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# A TUTORIAL ON LINEAR GAP METRICS FOR ROBUST CONTROL

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***Abstract.*** *Robust control theory aims to develop control systems that can effectively handle variations in system (i.e., mechanics and electrics) parameters, modeling errors, external disturbances, and other uncertainties. Linear gap metrics play a critical role in the field of robust control and in the general theory for similarity search, as they measure the dynamic distance between two Linear Time-Invariant (LTI) systems in the frequency and time domains. Gap metrics combined with stability margins or thresholds can guarantee the stability of a set of systems subjected to the same controller or the stability of a set of controllers subjected to the same system. The linear gap metrics are derived from functional analysis and have multiple calculation methods available in the literature; however, some of these methods can be difficult to implement. This paper presents a concise roadmap for calculating linear gap metrics, including the various steps in the process. By exploring these steps, we can better understand the different gap metrics, methodologies, and their implications. The main conclusions are as follows. 1) There are at least 11 linear gap metrics available in the current literature. 2) Only two of the 11 linear gaps are extensively applied in classic control problems. 3) The latest linear gap metric, called the K-gap metric, allows online analysis of the system's robustness through a data-driven algorithm applied to a state-space system representation, transfer functions in the $z$- domain, and Stable Kernel/Image realizations. 4) The K-gap is a big step in the gap metric theory, since the previous gaps are used for offline and frequency response analysis. 5) Linear gap metrics can be a practical and faster approach than the H-infinity and $\mu$-synthesis approach. 6) The K-gap metric may be a feasible path to approximate a distance measure between two nonlinear systems since it calculates a gap in the time domain. Measurement of the distance between two nonlinear systems is still an open problem in control theory and functional analysis.*

***Keywords:*** *linear gap metrics, tutorial, robust control, robust stability guarantee, functional analysis*

## 1. INTRODUCTION

(Robust) control theory benefits from the use of metric spaces as a means to measure the dynamic distance between systems or processes. A distance metric can be translated into how well one system approximates another, the sensitivity of the interconnected system to a change in its components, or the admissible uncertainties tolerated by the open loop without destroying stability when feedback is applied. The gap metric presents all these features. The gap metric measures the distance between two (possibly unstable) LTI systems in terms of dynamic response and exhibits robustness properties, which means that the second system can be considered as a perturbation or uncertainty regarding the first system; the gap between them indicates if the variation from one system to another preserves closed-loop stability. Generally, the range of values assumed by the gap is $[0, 1]$, $0$, which means that the systems are close and can receive the same control law, and $1$, which means that the systems are distant from each other and, therefore, it is difficult to simultaneously stabilize them by the same control law.

Given a candidate controller, the gap metric provides a stability guarantee bound to perturbed systems subject to the same controller. Stability guarantee is one of the features of the gap metric that concerns applications in real control problems. Systems where stability is a considerable requirement, such as cooperative systems and rehabilitation robotics, can benefit greatly from this aspect in terms of reliability and safety. Also, because the gap metric can quantify admissible uncertainties, systems with high nonlinearities, which is the case of chemical processes, and systems with several operating points, which is the case of the aerospace industry, can successfully use this mathematical tool for uncertainty analysis and controller design.

The usual control fields that use the linear gap metrics are multi-model control (Hariprasad *et al.*, 2012), decentralized control (Lee *et al.*, 2000), system identification (Geng *et al.*, 2015), which considers the gap metric as a generalization of the Theil index, and controller certification (Park and Bitmead, 2008). All of them are based on the discretization of continuous systems which comes from a linear analysis in the frequency response domain. For the nonlinear case, the similarity search between affine nonlinear systems is performed in the time domain by applying a stability guarantee test to systems that can be put into the state-space formulation, (Du *et al.*, 2009; Saki and Bolandi, 2022), although an analytical method is still lacking in the literature. Recent breakthroughs have been achieved for the calculation of linear gap metrics in the time domain, based on the subspace approach, for the field of signal processing (Jin *et al.*, 2021). This breakthrough could be explored in future research as a numerical framework to calculate nonlinear gap metrics. Furthermore, for the control of mechanical systems, in general, the state-of-the-art gap metric is based mainly on the data-driven gap metric for automated fault detection, diagnosis, monitoring, and control (Jin *et al.*, 2023),

This paper aims to show, in a concise way, the several linear gap metrics available in the literature following the timeline of their development. Simple examples of single-input-single-output (SISO) enable a better understanding while highlighting the advantages and disadvantages of each metric.

