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Dynamical analysis of vibrations with nonlinear damping using the Krylov-Bogoliubov method

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Collapsing vibrating mechanical systems due to oscillatory phenomena is a critical concern in mechanical design. One of the main causes of collapse in vibrating mechanical systems is the so-called fluttering, mathematically formulated by limit cycles, which linear models are not able to describe. The Van der Pol equation (VdP) is an example of a model that satisfactorily describes vibrations with nonlinear damping. However, it is known that this type of equation does not have an analytical solution, making the proper analysis more difficult. In this sense, the Krylov-Bogoliubov (K-B) method presents an approximate solution for certain differential equations, the VdP can be included, which contains dynamic information relevant to the model, and could indicate the characteristic of limit cycles. Therefore, this project seeks to analyze the dynamics of nonlinear vibrations modeled by Van der Pol equation using the K-B method, comparing the solutions with those obtained numerically. In this case, the system was linearized, based on the conditions required by the Hartman-Grobman theorem, which guarantees the validity of the linearization of dynamical systems with respect to hyperbolic points. Finally, the K-B method and the refined K-B method were used to find expressions for the unknown function VdP.

Keywords: *nonlinear vibrations, Krylov-Bogoliubov, Van der Pol oscillator, Limit cycle oscillation, modeling of dynamical systems.*

1. INTRODUCTION

One of the most significant fields of study in engineering is mechanical vibrations, as almost all structures and machines experience some form of vibration at various levels. The mathematical modeling of oscillatory phenomena is usually done through differential equations, which can be characterized by different reasons. In particular, it is possible to classify oscillatory systems as linear and nonlinear, depending on the forms of the equations that model them. Linear systems have a well-developed theory and features that facilitate their analysis. On the other hand, the study of nonlinear systems demands more sophisticated techniques (Thomson, 2018).

Among the most relevant equations that model nonlinear systems for the study of vibrations is the Van der Pol (VdP) equation (Marios, 2006). The dynamics of several systems that have nonlinear damping can be modeled by this equation. However, Boyce and DiPrima (1985) explain that usually nonlinear differential equations do not have analytical solutions, a fact that makes it necessary to use methods of approximated solutions in their analysis. An effective method for comprehending the behavior of this type of system is the Krylov-Bogoliubov method (Krylov and Bogoliubov, 1950), also called harmonic balance or averaging method. With this method, one may obtain approximate solutions for nonlinear equations, without changing their dynamic properties in the frequency domain.

2. OBJECTIVES

The main objective of this work is to analyze the vibration dynamics of nonlinear systems modeled by the Van der Pol equation using the Krylov-Bogoliubov method.

In order to achieve this, it is necessary to achieve specific objectives such as:

- To study the Van der Pol equation and other second-order non-linear differential equations that describe systems;
- To generate numerical solution of the Van der Pol equation.

- To develop the equations of the Krylov-Bogoliubov method.
- To use the K-B method to solve the Van der Pol equation.
- To perform numerical simulations of the equation for the K-B method.
- To compare the results obtained by numerical methods with those obtained by the Krylov-Bogoliubov method

3. METHODOLOGY

This work aims to analyze the dynamics of vibrations with nonlinear damping modeled by the Van der Pol equation using the Krylov-Bogoliubov method. In order to achieve this, firstly, the Hartman-Grobman theorem is used to analyze the system without the need to obtain a proper solution. Next, there is a search for particular solutions, which are obtained using the fourth-order Runge-Kutta method. These results will serve as a basis for comparison between different approaches.

3.1 Models for nonlinear vibrations

During the development of machines and structures, prediction about the dynamic behavior of the system is of fundamental importance, since unforeseen dynamic responses can lead to destructive vibration rates. Therefore, a good vibration analysis must be carried out to obtain a robust and reliable result as explained **?**. Analysis from the point of view of classical dynamics was considered satisfactory for a long time and continues to depend on the case studied. However, classical theory is based on linear models which do not present sufficient results when dealing with complex real systems, given that nature is essentially nonlinear, Lima (2008). As an alternative to this type of problem, several dynamic models are proposed, in particular we can deal with three particular cases, those that deal with parametric vibrations, nonlinear systems without and with nonlinear damping.

When at least one parameter of a system undergoes a relative change over time, it is said that this structural system is under the action of a parametric excitation. Mathematically, parametric excitations are characterized by the presence of inhomogeneity in the equations of motion, represented by a temporal variation in some of the parameters of the equation. An equation referring to this case is the Mathieu-Hill equation, shown in Eq. (1), which can have its variations in the equation parameters imposed both externally and in a self-excited manner, as said by Beraldo (2019).

$$m(t)\frac{d^2x}{dt^2} + c(t)\frac{dx}{dt} + k(t)x = 0, \tag{1}$$

where $m(t)$ represents the mass of the system as a function of time, $c(t)$ the undeservingness of the system as a function of time and $k(t)$ the stiffness of the system as a function of time. An interesting point is that cases of parametric vibrations only occur when it is removed from the equilibrium position.

