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A NEW EQUATION FOR THE POINT CLOSEST TO THE ORIGIN ON AN AXIS: A REVIEW OF CHASLES' THEOREM

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Abstract. The screw motion and some important theorems for rigid body motion are reviewed and new equations for parametrizing rigid body displacement are provided. The antisymmetric 3×3 *matrix adjoint to a three dimensional real vector is used to produce a new equation for a point on the screw axis closest to the origin which completes, in a novel fashion, the proof of the Chasles' Theorem. The use of sophisticated algebraic concepts aims to inspire students on the high value of mathematics. The new result, the closest point to the origin on a general screw axis, is itself of great relevance in the study of kinematics.*

Keywords: Chasles' Theorem, antisymmetric matrix, instantaneous screw axis, homogeneous transformations, parameter

1 INTRODUCTION

The study of rigid body motion, either finite or infinitesimal, started to get the contemporary form back in the eighteenth century with the screw axis defined by Giulio Mozzi in 1763 (Ceccarelli, 2000). A very interesting historical discussion on the development of the rigid body motion foundations is presented by Hunt (1990, p. 49) and extended by Gibson and Hunt (1990). Central to that study is the Chasles' Theorem which, in contemporary terms, states that *all proper rigid body motions in 3-dimensional space, with the exception of pure translations, are equivalent to a screw motion, that is, a rotation about a line together with a translation along the line* (Selig, 2005; Chasles, 1830); see Fig. 1. The original theorem[1], however, is not too different; the main difference is the exception absence.

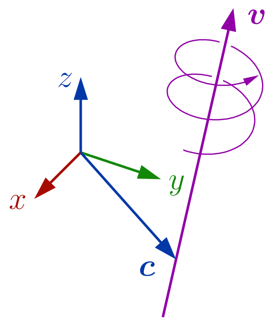


Figure 1: A screw motion

The problem of determining the location of the axis in space is hardly new. Chasles (1830) stated that *given three points of the body's new position, it is easy to determine the position and dimensions of the screw*, although he has not underlined a procedure for it. Hunt (1990) describes a geometrical procedure, again based on tracking three points. McCarthy and Soh (2011) derive some equations for determining the axis location given the screw motions description

[1]*... quand on a dans l'espace un corps solide libre, si on lui fait éprouver un déplacement fini quelconque, il existera toujours dans ce corps une certaine droite indéfinie, qui, après le déplacement, se retrouvera au même lieu qu'auparavant. Si on fait tourner le second corps (c'est à-dire le corps pris dans sa seconde position), autour de cette droite, il deviendra semblablement placé au 1er; et si, ensuite, on lui donne un mouvement de translation dans le sens de cette droite, il viendra se superposer sur le 1 corps; ce qui prouve que l'on peut toujours transporter un corps solide libre d'une position dans une autre position quelconque déterminée, parle mouve ment continu d'une vis à laquelle ce corps serait fixé invariablement.* (Chasles, 1830)

through a homogeneous transformation matrix. Here, we revisit the rigid body displacement, its description as a finite screw motion, its representation through a homogeneous transformation matrix and the inverse problem, namely, the determination of the motion parameters given the motion description as a transformation matrix.

In Section 2, the rigid displacement is described by a homogeneous transformation matrix of $SE\left(3\right)$, the Euclidean special group in three dimensions. Section 3 brings the study of the eigenvalues of orthogonal matrices, in particular, the group special orthogonal matrix in three dimensions, $SO\left(3\right)$. This is important because $SE\left(3\right)$ is the semidirect product of $SO\left(3\right)$, responsible for the rotational part of the motion, and $\mathbb{R}^3$, responsible for the translational part. Section 4 shows why the action of a rotation matrix $\boldsymbol{R_v}\left(\theta\right) \in SO\left(3\right)$ on a point of the $\mathbb{R}^3$ produces a rotation of that point of an angle θ about an axis in $\boldsymbol{v}$ direction through the origin. This is done by analysing the eigenvector of $\boldsymbol{R_v}\left(\theta\right)$. Section 5 depicts the antisymmetric matrix adjoint to a vector from $\mathbb{R}^3$ which product by another vector is equivalent to the vectors cross product. Those antisymmetric matrices are used throughout the paper in several algebraic manipulations. In Section 6 the inverse problem of rigid motion parametrization is addressed which proves the Chasles' Theorem. In Section 7, a new equation for the point in the axis closest to the origin is derived and that completes the proof. In Section 8 the geometric interpretation of that equation is given. Section 9 presents a different way of thinking about the location of a line in space and, in particular, of a screw axis. The same result found in Section 7 is obtained following a different line of reasoning. The special case of planar motion is treated in Section 10. Section 11 brings a numerical example of a PUMA robot motion and its analysis. Some conclusions are drawn in Section 12.