The paper can also be used as a bibliographical guide for researchers in many fields of science.

## 2. BACKGROUND OF THE GAP METRIC

The gap metric between two systems is defined as the distance between these systems in terms of the dynamic response. Geometrically speaking, the gap metric can be viewed as a distance quantified by the acute angle between the graphs of those same systems. It was initially developed for classic control theory as an extension of the operator norm to deal with unstable systems. This was an important step since the operator norm, as a distance metric, can only deal with bounded-input-bounded-output (BIBO) systems.

The mathematical background of the gap metric comes from functional analysis and the perturbation theory of linear operators. Although the gap metric dates to 1947 (Krein, 1947), it only found its application in control theory in the 1980s with the works of Zames (1980), El-Sakkary (1981), and Vidyasagar (1984a). Initially, the gap metric applied to control problems was used in the robust stabilization of the frequency response for single-input-single-output (SISO) systems (El-Sakkary, 1985), where the solution developed to extend the operator norm to unstable uncertain systems, which do not need to be proper, was to project them into a Riemann sphere and calculate this stereographic projection as seen in Fig. 1 (El-Sakkary, 1989).

El-Sakkary (1985) presented one of the first formulas to calculate the gap metric using transfer functions. Without loss of generality, let $P_1$ and $P_2$ be open-loop transfer functions, i.e. linear systems in the complex Hilbert space $\mathbb{H}$ (also called the Hardy space $\mathbf{H_2}$) that map, in the frequency domain, the inputs $U_1(s)$ and $U_2(s)$ to their respective outputs $Y_1(s)$ and $Y_2(s)$, where $s = a + j\omega$ is a frequency-dependent complex variable, with $a, \omega \in \Re$. For ease of notation, from this point until the end of the article, the dependency on the variable $s$ may be omitted.

Let $\pi_{P_1}$ and $\pi_{P_2}$ be projections mapping the pairs of Cartesian products $\mathbb{H} \times \mathbb{H}$ $(U_1, Y_1)$ and $(U_2, Y_2)$ onto the graphs $G(P_1)$, $G(P_2)$ of each respective linear system. This projection is called the *characteristic projection*, or *characteristic matrix*, of a linear unbounded operator (Stone, 1951a).

The gap metric $\delta(.)$ between $P_1$ and $P_2$ can also be defined as:

$$\delta(P_1, P_2) = \|\pi_{P_1} - \pi_{P_2}\|, \tag{1}$$

where the characteristic projection is given by:

$$\pi_P = \begin{pmatrix} (I + P^*P)^{-1} & P^*(I + PP^*)^{-1} \\ P(I + P^*P)^{-1} & PP^*(I + PP^*)^{-1} \end{pmatrix}, \tag{2}$$

with $P^*$ as the adjoint operator of $P$, that is, its conjugate complex transpose.

In the case where $P_1$ and $P_2$ are stable, the following inequality is true:

$$\delta(P_1, P_2) < \|P_1 - P_2\| \tag{3}$$

The operations described by Eq. (2) are based on the multiplication of the plant $P$ by its complex conjugate transpose $P^*$, the poles that excite unstable modes are removed (El-Sakkary, 1985), enabling the application of the maximum modulus principle:

$$\|\pi_P\| = \sup_{\omega \in \Re} |\pi_P(j\omega)|. \tag{4}$$

As an extension, for the Multiple-Input Multiple-Output (MIMO) case, the characteristic projection was used to map the Cartesian product of the Hilbert space into the graph of each system (Stone, 1951b). Both projections remove the poles that excite unstable modes.

An illustrative example of the characteristic projection for SISO systems, capable of removing unstable poles, is shown in Fig. 1, where the chordal distance between the stereographic projections of the frequency response systems onto the unitary Riemann sphere is equivalent to the gap topology.

There are a considerable number of gap metric extensions in the control literature, each one displaying advantages and disadvantages depending on the context where it is used. All of them are derived from the theory of linear unbounded operators, where the gap topology is the weakest topology that provides robustness to feedback stability.

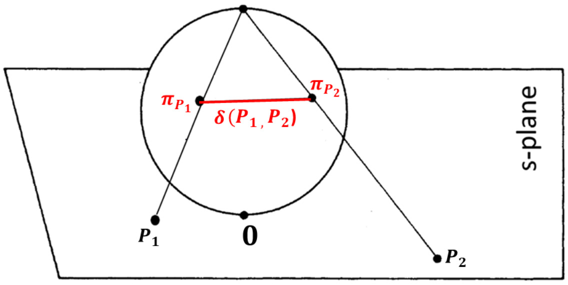


Figure 1: The characteristic projections $\pi_{P_1(s)}$ and $\pi_{P_2(s)}$ can be seen as a stereographic projection of systems $P_1$ and $P_2$ onto a Riemann sphere. Adapted from El-Sakkary (1981).