When dealing with a nonlinear system, but with linear damping, one of the best-known models is the Duffing equation, shown in Eq. (2), explains Siqueira and Saldanha (2019). ω_n is defined as the natural frequency, c the damping, α the nonlinearity term and F the forcing term. It is also common to study cases in which damping is negligible using this model.

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \omega_n^2 x \pm \alpha x^3 = Fcos(\omega t), \tag{2}$$

For situations in which the modeled vibrations present nonlinearity, the Van der Pol equation presents itself as one of the main alternatives. The unforced Van der Pol equation is a nonlinear second order differential equation whose form is presented by Eq. (3). This equation was initially proposed by Balthasar van der Pol in 1927 to model electrical engineering problems (Chura, 2019). Therefore, the parameter μ is originally related to characteristics of electrical circuits.

$$\frac{d^2x}{dt^2} + \mu(1-x^2)\frac{dx}{dt} + x = 0, \tag{3}$$

Still, an analogy of this model can be considered for a dynamic mass-spring-damper system, whose nonlinear term is related to damping. This fact is especially interesting, since for values of $\mu > 0$ it represents a negative damping for cases in which $|x| < 1$. It is also clear that this oscillator returns to a simple harmonic system if $x = 0$. Given the particular characteristics and wide range of applications that the Van der Pol model has, it will be used as the basis for all studies carried out in this article.

As a starting point for the initial analysis of the VdP, a methodology was chosen that allows the projection of the general behavior of the solutions. Therefore, it is necessary to define a hyperbolic equilibrium point and state the Hartman-Grobman theorem.

Definition 1 *Let S be a dynamic system described by* $\dot{\vec{x}} = f(\vec{x}), \vec{x}(0) = \vec{x}_0$*, for some continuously differentiable map* $f : \mathbb{R}^n \Rightarrow \mathbb{R}^n$. $\vec{\mathbf{x}_P}$ *is a equilibrium point if and only if* $f(\vec{\mathbf{x}}_P) = \vec{0}$. $\vec{\mathbf{x}}_P$ *will be called hyperbolic equilibrium point if and only if the Jacobian Matrix in this point,* $J(\vec{\mathbf{x}}_P) = [\partial f_i / \partial x_{P,j}]$*, is no singular and has no eigenvalues with real part equal to zero.*

Theorem 1 ***(Hartman-Grobman)*** *Let* $\vec{\mathbf{x}}_P \in \mathbb{R}^n$ *be a hyperbolic equilibrium point of a dynamic system S described by* $\dot{\vec{x}} = f(\vec{x}), \vec{x}(0) = \vec{x}_0$ *for some continuously differentiable map* $f : \mathbb{R}^n \Rightarrow \mathbb{R}^n$*. Then, there is a open neighborhood* $U \subset \mathbb{R}^n$ *of* $\vec{\mathbf{x}}_P$ *and a homeomorphism* $h : U \Rightarrow \mathbb{R}^n$ *such that* $h(\vec{\mathbf{x}}_P) = \vec{0}$ *and such that in the neighborhood U, S is topologically orbitally equivalent to the linearized system* $\dot{\vec{x}} = A\vec{x}, \vec{x}(0) = \vec{x}_0$*, where* $\vec{x} = [x_1\ x_2\ \ldots\ x_n]^T \in \mathbb{R}^n$ *and* $A = (\vec{\mathbf{x}}_P)$ *is the Jacobian matrix of the map f (and the system) applied at the equilibrium point* $\mathbf{x}_P$.

In other words, the Hartman-Grobman Theorem states that, in the vicinity of a hyperbolic equilibrium point, a nonlinear dynamic system exhibits behavior qualitatively equivalent to that of the corresponding linear system. If there is any eigenvalue of the Jacobian matrix with a zero real part, this theorem can not assert anything about the equivalence.