2 RIGID MOTION

Perhaps the most commonly used representation for a rigid displacement is through a 4×4 homogeneous transformation matrix. The most general displacement can be expressed by

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \tag{1}$$

where $\boldsymbol{R} = \boldsymbol{R_v}\left(\theta\right) \in SO\left(3\right)$ is a 3×3 rotation matrix parametrized by the rotation angle θ and the unit vector $\boldsymbol{v}$ which gives the rotation axis direction and $\boldsymbol{t}$ is the translational component of the motion. $\boldsymbol{G}$ can be interpreted in several ways: a rotation around an axis through the origin in the $\boldsymbol{v}$ direction followed by a translation given by $\boldsymbol{t}$; a translation given by $\boldsymbol{t}$ followed by a rotation around an axis through point initially coincident with the origin, but displaced by $\boldsymbol{t}$; a rotation about an arbitrary line together with a translation along the line. In the last case, we have Chasles' Theorem and

$$\boldsymbol{t} = \theta\, h\, \boldsymbol{v} + \left(\boldsymbol{I}_3 - \boldsymbol{R}\right)\, \boldsymbol{c} \tag{2}$$

where $\boldsymbol{I}_3$ is the 3×3 identity matrix, $\boldsymbol{c}$ is the position vector of some point on the axis and h is the screw pitch with relates the rotation and the translation. Geometrical interpretation of Eq. (2) is simple; it is just a conjugation: translate $\boldsymbol{c}$ to the origin perform the screw motion about a line through the origin and then translate the point at the origin back to $\boldsymbol{c}$. See Fig. 1. For a pure rotation, $h = 0$. For a pure translation, $\boldsymbol{R} = \boldsymbol{I}_3$ and the total displacement is $\theta\, h$ along vector $\boldsymbol{v}$. Note that h has the dimension of length, but its unit is given as the ratio of length and angle so that θ has to be measured in the same angular unit. Section 6 details the extraction of those parameters from $\boldsymbol{G}$.

3 EIGENVALUES OF A ORTHOGONAL MATRIX

An orthogonal real matrix $\boldsymbol{R} \in O\left(n\right)$ satisfies $\boldsymbol{R}\,\boldsymbol{R}^T = \boldsymbol{R}^T\,\boldsymbol{R} = \boldsymbol{I}_n$ in order to preserve the distance between points and the angle between vectors. All of its eigenvalues lie on the complex unit circle (Strang, 2016), see Fig. 2b. Since the coefficients of the characteristic equation of $\boldsymbol{R}$ are real numbers, the eigenvalues, which are the zeros of that equation, are either real or complex numbers in which case they always appear as complex conjugated pairs. Take, for example, the rotation about the z-axis in $\mathbb{R}^3$ given by

$$\boldsymbol{R}_{\hat{k}}\left(\theta\right) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

where θ is the rotation angle. The eigenvectors are the values of λ that satisfy the characteristic equation

$$\det\left(\boldsymbol{R}_{\hat{k}}\left(\theta\right) - \lambda\, \boldsymbol{I}_3\right) = 0 \tag{4}$$

where $\boldsymbol{I}_3$ is the 3×3 identity matrix. Equation (4) simplifies to

$$\left(1 - \lambda\right)\,\left(\cos\theta - \lambda\right)^2 + \sin^2\theta = \left(1 - \lambda\right)\,\left(\lambda^2 - 2\,\cos\theta\,\lambda + 1\right) = 0$$

and has solutions: $\lambda_1 = 1$ and $\lambda_{2,\,3} = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm \sqrt{i^2\,\sin^2\theta} = \cos\theta \pm i\,\sin\theta$. All of the three eigenvalues lie on the unit circle, see Fig. 2a. The eigenvector associated with λ_1 is $\hat{k}$. Since any rotation can