The practical use of the gap metric in control theory comes from the robust stability guarantee theorem presented below:

**Theorem 1** *A control transfer function, $C$, stabilizes both $P_1$ and $P_2$ if:*

$$\delta(P_1, P_2) < \gamma(P_1, C), \tag{5}$$

*with $\gamma$ as a stability margin threshold and $[P_1, C]$ defined as a stable output feedback configuration.*

There is a dual formulation of **Theorem 1** where two controllers $C_1$, and $C_2$ may stabilize the same plant $P$ if the inequality of Eq. (5) holds. In the control literature there are a few stability factors that satisfies 1, however the most used is the generalized stability margin $\gamma := b$.

Given the sensitivity function:

$$T(P,C) = \begin{pmatrix} P(I+CP)^{-1}C & P(I+CP)^{-1} \\ (I+CP)^{-1}C & (I+CP)^{-1} \end{pmatrix} \tag{6}$$

Then the generalized stability margin $b(P, C)$ is:

$$\begin{cases} \|T(P,C))\|_{\infty}^{-1}, \ if \ [P,C] \ is \ stable, \\ 0, \ else. \end{cases} \tag{7}$$

An easy way to calculate Eq. 5 is presented in (Cantoni and Vinnicombe, 1999). Both the gap metric and the generalized stability margin have values between 0 and 1.

## 3. LINEAR GAP METRICS ROADMAP

A linear gap metric is a mathematical framework that provides a way to measure the distance between two linear systems. This distance is defined as the maximum deviation between the two systems, given a certain class of input signals. Linear-gap metrics are typically used to assess the robustness of control systems, i.e., their ability to withstand perturbations and uncertainties.

This section will explore the basics of linear gap metrics and some of their applications in control theory.

### 3.1 Graph metric

The graph metric $\delta_{graph}$ (Vidyasagar, 1982, 1984b), has this name because it uses the graph topology, which is the weakest topology (equivalent to the gap topology), that is, the primary topology of the gap metrics in which feedback stability is robust. It is defined by a set of BIBO stable transfer functions. For MIMO systems, this set denotes the matrices of BIBO transfer functions.

This metric aims to express the stability boundary between two unstable systems rather than focusing on the numerical results of their distance. Such expressions can then be used for robust analysis and the design of stabilizing controllers.

Moreover, if the disturbed system has the same number RHP poles as the nominal system, then it is possible to obtain sufficient conditions for robustness concerning a given class of (parametric) perturbations.

**Definition 1.** Let the plants $P_1$ and $P_2$ have the same dimensions, and the normalized right coprime factors (NRCFs) $\{N_1, M_1\}$ and $\{N_2, M_2\}$, respectively. The directed graph metric $\vec{\delta}_{graph}$ is then given by:

$$\delta_{graph}(P_1, P_2) = \inf_{\|Q\|_{\infty} \leq 1,\ Q \in \mathbf{H}_{\infty}} \left\| \begin{pmatrix} N_1 \\ M_1 \end{pmatrix} - \begin{pmatrix} N_2 \\ M_2 \end{pmatrix} Q \right\|_{\infty}, \tag{8}$$

Note that the solution of Eq. (8) lies on the achievement of a functional $Q \in \mathbf{H}_\infty$ that maximizes the above optimization problem, which involves the field of variational calculus (Vinnicombe, 1993) and cannot be precisely evaluated (Georgiou, 1988). Instead, it is possible to calculate an upper bound $v$ for $\delta_{graph}$ (Vidyasagar, 1984b). Using the same NRCF for $P_1$ and the RCF $\{N_2', M_2'\}$ for $P_2$, the graph metric is bounded by $v$ such as:

$$\delta_{graph}(P_1, P_2) < \frac{2v}{1-v}, \tag{9}$$

with

$$v = \left\| \begin{pmatrix} N_1 \\ M_1 \end{pmatrix} - \begin{pmatrix} N_2' \\ M_2' \end{pmatrix} \right\|_\infty < 1. \tag{10}$$

