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THIN FILM THEORY APPLIED TO MATHEMATICAL MODELING OF FLUIDYNAMIC BEARINGS

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Abstract. *The modeling of fluid dynamic bearings has been developed and used based on the classic Reynolds model, with which one can calculate the load that can be sustained and the position of the axle relative to a given load. This theory lends it'self to simulation in a permanent regime or even a pseudo transient, in the sense that the load is supplied and the position that is determined to obtain equilibrium. In the present work, a new proposal for modeling this dynamic system is presented, based on the thin film theory. The proposed model is one-dimensional for velocity, for pressure and for temperature. What is obtained is the average information in the radial direction, but a function of the tangential coordinate and of the time. The axle movement is transmitted to the fluid via modeling the stress between the fluid and the shaft and between the fluid and the bearing. This transient modeling was developed in the present work. The axis starts from rest and accelerates according to a generic function of time. In the proposed model the injected mass flow is incorporated, which results from the extension of the model of two-dimensional flows by having movement according to tangential and axial coordinates. In the present work, the details of the modeling will be presented, but the results are still related to tangential flows. Comparisons with the results of other methods are presented. The promise of this proposal is the modeling and simulation of dynamics of axle-bearing systems in transient regime. Low computational cost is also expected, since the model is one dimensional. The results to be presented are promising.*

1. INTRODUCTION

According to White (2022), lubrication plays a crucial role in reducing friction between closely contacting bodies. The use of a viscous fluid in a narrow and variable gap between the bodies allows for smooth sliding, minimizing friction and wear.

Newton (1687) formulated the laws of motion and gravity, establishing the foundations of classical physics. Although he did not directly address fluid mechanics, his laws are applied in the study of fluid behavior.

Significant contributions to applied mathematics in physics and engineering were made by Euler (1755). Euler formulated the conservation of mass equation and applied differential equations to analyze fluid flow.

Expanding on Euler's work, Navier (1822) considered the viscosity of fluids and formulated the Navier-Stokes equations, which describe the motion of viscous fluids, taking into account the viscous stress between fluid layers. Additionally, Stokes (1845) reformulated the Navier-Stokes equations for fluids with low Reynolds numbers. He developed the Incompressible Stokes Navier-Stokes Equations, which simplify the equations for laminar and incompressible flows of fluids with low velocities or viscosities.

The theoretical basis for hydrodynamic bearings was established by Reynolds (1886), who derived the Reynolds equation to describe the pressure profile between lubricated moving surfaces. Reynolds also explored the approximation for infinitely long bearings. This work was fundamental to advancements in the theory of hydrodynamic bearings and influenced the development of modeling and simulation techniques.

Although Sommerfeld (1904) obtained an explicit analytical expression for infinitely long bearings, their boundary conditions did not consider film rupture and exhibited negative pressure distribution in the divergent zone of the wedge. Swift (1932) and Stieber (1933) introduced new boundary conditions that represent the lubricant outlet, which are still used today for calculations of bearings with constant load. Ocvirk (1952) developed a detailed solution for infinitely short bearings, which is still widely used. Pinkus (2022) was the first to use computers for numerical calculations of pressure

in circular, elliptical, and lobed bearings.

The basic concepts of hydrodynamic bearings are addressed in the works of Hamrock (1994), J Frene (1997), and more recently, Ishida and Yamamoto (2013). These works utilize approximate equations for both short and infinitely long bearings.

A more precise modeling of bearing geometry is proposed by Mota *et al.* (2022).

This article presents a new and more effective proposal for solving the pressure and velocity fields in bearings. The finite difference method, proposed by Cauchy (1829) [1], will be used to discretize the equations of mass conservation and linear momentum, allowing for a more detailed and transient description of the problem. The obtained solution will be compared with the classical Reynolds model. The proposed computational model will be analyzed for cylindrical bearings with predetermined attitude angle and journal eccentricity.

The present work is divided into different sections. Section 2 presents the assumptions (simplifications) of the problem, as well as the differential mathematical modeling, including the continuity equation and the equation of linear momentum balance. Section 3 performs the discretization to obtain the pressure field and velocity field using the finite difference method. The results and the respective method comparisons are presented in Sec. 4. Finally, the conclusions are presented in Sec. 5.

