS MATRIX

Table 5: P-HIERARCHICAL FEM: MASS MATRIX

Number of Levels	1	2	3	4	5	6	7	8	9	10
Number of Significant Digits	1	2	2	2	2	2	2	2	3	3
Order of Condition Number	10^1	10^2	10^2	10^3	10^3	10^3	10^3	10^4	10^4	10^4

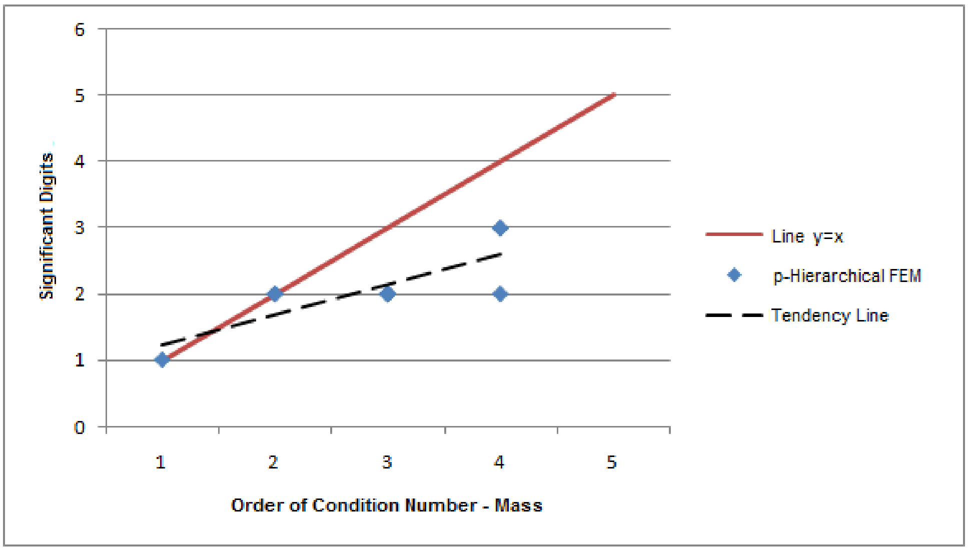


Figure 12: CONDITION NUMBER × SIGNIFICANT DIGITS - P-HIERARCHICAL FEM: MASS MATRIX

Table 6: P-HIERARCHICAL FEM: STIFFNESS MATRIX

Number of Levels	1	2	3	4	5	6	7	8	9	10
Number of Significant Digits	1	1	1	2	3	3	3	5	5	6
Order of Condition Number	10^0	10^0	10^0	10^0	10^0	10^0	10^0	10^0	10^0	10^0

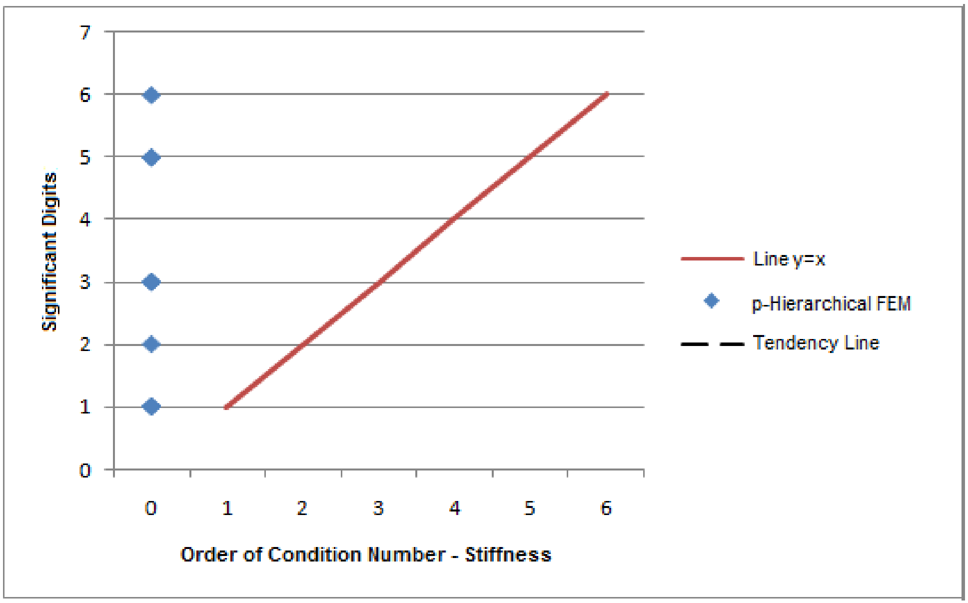


Figure 13: CONDITION NUMBER × SIGNIFICANT DIGITS - P-HIERARCHICAL FEM: STIFFNESS MATRIX

Note that the stiffness matrix of the p-Hierarchical FEM (figure 13) is well conditioned for the first 10 levels.

5. CONCLUSIONS

The objective of this work was to analyze the generalized eigenvalues problem and the perturbation generated by the approximations of the Generalized Finite Element Method (GFEM) matrices proposed by [5], the modification presented by [15] and the p-Hierarchical FEM. Even with all the mathematical theoretical basis ensuring that the generalized eigenvalue problem of the free vibration of the bar is well conditioned, there is a great numerical sensitivity in the problem of Eq.(3). This sensitivity is directly related to the approximation errors generated by the methods in the construction of mass and stiffness matrices, which eventually cause perturbations in the eigenvalues. From the analysis performed, it is concluded that the numerical and analytical approximations are good, despite the sensitivity of the problem.

A direct correlation between the power of the order of the mass matrix condition number and the number of significant digits (precision) necessary for the mass matrix to become numerically positive definite can be observed. Thus, the order of magnitude of the condition number of the mass matrix can be used to estimate the precision needed in the construction of the mass matrix and therefore it also be used in the numerical stability comparison of different enrichment function for the GFEM and other methods.

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