In terms of robust stabilization, consider a stable feedback configuration $[P_1, C_1]$ of the nominal plant $P_1$ with a controller $C_1$. Defining the quantity $l$ as follows:

$$l = \sqrt{1 + \|[P_1, C_1]\|_\infty^2}, \tag{11}$$

and considering a simultaneously perturbed plant $P_2$ and controller $C_2$, the following robust stability guarantee criterion can be defined:

$$max\left\{\delta_{graph}(P_1, P_2), \delta_{graph}(C_1, C_2)\right\} < b_{graph}(P_1, C_1) \tag{12}$$

with

$$b_{graph}(P_1, C_1) := \frac{1}{2+4l}. \tag{13}$$

Observe that Eq. (12) calculates both the graph metric of plants $P_1$ and $P_2$ and the graph metric of controllers $C_1$ and $C_2$. Then, it compares the largest value between them with a stability margin $b_{graph}$ to guarantee stability, similar to **Theorem 1.** One can also consider only one disturbed system, that is, a disturbed plant ($C_1 = C_2$) or a disturbed controller ($P_1 = P_2$).

*Example 3.1:*

Although no analytical formula is available in the literature to calculate the graph metric, a robust analysis can still be performed. The inequality of Eq. (14) is valid to project a more conservative control system:

$$b_{graph}(P_1, C_1) \leq b(P_1, C_1) \tag{14}$$

### 3.2 Giorgiou's Gap Metric

Giorgiou proposed (Georgiou, 1988) a gap metric $\delta_g$ that is computable and overcomes the mathematical complexity of the graph metric $\delta_{graph}$. The gap metric $\delta_g$ is the second most used gap metric in the control literature, losing just to Vinnicombe's $\nu$-gap metric.

**Definition 2.** Assuming two systems represented by rational transfer functions with NRCFs $\{N_1, M_1\}$ and $\{N_2, M_2\}$, the gap $\delta_g$ between the two systems is defined as follows:

$$\delta_g(P_1, P_2) = \inf_{Q \,\in\, \mathbf{H}_\infty} \left\| \begin{pmatrix} N_1 \\ M_1 \end{pmatrix} - \begin{pmatrix} N_2 \\ M_2 \end{pmatrix} Q \right\|_\infty. \tag{15}$$

Note that the only difference between $\delta_{graph}$ and $\delta_g$ is that the latter *does not limit* the norm $\|Q\|_\infty \leq 1$, as can be seen in Eq. (15). Equation (15) can be solved by interpolation theory and the commutant lifting theorem. (Georgiou, 1988) also presents the following mathematical relation:

$$\delta_g \leq \delta_{graph} \leq 2\delta_g \tag{16}$$

The robustness analysis for Giorgiou's gap metric can be done indirectly with the use of $\delta_{graph}$ and its boundary shown in Eq. (9), that is,

$$\delta_g \leq \delta_{graph} < \frac{2v}{1-v} \tag{17}$$

Georgiou and Smith (1990) showed that $\delta_g$ can also be used to infer the stability of a closed-loop plant-controller pair $[P_1, C_1]$, as in **Theorem 1.**

*Example 3.2:*

Consider the mass-spring-damper system $P(s)$ described by Eq. (18) that is a second-order transfer function representing a wide variety of mechanical and electrical systems (i.e. mechanical and electrical actuators for legged robotics) :

$$P(s) = \frac{1}{ms^2 + bs + k}, \tag{18}$$

with low damper and spring coefficients $b = 0.3$ and $k = 0.4$, respectively, and an uncertain parameter $m \in [0, 5]$. The nominal value of the mass is 1. The gap metric $\delta_g\left(P_{nom}, P_\Delta\right)$ as a function of the mass variation is calculated using the MATLAB command *gapmetric* and shown in Figure 2.

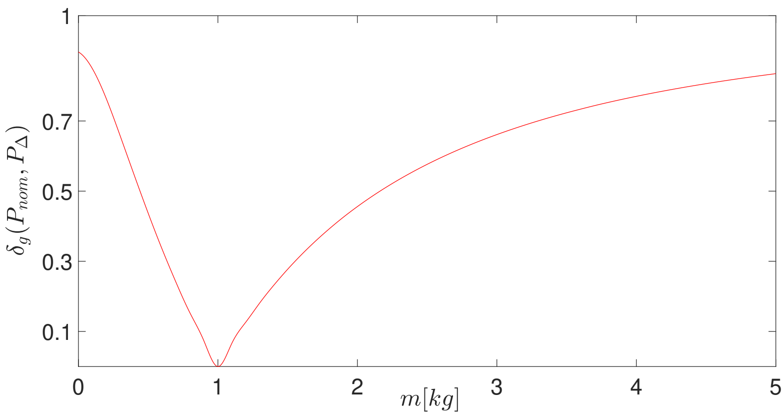


Figure 2: The gap metric between the nominal plant and the disturbed plant exhibits a proportional response to the parametric variation of the mass. However, the gap metric assumes higher values more quickly for a decrease in the plant's mass. Therefore, designing a controller for the mass decrease scenario would be more difficult.