2. DIFFERENTIAL MATHEMATICAL MODEL

This article provides an analysis of the fluid flow dynamics in the annular space between two cylinders. While the main focus is on hydrodynamic bearings, the ideas discussed here can be extrapolated to other applications, such as seals. Figure 1 schematically illustrates the geometries and configuration of an eccentric bearing.
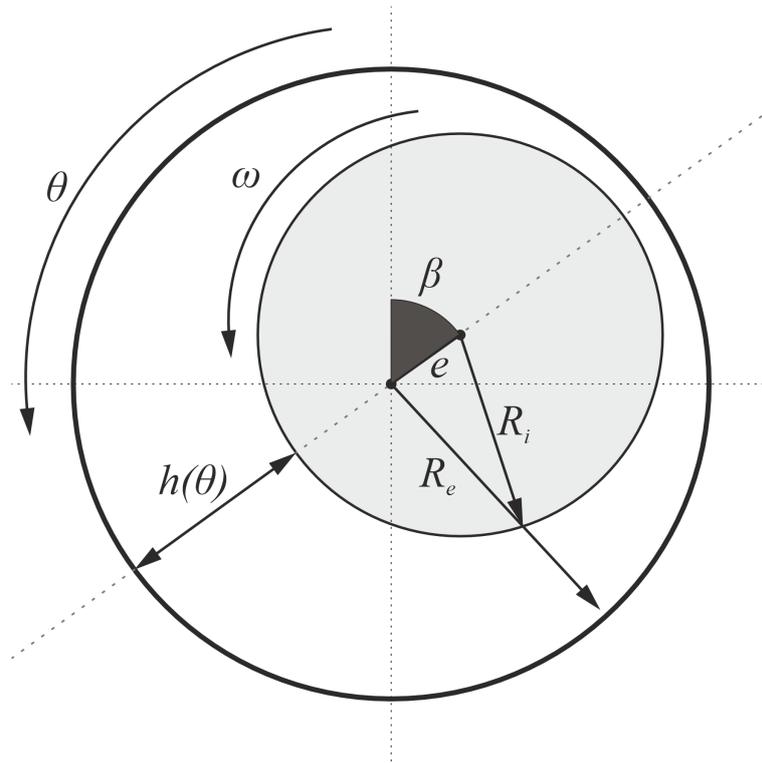

Figure 1. Schematic diagram of a bearing with an eccentric shaft relative to the bearing.

Reynolds describes the distribution of angular pressure between two cylindrical bodies as a function of the variation of the gap $h(\theta)$, the journal radius R_i, the fluid viscosity μ, and the rotational speed of the journal ω. Equation 1 shows how these terms are related:

$$\frac{\partial}{\partial \theta}\left(h(\theta)^3 \frac{\partial P}{\partial \theta}\right) = 6\mu\omega R_i^2 \frac{\partial h(\theta)}{\partial \theta}. \tag{1}$$

The article by Mota *et al.* (2022) also describes an equation for the variation of angular velocity along the radius of the bearing. The Reynolds model has a significant limitation imposed by the Reynolds number of the operating system, which

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[1]Although credited as the creator of the method, earlier works such as Takakazu (1674) used similar methods.

must be such that the modified Reynolds number (White, 2022) is $Re^* \ll 1$. Although it has not been evaluated, it is estimated that the new proposal presented in this article also overcomes this limitation.

2.1 Assumptions

The standard modeling describes angular velocity as a two-dimensional component $(v_\theta(r, \theta))$. This study proposes that this velocity profile can be calculated as an average along the radius of the bearing, since the gap between the journal and the bearing is very small. Thus, the angular velocity becomes an average over r and becomes one-dimensional, varying only in θ $(\bar{v}_\theta(\theta))$.

This study does not consider the weight force exerted by the journal on the oil film, which means that the calculated pressure gradient is solely caused by the narrowing of the gap. This pressure, similar to what Reynolds calculated, does not vary along the radial component of the bearing. The gap h varies along the angular component; however, the variation is zero over time.

2.2 Balance equations

The proposed modeling approach relies on two balance equations: the equation of mass conservation and the equation of linear momentum balance. Both equations are derived using the concept of series expansion introduced by Taylor (1715).