If one starts from the Van der Pol equation in its simplest form, the first step was to find the equilibrium points of the system. In this case, the change of variables shown in Eq. (4) was made, with the intention of transforming Eq. (3) into a system of two 1st order equations.

$$\begin{aligned} y_1 &= x, \\ y_2 &= \dot{x}. \end{aligned} \tag{4}$$

After this change of variables, the following system obtained is:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= \mu(1 - y_1^2)y_2 - y_1. \end{aligned} \tag{5}$$

This system of two first-order differential equations, represented by Eqs. (5) is equivalent to Eq. (3). In this way, to find the equilibrium points of the system, it is enough to equate each equation in (5) to zero and isolate y_1 and y_2.

$$\begin{aligned} \dot{y}_1 &= 0 \Rightarrow y_2 = 0, \\ \dot{y}_2 &= 0 \Rightarrow \mu(1 - y_1^2)y_2 - y_1 = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} y_1 &= 0, \\ y_2 &= 0. \end{aligned} \tag{7}$$

In this case, the only equilibrium point found was the point (0, 0). The next step was to calculate the Jacobian matrix of the system represented by Eqs. (5) and apply the equilibrium point to the values of y_1 and y_2.

$$J = \begin{bmatrix} 0 & 1 \\ -1 - 2\mu y_1 y_2 & -\mu(y_1^2 - 1) \end{bmatrix}, \tag{8}$$

Applying the equilibrium point (0,0) in the Jacobian, we have:

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}. \tag{9}$$

Note that the determinant of the matrix $J(0,0) > 0$, therefore, already satisfies one of the conditions of the Hartman-Grobman theorem. To check the second condition it is necessary to calculate the eigenvalues of the matrix $J(0,0)$, which result in the values expressed in Eq. (10):

$$\lambda_{1,2} = \frac{\mu}{2} \pm \frac{\sqrt{(\mu^2 - 4)}}{2}. \tag{10}$$

Therefore, for the real part of λ to be different from zero, it is necessary that $\mu \neq 0$. In this way, one can apply the Hartman-Grobman theorem to the Van der Pol equation and then study the behavior of the system solutions for different values of μ and predict their critical points depending only on the value of their eigenvalues.

The classification of the equilibrium point of a linear system can be done according to the topology and stability of its phase plane. These classifications are given according to the signs of their eigenvalues, being better expressed using the trace T and the determinant Δ of the matrix that represents the Monteiro (2002) system. Linear systems can be classified as:

- If Δ <0, then λ_1 and λ_2 are real numbers with opposite signs. In this case, the equilibrium point will be called the saddle.
- If $\Delta > 0$ with $T^2 - 4\Delta > 0$, then λ_1 and λ_2 are real numbers with the same sign. For $T > 0$ the equilibrium point is an unstable node, whereas for $T < 0$ it is an asymptotically stable node.
- if $\Delta > 0$ with $T^2 - 4\Delta < 0$, then λ_1 and λ_2 are conjugate complex numbers. In this case, if $T > 0$ the equilibrium point is called an unstable focus, if $T < 0$ it is an asymptotically stable focus and if $T = 0$, a stable or neutrally stable center.

According to these classifications, the Van der Pol equation presents four types of critical points, they are:

a) If $\mu \leq -2$, then $\lambda_1, \lambda_2 \in \mathbb{R}$: Asymptotically stable node;

b) If $-2 < \mu < 0$, then $\lambda_1, \lambda_2 \in \mathbb{C}$: Asymptotically stable focus;

c) If $0 > \mu > 2$, then $\lambda_1, \lambda_2 \in \mathbb{C}$: Unstable focus;

d) If $\mu \geq 2$, then $\lambda_1, \lambda_2 \in \mathbb{R}$: Unstable node.

3.2 Solution for numerical methods

Another possible approach is to directly search for approximate solutions to the equation. In this case, numerical methods are the most used. In this work, the fourth-order Runge-Kutta (R-K) method was implemented to obtain numerical solutions for the VdP, which will serve as a basis for comparison. Developing the equations of the Runge-Kutta method for the Van der Pol equation, we have:

For y_1:

$$\begin{aligned} k_1 &= y_2(i), \\ k_2 &= y_2(i) + \frac{h}{2}k_1, \\ k_3 &= y_2(i) + \frac{h}{2}k_2, \\ k_4 &= y_2(i) + hk_3, \end{aligned} \tag{11}$$

$$y_1(i+1) = y_1(i) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{12}$$

For y_2:

$$\begin{aligned} k_1 &= \mu\left(1 - y_1^2(i)\right) y_2(i) - y_1(i), \\ k_2 &= \mu\left(1 - \left(y_1(i) + \frac{h}{2}k_1\right)^2\right)\left(y_2(i) + \frac{h}{2}k_1\right) - \left(y_1(i) + \frac{h}{2}k_1\right), \\ k_3 &= \mu\left(1 - \left(y_1(i) + \frac{h}{2}k_2\right)^2\right)\left(y_2(i) + \frac{h}{2}k_2\right) - \left(y_1(i) + \frac{h}{2}k_2\right), \\ k_4 &= \mu\left(1 - (y_1(i) + hk_3)^2\right)(y_2(i) + hk_3) - (y_1(i) + hk_3), \end{aligned} \tag{13}$$

$$\begin{aligned} y_2(i+1) &= y_2(i) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ t(i+1) &= t(i) + h. \end{aligned} \tag{14}$$

This development allows the use of software for the graphical representation of particular solutions and, consequently, the VdP phase plan, depending only on the definition of a value for μ and initial conditions.