A quick analysis of Fig. 2 gives information about the robust stability of the plant. This practical verification is one of the appealing features of the gap metric and the **Theorem 1**. A controller $C$, considered to have great performance and robustness, yields a generalized stability margin $b(P_{nom}, C)$, of at least 0.4. Figure 2 shows that a secure region that guarantees stability in the face of mass change would be $m = [0.56, 1.52]$.

For the same stability margin value, but with a different nominal mass value $m = 3$, new values of the gap metric are calculated:

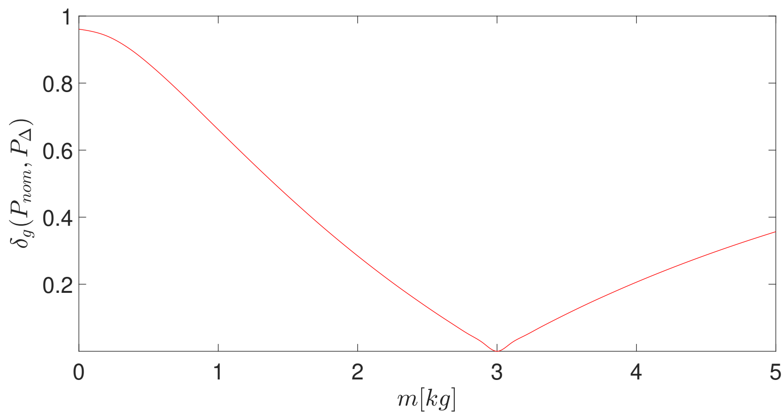


Figure 3: Gap metric between the nominal plant, $m = 3$ and the disturbed plant $m \in [0, 5]$

Figure. 3 compared to Fig. 2 shows better values of the gap metric by choosing a different nominal mass. From Figure 3 and considering the generalized stability margin of 0.4, a different range of mass variability is covered. The controller can stabilize $m \in [1.7, 5]$, which is a larger mass range than $m \in [0.56, 1.52]$ in Fig. 2.

### 3.3 T-gap metric

The T-gap metric (Georgiou and Smith, 1989), together with the work of Sefton and Ober (1993), shows that the uncertainty sphere in the gap metric is equal to a sphere of uncertainty of equal radius defined by perturbations of a normalized coprime factor. A general investigation of the T-gap is carried out using a normalized left coprime factor (NLCF) in *transpose* form as a measurement of the distance between plants, justifying the name *T-gap*.

**Definition 3.** Let: $\left\{\tilde{N}_1, \tilde{M}_1\right\}$ and $\left\{\tilde{N}_2, \tilde{M}_2\right\}$ be the NLCFs of the systems $P_1$ and $P_2$, respectively. The T-gap $\delta_T$ between these two frequency response systems is:

$$\delta_T\left(P_1, P_2\right) = \inf_{Q \,\in\, \mathbf{H}_\infty} \left\| \begin{pmatrix} -\tilde{M}_1 & \tilde{N}_1 \end{pmatrix} - Q \begin{pmatrix} -\tilde{M}_2 & \tilde{N}_2 \end{pmatrix} \right\|_\infty . \tag{19}$$

An important remark is that $\delta_T$ is transpose invariant, that is, $\delta_T\left(P_1, P_2\right) = \delta_T\left(P_1^T, P_2^T\right)$.

Furthermore, in the development of the T-gap metric, another topic has started to be discussed: the possibility of finding an optimal generalized stability margin in analytical form. The answer to that question was positive, and some techniques were able to deal with this problem. The following equations translate the statement of finding the optimal generalized stability margin for a certain plant $P$ given a fixed controller $C$:

$$b_{opt}\left(P\right) = \sup_{C}\; b\left(P, C\right). \tag{20}$$

An important remark on the general solution $b_{opt}(P)$ is that it does not answer how to find the controller structure that produces such an optimal generalized stability margin. This is still an open mathematical question. An analytical formulation of Eq. (20) is defined below:

$$b_{opt} = \sqrt{1 - \left\| \left\{ \tilde{M}, \tilde{N} \right\} \right\|_{\breve{\mathtt{H}}}^2} = sup_{\;all\ stabilizing\ C}\; inf\; b\left(P, C\right). \tag{21}$$

where $\breve{\mathtt{H}}$ is the Hankel operator (Koenings *et al.*, 2017).