Based on the assumptions and concepts established by the prominent figures mentioned in Sec. 1, we can write the simplified equations for mass conservation and linear momentum balance for an incompressible fluid with a specific mass ρ. These equations are presented as Eq. 2 and Eq. 3, respectively:

$$\frac{\partial}{\partial \theta}\left(h\bar{v}_\theta\right) = 0, \tag{2}$$

and

$$\frac{\partial \bar{v}_\theta}{\partial t} + \frac{2}{2R_e - h}\bar{v}_\theta\frac{\partial \bar{v}_\theta}{\partial \theta} = \frac{2h}{\rho(2R_e - h)}\left(-\frac{\partial hP}{\partial \theta} + \frac{\partial h\tau_{\theta\theta}}{\partial \theta}\right) + \frac{F_{ext}}{\rho}. \tag{3}$$

The Navier-Stokes equations include terms for viscous stresses. The term $\tau_{\theta\theta}$ represents the normal stresses, also known as "Stokes stresses". It is calculated as the pointwise variation of velocity multiplied by the fluid viscosity μ and divided by the evaluated value of r, as shown in Eq. 4:

$$\tau_{\theta\theta} = \frac{2\mu}{R_e - h/2}\frac{\partial \bar{v}_\theta}{\partial \theta}. \tag{4}$$

The Navier-Stokes equations also include a term for tangential stress $\tau_{r\theta}$, which depends on the variation of angular velocity along r. Knowing the velocities of the journal and the bearing wall, we have the boundary conditions for determining the velocity. The present work does not use discretization in the r direction, so these stresses must be modeled in a non-conventional way to impose the no-slip conditions on the system. The term F_{ext} in Eq. 3 models these stresses to satisfy the same conditions. Equation 5 uses an analogy to the conventional modeling used for determining stresses on bearing walls. It represents the difference in stresses applied to both walls, adjusted by a function α dependent on the eccentricity factor ϵ:

$$F_{ext} = \alpha(\epsilon)\left(\tau_{r\theta}\big|_{r=R_e} - \tau_{r\theta}\big|_{r=R_e-h}\right) = \alpha(\epsilon)\mu\left(\left(\frac{\partial v_\theta}{\partial r} - \frac{\bar{v}_\theta}{r}\right)\Bigg|_{r=R_e} - \left(\frac{\partial v_\theta}{\partial r} - \frac{\bar{v}_\theta}{r}\right)\Bigg|_{r=R_e-h}\right). \tag{5}$$

Equation 6 provides a clearer visualization of the factor ϵ, which is the ratio between the eccentricity e and the radial clearance:

$$\epsilon = \frac{e}{R_e - R_i}. \tag{6}$$

The function $\alpha(\epsilon)$, shown in Eq. 7, was obtained empirically. Results obtained using a value of α equal to unity were compared with the results extracted from the Reynolds model (Eq. 1). The ratio between the results obtained in both solutions was obtained for a series of eccentricity configurations, and a fitting was performed using the Python programming language. Figure 2 graphically represents this fitting.
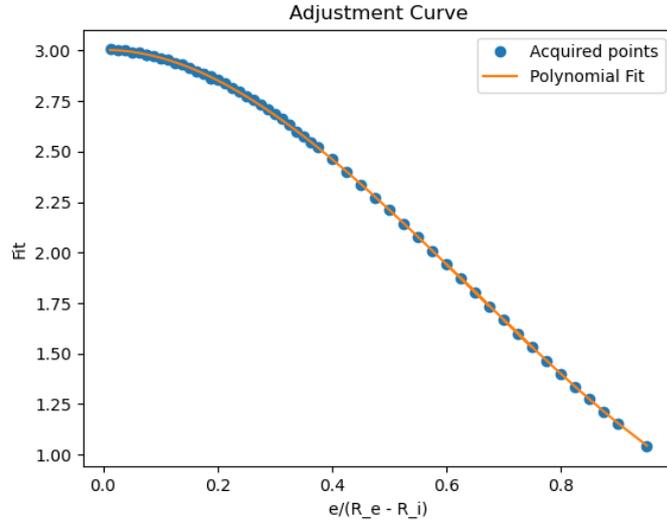
Figure 2. Polynomial fitting of $\alpha(\epsilon)$ function.