3.3 Krylov-Bogolyubov method

The Krylov-Bogoliubov method (K-B) Krylov and Bogoliubov (1950) is a method that allows obtaining approximate solutions for a class of nonlinear differential equations called quasi-harmonic autonomous systems. The equations belonging to this class are characterized by the following form:

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right), \tag{15}$$

where ε is a positive number, but $\varepsilon << 1$, ω is an arbitrary constant and f is a function that depends on x and $\frac{dx}{dt}$. The K-B method then proposes that equations similar to Eq. 15 have an approximate solution described by:

$$x = a(t)sen(\omega t + \theta(t)), \tag{16}$$

provided that the functions $a(t)$ and $\theta(t)$ vary slowly with time and that their respective derivatives $\dfrac{da}{dt}$ and $\dfrac{d\theta}{dt}$ are proportional to the parameter ε. The functions $a(t)$ and $\theta(t)$ can be obtained from the following system:

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{2\omega\pi}\int_0^{2\pi} f(asen(\theta), a\omega cos(\theta))cos(\theta)\,d\theta\ , \\ \frac{d\theta}{dt} &= \frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(asen(\theta), a\omega cos(\theta))sen(\theta)\,d\theta\ . \end{aligned} \tag{17}$$

The Van der Pol equation is part of the 15 group of equations, so the K-B method can be applied to it. In this way, it is noted that

$$f\left(x, \frac{dx}{dt}\right) = (x^2 - 1)\frac{dx}{dt}. \tag{18}$$

Considering $x = a(t)sen(t + \theta(t))$ as the method proposes, when deriving x with respect to time we obtain:

$$\frac{dx}{dt} = \frac{da}{dt}(t)sen(t + \theta(t)) + a(t)cos(t + \theta(t))\left(1 + \frac{d\theta}{dt}(t)\right). \tag{19}$$

In order to be able to apply the K-B method, the functions $a(t)$ and $\theta(t)$ must vary slowly with time, this means that, as an approximation, $\dfrac{da}{dt}(t)$ and $\dfrac{d\theta}{dt}(t)$ can be considered zero. This implies that:

$$\frac{dx}{dt} = a(t)cos(t + \theta(t)). \tag{20}$$

So, replacing the values of x and $\dfrac{dx}{dt}$ by $a(t)sen(t + \theta(t))$ and $acos(t + \theta)$ in the function f and applying these values in Eq. 17 we obtain the system 21.

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{2\pi}\int_0^{2\pi} (a^2 sen(\theta)^2 - 1)acos(\theta)cos(\theta)\,d\theta\ , \\ \frac{d\theta}{dt} &= -\frac{\varepsilon}{2\pi a}\int_0^{2\pi} (a^2 sen(\theta)^2 - 1)acos(\theta)sen(\theta)\,d\theta\ . \end{aligned} \tag{21}$$

Solving the integrals present in Eqs. 21 we obtain the system of Eq. 22. The solution to $\dfrac{da}{dt}$ can be obtained by simple techniques for solving ordinary differential equations, such as by separation of variables. The solution of $\dfrac{d\theta}{dt}$ is trivial. The solutions for both equations are explained in the system represented by Eq. 23, where a_o and θ_o are constants that depend on the initial conditions of the equation.

$$\begin{aligned} \frac{da}{dt} &= \frac{\mu a}{2}(1 - \frac{a^2}{4}), \\ \frac{d\theta}{dt} &= 0. \end{aligned} \tag{22}$$

$$\begin{aligned} a^2 &= \frac{a_o^2 e^{\mu t}}{1 + \frac{1}{4}a_o^2(e^{\mu t} - 1)}, \\ \theta &= \theta_o, \end{aligned} \tag{23}$$

Therefore, a first approximation to the solution of Eq. 3 using the Krylov-Bogoliubov method is as follows:

$$x = \frac{a_o e^{\frac{\mu}{2}t}}{\sqrt{1 + \frac{1}{4}a_o^2(e^{\mu t} - 1)}} sen(t + \theta_o), \tag{24}$$

The amplitude $a(t)$ can be described by Eq. 25, considering that $k = \frac{4}{a_o^2} - 1$.

$$a(t) = \frac{2}{\sqrt{ke^{-\mu t} + 1}}. \tag{25}$$

This way of representing the amplitude is particularly interesting when analyzing the limit of $a(t)$ where $t \to \infty$.

$$\lim_{t\to\infty} \frac{2}{\sqrt{ke^{-\mu t} + 1}} = \frac{2}{\sqrt{k \cdot 0 + 1}} = 2, \tag{26}$$

this result implies that, for values of μ close to zero, the amplitude of the solutions of the Van der Pol equation tends to 2, regardless of the initial conditions of the system.