*Example 3.3:*

Consider the disturbed plant $P$ of *Example 1.2*, with $m = 3$. Eq. (21) calculates the optimal stability margin, $b_{opt}(P)$. First, the LCFR of $P$ is obtained from the MATLAB command *fact = lncf(.)* :

$$P := \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} \frac{(s^2+0.1s+0.1333)}{(s^2+0.6792s+0.359)} \\ \frac{0.33333}{(s^2+0.6792s+0.359)} \end{pmatrix} \tag{22}$$

Then, Eq. (21) becomes:

$$b_{opt} = \sqrt{1 - \left\| \begin{pmatrix} \frac{(s^2+0.1s+0.1333)}{(s^2+0.6792s+0.359)} \\ \frac{0.33333}{(s^2+0.6792s+0.359)} \end{pmatrix} \right\|_{\breve{\mathtt{H}}}^2}. \tag{23}$$

Finally, the MATLAB command *sigma(.)* applied to Eq. (23) returns the value $b_{opt} = 0.1$.

### 3.4 Extended graph metric

The extended graph metric, also known as the *ngraph* metric $\delta_{ngraph}$ Meyer (1990), was developed to overcome the biggest obstacle of the graph metric, that is, the existence of a constraint in the optimization problem represented by $\|Q\|_\infty \leq 1$. The extended graph metric preserves the graph topology, but has some features; that is, it gives a feasible optimization problem compared to the graph metric and generates a distance value in the range of $[0, \infty]$. To our knowledge, this is the only gap metric that works with an upper bound greater than 1. All other gap metrics in the literature are limited by $[0, 1]$, or in the form $arcsin(.)$ of these bounds, $[0, \frac{\pi}{2}]$. The extended graph is based on the concept of nearly normalized coprime factorization (NNCF).

**Definition 4.** Given a NRCF: $\{N_1, M_1\}$ of $P_1$, this factorization is called NNRCF if, in addition to the Bezout identity, the following conditions are satisfied:

$$N^T\left(-s\right) N\left(s\right) + M^T\left(-s\right) = I + M^T\left(s\right) M\left(s\right), \;\; \forall s, \tag{24}$$

and

$$M(\infty) = I. \tag{25}$$

Then the $\delta_{ngraph}$ metric between two frequency response systems is:

$$\delta_{ngraph}(P_1, P_2) = \left\| \begin{pmatrix} N_1 - N_2 \\ M_1 - M_2 \end{pmatrix} \right\|_\infty . \tag{26}$$

### 3.5 $\nu$-gap metric

Vinnicombe's $\nu$-gap metric $\delta_\nu$ is a frequency-dependent gap metric and it is based on a homotopy argument called the winding number condition (WNC), differing, therefore, on an important point from the other gap metrics induced by the gap topology. The $\nu$-gap metric has a frequency-wise interpretation; that is, it can measure the distance between linear systems directly from the frequency response values, allowing for a wider robustness and performance analysis (Vinnicombe, 1991, 1992). It is also important to note that this metric has the tightest boundary of all linear gap metrics.

**Definition 5.** A frequency-wise formulation was introduced in (Vinnicombe, 1993) and is given by:

$$\delta_\nu(P_1, P_2) = \begin{cases} \bar{\zeta}(P_1, P_2), & if\ WNC^*\ holds \\ 1, & otherwise \end{cases} \tag{27}$$

with the WNC defined, in this case, as

$$\begin{aligned} WNC^* &:= det(I + P_2^* P_1)(j\omega) \neq 0,\ \forall\omega,\ and \\ & wno[det(I + P_2^* P_1)] + \eta(P_1) - \eta(P_2) = 0, \end{aligned} \tag{28}$$

where $\zeta(P)$ is the number of open RHP poles of $det(P)$, and $\zeta(.)$ is the frequency-wise $\nu$-gap metric function defined as follows:

$$\begin{aligned} \bar{\zeta} &= sup_{\omega \in \Re}\, \zeta(j\omega) \\ \zeta(P_1, P_2) &= \sigma\left( (I + P_2 P_2^*)^{-\frac{1}{2}} (P_1 - P_2)(I + P_1^* P_1)^{-\frac{1}{2}} \right). \end{aligned} \tag{29}$$