That way, the function $\alpha(\epsilon)$ can be calculated using the polynomial Eq. 7:

$$\alpha(\epsilon) = 2.2781\epsilon^3 - 4.3676\epsilon^2 + 0.0313\epsilon + 3.0032. \tag{7}$$

Finally, the function $h(\theta)$ represents the distance between any point R_θ (Mota *et al.*, 2022) on the surface of the journal and the surface of the fixed bearing. Therefore, the gap function can be calculated using Eq. 8:

$$h(\theta) = R_e - \left(e \sin \left(\theta + \beta \right) + \sqrt{R_i^2 - e^2 + e^2 \sin^2 \left(\theta + \beta \right)} \right). \tag{8}$$

3. DISCRETE MATHEMATICAL MODEL

Once the governing differential equations are defined, we discretize them in order to solve them computationally using the Fortran programming language. As mentioned earlier, this work differs from others by treating two dimensions as one, which reduces computational cost. Thus, the computational mesh is divided only along the θ component. Figure 3 provides a detailed illustration of the significant difference between the two methods.

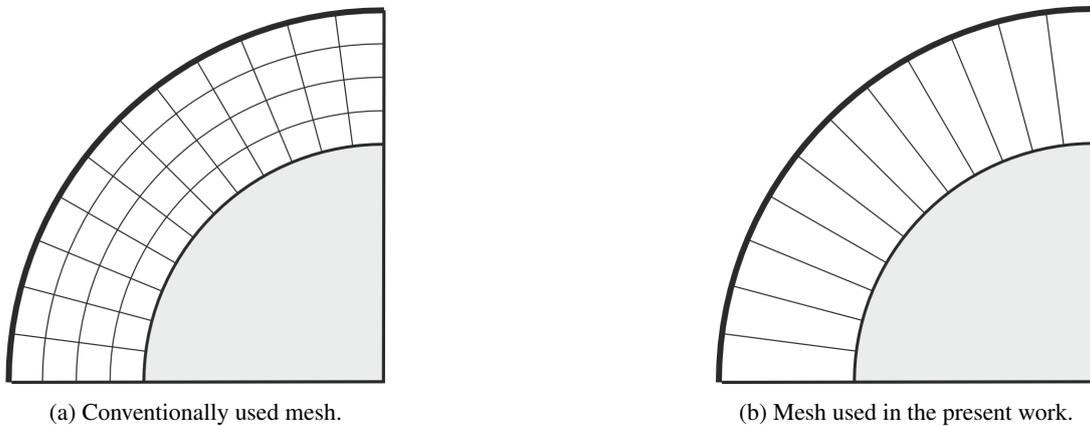(a) Conventionally used mesh.   (b) Mesh used in the present work.

Figure 3. Difference in used meshes.

Thus, applying the finite difference method, along with the concept of staggered grids (Harlow and Welch, 1965), for spatial discretization and the method by Euler (1768) for temporal discretization, we rewrite Eq. 3 in a discretized form:

$$\frac{\bar{v}_{\theta j}^n - \bar{v}_{\theta j}^{n-1}}{\Delta t} + \frac{2\bar{v}_{\theta j}^{n-1}}{2R_e - h_j} \frac{\bar{v}_{\theta j+1}^{n-1} - \bar{v}_{\theta j-1}^{n-1}}{2\Delta\theta}$$
$$= \frac{2h_j}{\rho(2R_e - h_j)} \left(-\left(\frac{h_i P_i^n - h_{i-1} P_{i-1}^n}{\Delta\theta} \right) + \left(\frac{h_i \tau_{\theta\theta i}^{n-1} - h_{i-1}\tau_{\theta\theta i-1}^{n-1}}{\Delta\theta} \right) \right) + \frac{F_{ext}^{n-1}}{\rho}, \tag{9}$$

where, according to Eq. 4, we have:

$$\tau_{\theta\theta i}^{n-1} = \frac{2\mu}{R_e - h_i/2} \frac{\bar{v}_{\theta j+1}^{n-1} - \bar{v}_{\theta j}^{n-1}}{\Delta\theta}, \tag{10}$$

and the term F_{ext} after some simplifications becomes:

$$F_{ext}^{n-1} = \alpha(\epsilon)\mu \left(\frac{2(\omega R_i - 2\bar{v}_{\theta j}^{n-1})}{h_j} - \frac{\bar{v}_{\theta j}^{n-1}}{R_e - h_j/4} + \frac{\bar{v}_{\theta j}^{n-1}}{R_e - 3h_j/4} \right), \tag{11}$$

where the subscripts i and j represent the position of the node in the grid (except for R_i), where i represents the position of a point where pressure is calculated and j represents a point where velocity is calculated, and $\Delta\theta$ is the size of the spatial division. Fig. 4 represents the scheme of the shifted grids. The superscripts n and $n-1$ represent the time iteration, where n is the iteration at the current time step when the variables are being calculated, and $n-1$ is the previous iteration where the values of the variables are already known. Another important point to highlight is the periodic nature of the system. That is, for the mesh shown in Fig. 4, the velocity at $j-1$ is equal to the velocity calculated at $j+2$.
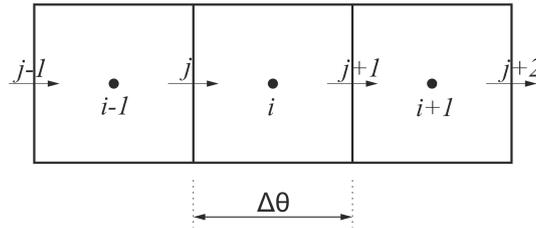

Figure 4. Nodal scheme of the mesh.

The equation involves two variables to be solved at time step n, which are interdependent. The `SIMPLE` method proposed by Patankar and Spalding (1972) was employed to obtain the velocity and pressure fields. For the efficient solution of the linear system, the `Bi-CGSTAB` method (Van der Vorst, 1992) was implemented.

4. RESULTS AND DISCUSSIONS

The proposed method in this study has shown to be convergent and stable. Within the limitations imposed by the Reynolds model ($Re^* << 1$), the full potential of the new model has not been fully explored. Several bearing configurations were simulated, including different speeds, eccentricities, clearances, and shaft dimensions, and compared to the conventional model. Figure 5 presents some comparisons between the pressure field results obtained using the proposed model. Table 1 presents the configurations used for each of the cases shown in Fig. 5.

Table 1. Configurations of the presented bearings.

Case	R_e [m]	R_i [m]	ϵ	ω [RPM]	β [°]	μ [Pa.s]	ρ [kg/m^3]	Re
Figure 5a	50.00E-3	48.00E-3	0.50	10000	90	1.0E2	1000.0	1.50E-2
Figure 5b	60.00E-3	55.00E-3	0.25	50000	90	1.7E2	500.0	9.57E-2
Figure 5c	60.00E-3	58.00E-3	0.75	50000	90	1.7E2	776.0	4.66E-2
Figure 5d	52.00E-3	51.00E-3	0.85	100	180	8.1E-2	914.0	6.43E-2