The solutions produced by the K-B method can be refined by adding a new term to Eq. 16 as presented by Krylov and Bogoliubov (1950). By the refined K-B method, class equations,

$$\frac{d^2x}{dt^2} + \omega^2 x + \varepsilon f(x)\frac{dx}{dt} = 0, \tag{27}$$

have approximate solutions of the form:

$$x = a(t)cos(\omega t + \theta) - \frac{\varepsilon}{\omega}\sum_{n=1}^{\infty} F_n(a)sen(n\omega t + n\theta)), \tag{28}$$

where θ is an arbitrary constant and the functions $a(t)$ and $F_n(a)$ can be obtained through the following equations:

$$F(x) = \int_0^x f(x)\,dx, \tag{29}$$

$$F(acos(\phi)) = \sum_{n=1}^{\infty} F_n(a)cos(n\phi), \tag{30}$$

$$\frac{da}{dt} = -\frac{\varepsilon}{2}F_1(a). \tag{31}$$

Note that the Van der Pol equation belongs to the form presented by the equation 27, therefore it is possible to apply the refined K-B method to VdP. To develop the method for the VdP, we must find the function $F_n(a)$, and for that we did,

$$\begin{aligned} f(x) &= x^2 - 1, \\ F(x) &= \int_0^x (x^2 - 1)\,dx = \frac{x^3}{3} - x, \end{aligned} \tag{32}$$

$$F(acos(\phi)) = \frac{a^3}{3}cos(\phi)^3 - acos(\phi) = \left(\frac{a^3}{4} - a\right)cos(\phi) + \frac{a^3}{12}cos(3\phi). \tag{33}$$

Comparing Eq. 33 with Eq. 30, one can see the following correlation:

$$\begin{aligned} F_1(a) &= a\left(\frac{a^2}{4} - 1\right), \\ F_3(a) &= \frac{a^3}{12}, \\ F_n(a) &= 0,\ \forall\, n \neq 1, 3. \end{aligned} \tag{34}$$

Consequently, $a(t)$ can be found by the relation 31 as follows:

$$\frac{da}{dt} = -\frac{\varepsilon}{2}F_1(a) = -\frac{\varepsilon}{2}a\left(\frac{a^2}{4} - 1\right), \tag{35}$$

This equation is the same one used to find $a(t)$ for the first approximation and has a known solution. Therefore, the approximate solution to the Van der Pol equation by the refined K-B method is:

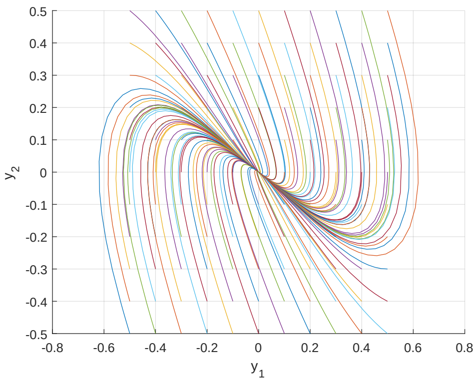
$$x = cos(t + \theta) - \frac{\mu a^3}{32}sen(3t + 3\theta). \tag{36}$$

4. Results

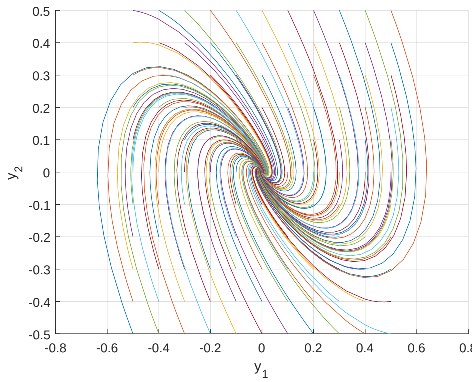
This section presents the phase plans obtained through the three tools analyzed. Firstly through the Runge-Kutta method, secondly through the K-B method and finally the refined K-B method. For the simulations referring to the fourth-order R-K method, different values for μ were tested in order to exemplify each of the possible regimes observed when using the Hartman-Grobman theorem. For the other simulations, values of μ were chosen in order to satisfy a necessary condition for the K-B method, that is, $0 < \mu \ll 1$.