For the SISO case, the Eq. (29) becomes Eq. (30):

$$\zeta(P_1, P_2) = (1 + P_2 P_2^*)^{-\frac{1}{2}} |P_1 - P_2| (1 + P_1^* P_1)^{-\frac{1}{2}} \tag{30}$$

The function $\zeta$ from 29 is called the $\mathbf{L_2}$ gap metric, that is, the distance between the $\mathbf{L_2}$ graph spaces. Initially, only a lower bound was obtained for the gap metric $\mathbf{L_2}$ by (Zhu *et al.*, 1989). Later, (Vinnicombe, 1993) showed that the $\mathbf{L_2}$ gap metric is equal to the $\nu$-gap metric when the WNC is satisfied. A frequency-wise version, $\rho(.)$, of the generalized stability margin, $b(.)$, can be calculated following the same structure as in Eq. (29).

*Example 3.4:*

Consider an unstable plant, $P_1 = \frac{1}{s-1}$, and a controller, $C = 2$ which stabilizes $P_1$. The objective is to find out if the controller $C$ can also stabilize a second plant $P_2 = \frac{1}{s-2}$. Figure 4 shows the frequency-wise analysis considering two stability margin thresholds.

The stability thresholds $b(.)$ and $b_{opt}$, are used in controller certification methodologies to define levels of certification $\alpha$ and $\beta$, respectively.

### 3.6 Analytical gap metrics

The next linear gap metrics have complex mathematical formulations and are not currently being used in control problems due to the troublesome implementation, however, they worth being mentioned because of some of their features.

1. The pointwise gap metric, $\delta_p$ Qui and Davison (1992) is a frequency-wise gap metric (equivalent to the $\nu$ - gap) restricted to stable systems

2. The angular gap metric, $\delta_{\rho_\theta}$, (Zhang and Qiu, 2007) is designed to measure the distance between systems with different dimensions

3. (Hosseini *et al.*, 2011) presented an extension of the $nu$-gap $\delta_\nu$ based on the $\mathbf{H_2}$ norm, which shows the energy of a given system. Quantitatively, this new metric $\delta_{h\nu}$ was shown to be smaller than the traditional $\delta_\nu$, however, this metric does not provide a robust stability guarantee.

4. An extension of the $\nu$-gap metric allows the calculation of the distance between infinite-dimensional systems with rational and nonrational fractions (Ball and Sasane, 2010; Rupp and Sasane, 2013). It is called the abstract $\nu$-gap metric $\delta_{d_\nu}$.

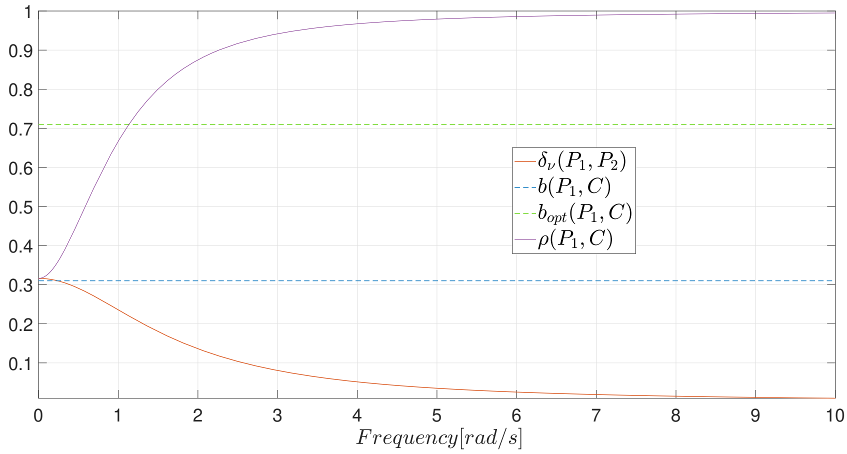


Figure 4: The controller cannot stabilize the plant $P_2$ in steady state; therefore, an overall analysis would not certify this controller. However, two major inferences can be made. (1) The controller is adequate for frequencies greater than $0.5\ [rad/s]$; (2) The optimal generalized stability margin shows that better controllers can be designed.