The results show that the two methods differ slightly, as expected, since the proposed method includes inertial forces in its equations along with empirically modeled stresses. New proposals for modeling these stresses will be discussed in the future to improve the convergence of the model.
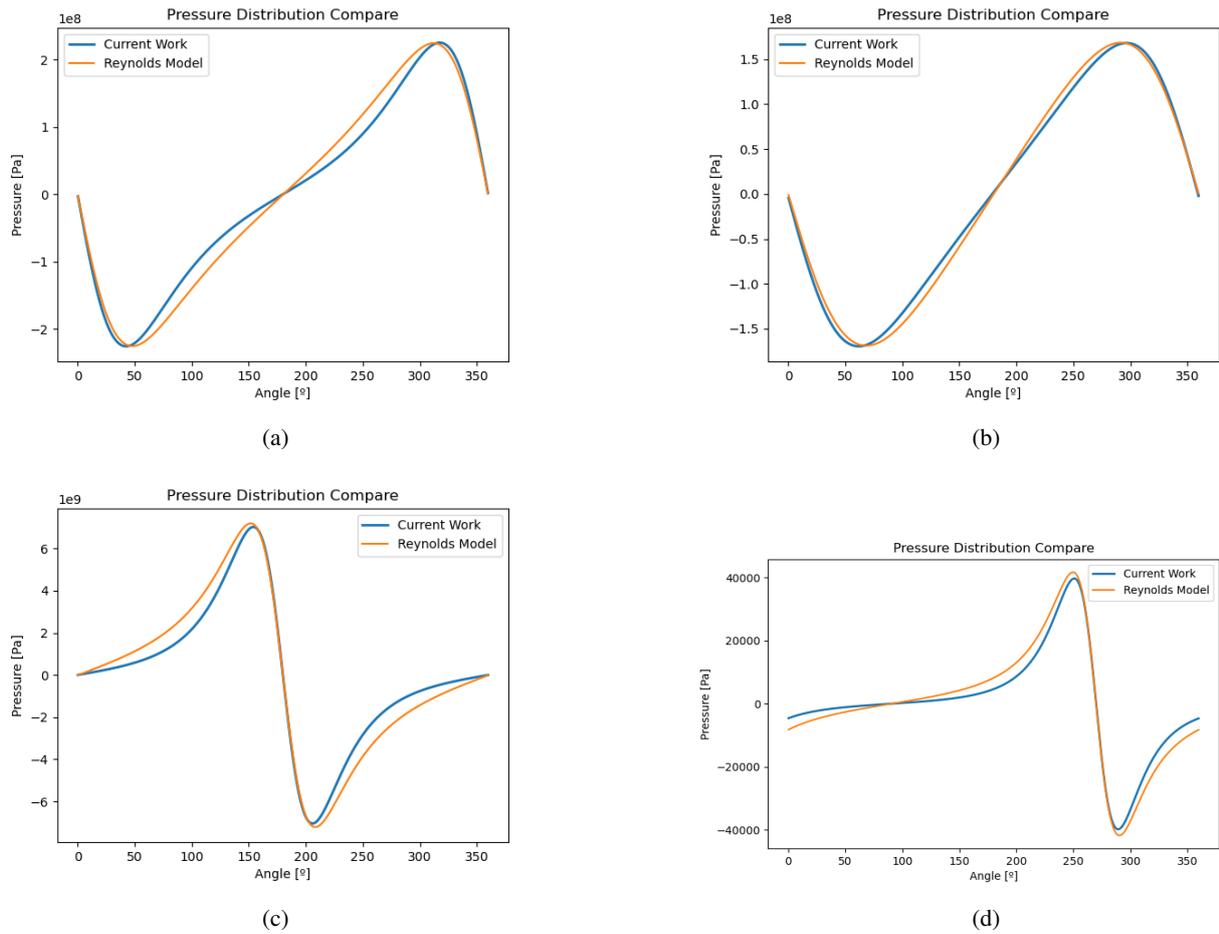
(a)

(b)

(c)

(d)

Figure 5. Pressure distribution for different configurations.

Figure 6 shows, although not validated, the profiles of average velocity ($\bar{v}_\theta$). The sequence of images is the same as presented in Fig. 5. The velocities are somewhat contradictory when evaluated with respect to Bernoulli's principle (Bernoulli, 1738), but it is important to note that Bernoulli's principle is valid only for inviscid flows, unlike the case evaluated here, which has $Re^* << 1$. In other words, viscous forces prevail over inertial forces.

The method was also evaluated in channels with variable height in the Cartesian plane, where velocity profiles were validated, yielding extremely satisfactory results. This indicates that the profiles shown in Fig. 6 are also correct.

5. CONCLUSION

The proposed method for modeling and solving the pressure and velocity fields in hydrodynamic bearings yields satisfactory results, considering the adopted simplifications and modeling approaches. The use of finite difference method, in conjunction with the concept of shifted grids, has proven to be efficient and rapidly convergent.

The simulation results showed good agreement with the conventional Reynolds model, highlighting the validity and potential of the new method. Comparative analysis between the two models revealed subtle differences, particularly due to the inclusion of inertial forces in the equations and the empirical modeling of stresses. It is suggested that future research focuses on optimizing the modeling of the F_{ext} term, which could lead to improved convergence of the model.

Furthermore, it is recommended to explore more stable and computationally efficient alternatives for solving the equations, such as the Runge-Kutta method (Runge and Kutta, 1901). This could further enhance the accuracy and effectiveness of the proposed method.

This pioneering work paves the way for applying the method to more complex problems, such as two-phase flows and complete modeling of bearings, considering the weight force of the shaft and transient motion towards equilibrium position. These potential extensions hold promise for advancing the field and warrant further investigations.