4.1 Simulations by Runge-Kutta

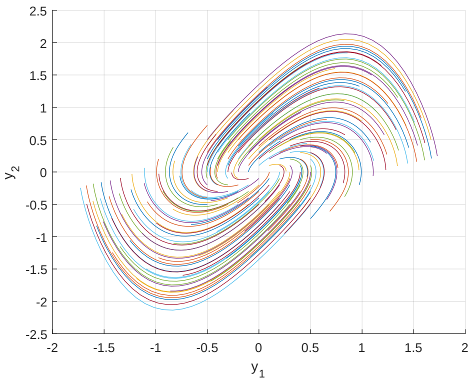
Figure 1(a) exemplifies the behavior of the phase portrait with $\mu = -2.2$, which presents an inappropriate node, due to its format in which the solutions tend to a limiting direction, with the exception of two from them. In particular, it is an asymptotically stable node, as solutions tend to the origin as time progresses. For $-2 < \mu < 0$, we have Fig. 1(b), which shows a critical point with $-2 < \mu < 0$ where the trajectories spiral towards the origin (equilibrium point) as time tends to infinity, characterizing an asymptotically stable focus. A similar behavior occurs at Fig. 1(c), however, instead of the solutions moving towards the origin, in this case where $0 > \mu > 2$, they move away from the equilibrium point , forming an unstable focus. Finally, Figure 1(d) exposes the phase portrait for μ =2.2, showing the behavior of an unstable node, in which the solutions tend towards the same direction and move away from the point of balance.



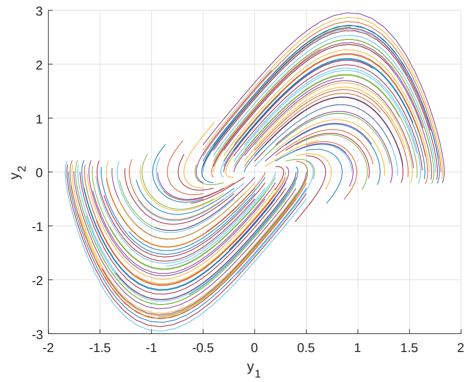
a) Case for $\mu = -2.2$.



b) Case for $\mu = -1.5$.



c) Case for μ =1.5.



d) Case for μ =2.2.

Figure 1: Fourth-order Runge-Kutta phase plan.

When one observes the phase portrait in a broader way than that covered by the Hartman-Grobman theorem, we notice an interesting behavior. Note that, for $\mu > 0$, the solutions of the Van der Pol equation tend to stabilize throughout a cycle according to $t \to \infty$, regardless of whether the trajectories start inside or outside this cycle, as it is easily seen in Fig. 2. This behavior characterizes the existence of a stable limit cycle, a result that is also commented on by /citeonlinepereira, for instance.

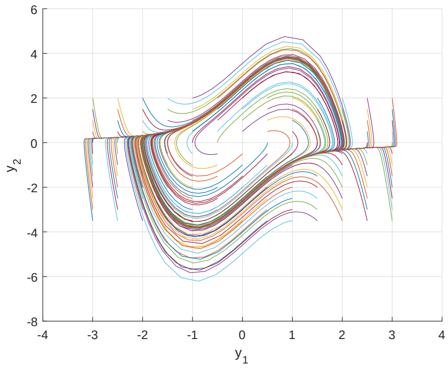
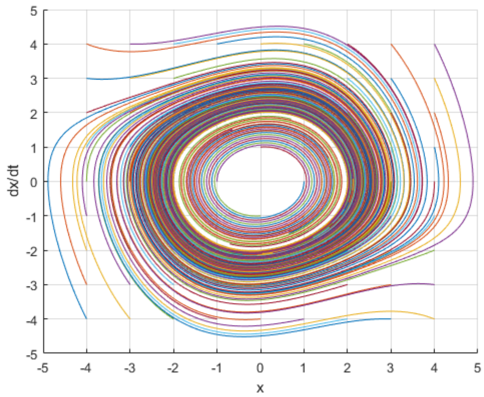


Figure 2: Phase portrait of the Van der Pol equation for $\mu = 2$.

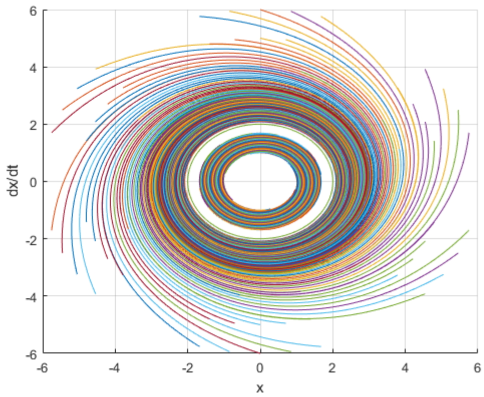
4.2 Simulations of the Krylov-Bogoliubov method

Based on the solution given by Eq. (16) presented in section 3.3 it is possible to obtain particular solutions of the VdP by defining the values of parameters a_o and θ_o. Thus, it is possible to compare the results obtained by the Runge-Kutta numerical method with those obtained by the Krylov-Bogoliubov method. In order to accomplish this, different parameters, as indicated in section 4. and initial conditions are chosen so that so that both solutions are equivalent.