### 3.7 SBM: A data-driven gap metric framework

Subspace-based methods (SBMs) are an important tool for data representation and analysis. They provide a way to simplify complex data and make it easier to analyze while maintaining its essential properties. With their ability to reduce dimensionality, improve performance, and have a wide range of applications, subspace-based methods are an essential component of important areas such as system identification, computer vision, and pattern recognition (Katayama *et al.*, 2005). The kernel-gap metrics are based on stable image representations (SIRs) and stable kernel representations (SKRs) and can be calculated offline and online in the time domain through the use of the $z$ - transformation.

A SIR, $Im$, is presented in Eq. (31):

$$Im = \begin{bmatrix} -\tilde{N}(z) & \tilde{M}(z) \end{bmatrix}. \tag{31}$$

A SKR, $K$, is presented in Eq. (32):

$$K = \begin{bmatrix} M(z) \\ N(z) \end{bmatrix}. \tag{32}$$

Based on the equivalence between SKRs/SIRs in the $s$ domain and in the $z$ domain, (Ding *et al.*, 2014) established data-driven realizations for Stable Kernel and Image Representations. A data-driven gap metric, or SBM-gap, was first defined in (Ding, 2015) as the $K$-gap metric. Both works, mentioned previously, focus on faulty signal detection and measurement noise. More recent research also has the detection bias of faulty signals (Li *et al.*, 2018; Wang *et al.*, 2021), although some of them incorporate the $K$-gap metric into a fault-tolerant control framework. According to (Koenings *et al.*, 2017), for further research, an iterative scheme that allows a simpler online calculation of the gap can be considered.

According to Ding (2015) the $K$-gap and the $T$-gap are equivalent metrics in the tie domain. Equation. (33) presents the definition of the $K$-gap:

$$\delta_K(K_1, K_2)(z) = inf_{Q \in H_\infty} \left\| \begin{bmatrix} \tilde{M}_1 & -\tilde{N}_1 \end{bmatrix} - Q \begin{bmatrix} \tilde{M}_2 & -\tilde{N}_2 \end{bmatrix} \right\|_\infty (z), \tag{33}$$

where $\left\{ M, N, \tilde{M}, \tilde{N} \right\}$ are the stable representations described by Eqs. (32) - (31).

(Li and Ding, 2020) presents two definitions of the SBM gap metric. One is equivalent to Giorgiou's gap metric, and the second is equivalent to the $\nu$-gap metric. This second metric is called the $L_2$ kernel gap metric:

$$\delta_{L_2}(\bar{K}_1, \bar{K}_2)(z) = \left\| \tilde{M}_1 N_2 - \tilde{N}_1 M_2 \right\|_\infty (z) \tag{34}$$

An important remark is that for SISO systems, the T-gap and Giorgiou's gap are equivalent. A restriction from Eqs. (33) - (34) is that the gaps are calculated for the worst-case scenario by assuming the infinity norm.

An inverse z-transform of $L_2(\bar{K}_1, \bar{K}_2)(z)$ gives a discrete gap metric:

$$iztrans\left(L_2(\bar{K}_1, \bar{K}_2)(z)\right) := \delta(k,k)\; L_2(K_1, K_2) = \delta_{\zeta,k}\left(K_1, K_2\right)(k), \tag{35}$$

where $k$ is a fixed number of sampling data and $\delta(k,k)$ is the Kronecker delta and:

$$\delta_{\zeta,k\to\infty}(.) \to \delta_{\nu}(.) \tag{36}$$

To obtain an adequate version of Eq. (35) in the time domain, the following hypotheses are assumed:

- The time spent in the sampling process is negligible
- The Analog/Digital converter resolution is big enough to assume:

$$TF_{analog}(k) = TF_{digital}(kT) \tag{37}$$

Equation. (35) gives the kernel gap metric in the time domain with $kT = \Delta t$, based on the Empirical Transfer Function Estimate (ETFE). Similarly to Eq. (35), $b_{k\to\infty}(.) \to\; b(.)$ and $b_{opt,\; k\to\infty}(.) \to\; b_{opt}$ follows from the same procedure.

## 4. CONCLUSION

This tutorial has provided a comprehensive overview of linear gap metrics and their applications in linearized mechanical and electrical systems represented by strictly proper first and second-order transfer functions. On a timeline, we have explored the concept of linear gaps, their importance in measuring the dynamic distance between systems, and the different types of linear gap metrics commonly used. We have demonstrated how these metrics can be implemented and interpreted through numerical examples and inferences. The robust stability guarantee theorem is the main mathematical tool used to solve control problems, and therefore it depends intrinsically on stability margin thresholds. Future work is needed to use the analytical gap metrics in more engineering problems and extend the linear gap metrics to measure the distance between nonlinear systems.

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