In summary, the results obtained so far indicate that the proposed method is a promising approach for the analysis of
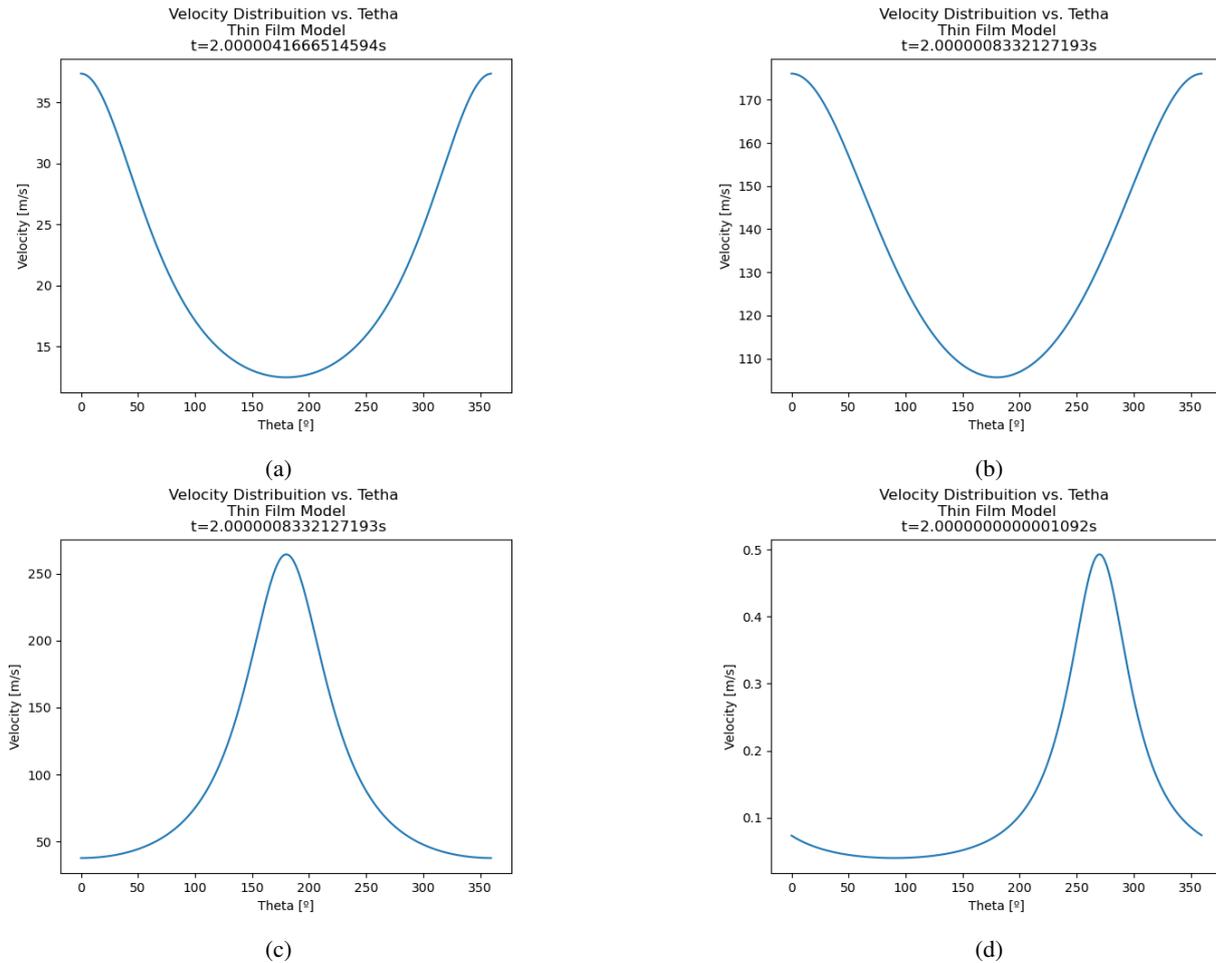
(a)

(b)

(c)

(d)

Figure 6. Velocity distribution for different configurations.

hydrodynamic bearings. However, additional research is needed to further refine and validate the model, aiming for its application in more challenging engineering scenarios.

6. ACKNOWLEDGEMENTS

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