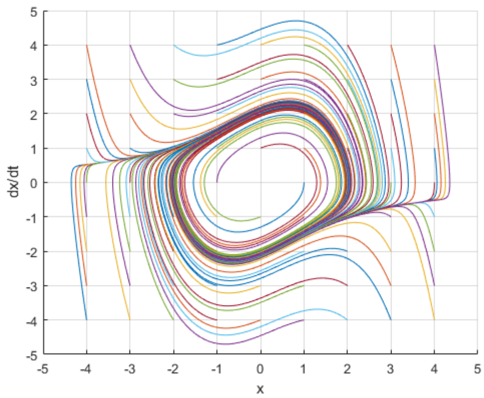
On the following simulations, the values used for μ were 0.1 and 0.5, due to the condition required by the method that ε (in this case μ) be a positive value, but much smaller than one.



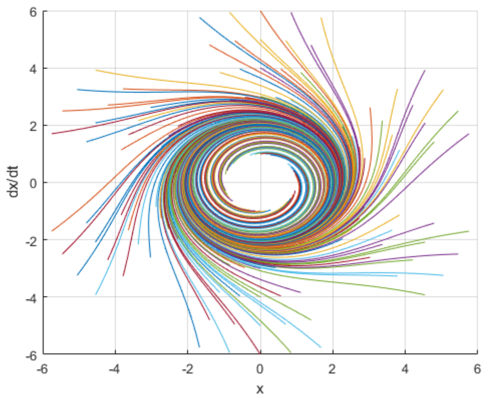
a) Phase plan by R-K for $\mu = 0.1$.



b) Phase plan by K-B for $\mu = 0.1$.



c) Phase plan by R-K for $\mu = 0.5$.



d) Phase plan by K-B for $\mu = 0.5$.

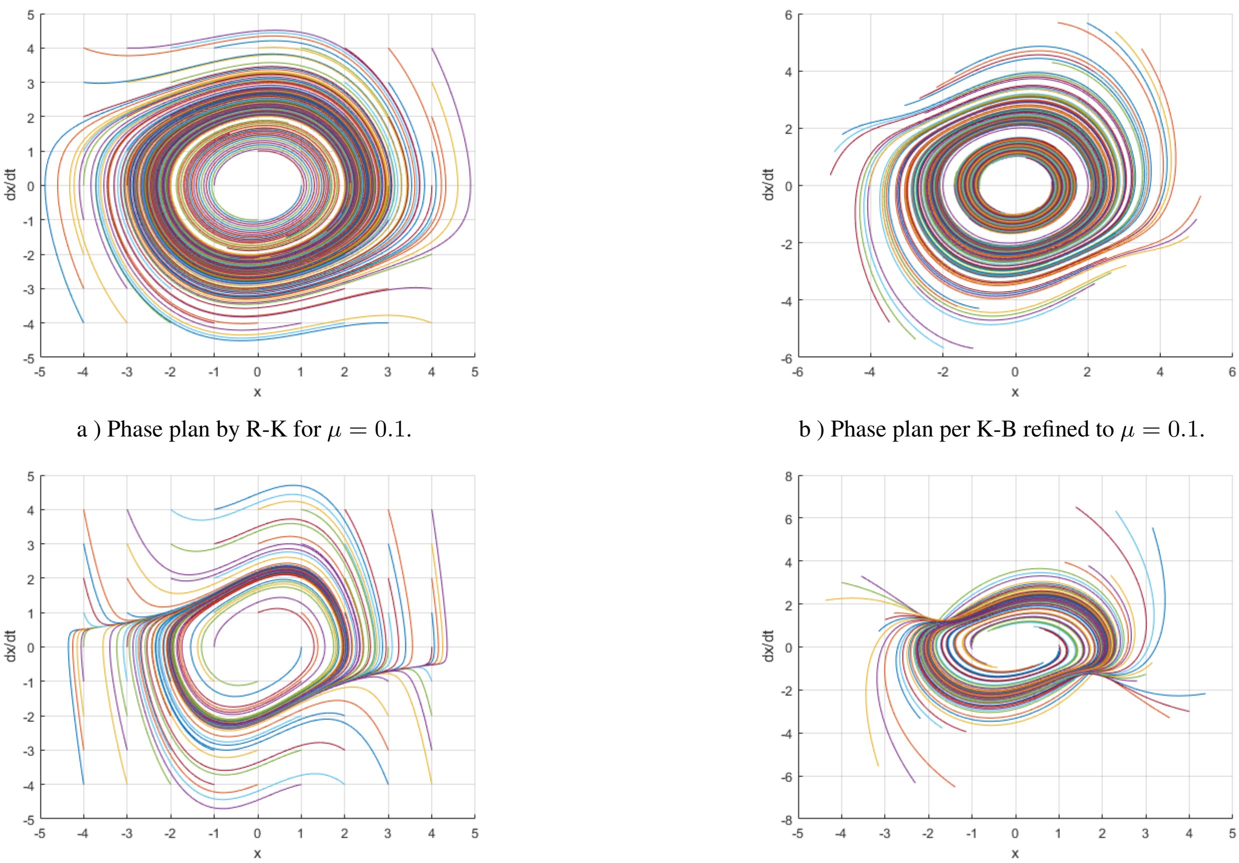
Figure 3: Comparison between the phase planes obtained by the R-K method and the K-B method.

When the phase planes presented in Fig. 3 are compared, it can be seen that there is a great similarity between the internal trajectories of the limit cycle, both presenting the shape of an unstable focus. For those that start outside this region, the behavior presented by the K-B method differs from that presented by the R-K method. This observation reveals that solutions obtained by the Krylov-Bogoliubov method tend to have a smaller error the closer the initial conditions are to the origin.

The value of μ also directly influences the error of the solutions produced by the Krylov-Bogoliubov method, due to the consideration that the method imposes that the value of μ is close to zero. Figs. 3(c) and 3(d) exemplify this characteristic, in which a considerably different behavior can be observed, mainly regarding the shape of the limit cycle.

4.3 Simulations of the refined Krylov-Bogoliubov

The dynamics of the system can be better represented, when the solution produced by the refined Krylov-Bogoliubov method, explained by Eq. (36) is used. Fig. 4 exemplifies this improvement, in which the trajectories that start in the region outside the limit cycle present a more adequate behavior with those presented by the R-K method when compared to those portrayed in Fig.3.



a) Phase plan by R-K for $\mu = 0.1$.

b) Phase plan per K-B refined to $\mu = 0.1$.

c) Phase plan by R-K for $\mu = 0.5$.

d) Phase plan per K-B refined to $\mu = 0.5$.

Figure 4: Comparison between the phase planes obtained by the R-K method and the refined K-B method.

For values of μ further from zero, the solutions generated by the refined method tend to have their behavior better represented, as can be seen in Fig. 4, in which the shape of the limit cycle and the trajectories in Figs. 4(c) and 4(d) that start external to it are better described when compared to those presented in Fig. 3.

An important characteristic of the dynamics of the Van der Pol equation is that, for values of μ close to zero, the oscillation amplitude of the solutions tends to 2 as $t \to \infty$, independently of values from initial conditions, as commented in the section 3.3This characteristic helps to highlight the existence of limit cycles for VdP within the conditions of the K-B method. Such behavior can be noticed in all simulations presented in this section, whether obtained by the numerical method or by K-B.

5. Conclusions

From what has been presented here, the exact linearization technique, that obeys the hypotheses of the Hartman-Grobman theorem, can be considered to be an efficient way of obtaining estimates regarding the shape of critical points and regarding the study of the stability of nonlinear oscillatory systems, such as Van der Pol's discovery. However, for a more in-depth study, some important behaviors for the system dynamics may go unnoticed due to the its limitation, which is only capable of approximating the real behavior close to the equilibrium point. This detail is illustrated, for example, by the existence of the VdP limit cycle presented in Fig. 2.

The algorithms generated to obtain particular solutions and to present the phase portrait of the solution proved to be effective for different values of μ or values of initial conditions y_1 and y_2, both of which can also be modified in terms of solutions and unique time intervals restricted by the processing capacity of the *hardware* that is executing the code.

The Krylov-Bogoliubov method proved to be quite efficient in approximating VdP solutions for different values of μ, and initial conditions close to zero, regarding a general behavior of the phase portrait. In comparison to the numerical methods, the results are quite similar, which also agrees with the Hartman-Grobman theorem.

For cases in which the value of μ and the initial conditions are far from zero, the trajectories disagree visibly for the simple and refined methods, however, depending on the need for precision in the analysis of models of this type, the method can be considered as a study alternative.

An advantage of the K-B method in relation to the numerical method is the low computational cost, due to the fact that K-B solutions can be simulated using a single expression, unlike methods such as Runge-Kutta, which normally require several consecutive iterations to obtain each solution point. Another advantage of the Krylov-Bogoliubov method is the evidence of the characteristics of the system studied more broadly.

The use of approximate methods has considerable relevance due to its power of dynamic analysis, in addition to making it possible to apply control tools for closed-loop systems that can be applied to a wide set of engineering problems, as explained in the works of Landau et al. (2008), in that of Alhejaili et al. (2023) and Salas et al. (2022). These purposes could not be achieved through numerical or purely topological methods.

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