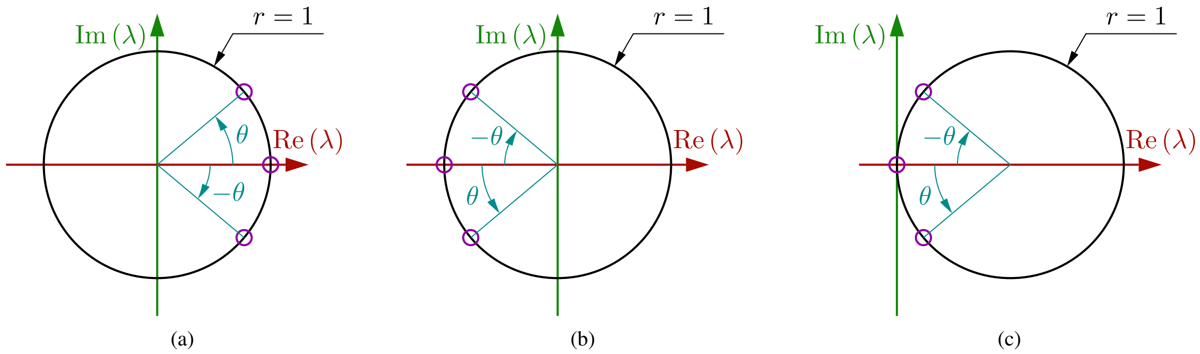


Figure 2: Eigenvalues of: (a) a rotation matrix, $\boldsymbol{R} \in SO\,(3)$, (b) $-\boldsymbol{R}$ and (c) $\boldsymbol{I}_3 - \boldsymbol{R}$

be constructed by applying a similarity transformation in Eq. (3) and because this transformation does not change the eigenvalues (Strang, 2016), the general $\boldsymbol{R} \in SO\,(3)$ that appears in Eq. (1) has the same eigenvalues that Eq. (3). In this case, however, the axis direction, $\boldsymbol{v}$, is the eigenvector of $\boldsymbol{R}$ associated with the unit eigenvalue because $\boldsymbol{v}$ itself is not subject to the rotation, $\boldsymbol{R}\,\boldsymbol{v} = \boldsymbol{v}$. The other two eigenvalues are complex conjugates of the form $\cos\theta \pm i\,\sin\theta$. Another way of reaching the same conclusion is by remembering that the determinant of a matrix can be computed by the product of all its eigenvalues, including the repeated values if any. Therefore, the determinant of $\boldsymbol{R} \in O\,(n)$ is ± 1 since the product of all complex conjugate pair is always one. If $\boldsymbol{R}$ has an odd number of repeated real eigenvalues at minus one, it is a reflection matrix and $\det\,(\boldsymbol{R}) = -1$. Otherwise, it is a rotation matrix of the special orthogonal group, $SO\,(n)$, and $\det\,(\boldsymbol{R}) = 1$.

Moreover, if a matrix is multiplied by a constant, its eigenvalues are also multiplied by the same constant. Thus, the eigenvalues of $-\boldsymbol{R}$ are opposites of the eigenvalues of $\boldsymbol{R}$, as shown in Fig. 2b. Also, if a $n \times n$ matrix is added to $k\,\boldsymbol{I}_n$, its eigenvalues are added to k. So, the eigenvalues of $\boldsymbol{I}_3 - \boldsymbol{R}$ are, zero and $1 - \cos\theta \pm i\,\sin\theta$, see Fig. 2c. A zero eigenvalue implies that the matrix in Eq. (2) is singular. However, for $\theta \neq \pm\pi\,\mathrm{rad}$, only one eigenvalue is zero and $\boldsymbol{v}$ is the only vector in the null space of $\boldsymbol{I}_3 - \boldsymbol{R}$:

$$(\boldsymbol{I}_3 - \boldsymbol{R})\,\boldsymbol{v} = \boldsymbol{I}_3\,\boldsymbol{v} - \boldsymbol{R}\,\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v} = \boldsymbol{0}. \tag{5}$$

Note that the transpose of Eq. (5) also holds and the reason why is because $\boldsymbol{v}^T$ is in the left null space of $\boldsymbol{I}_3 - \boldsymbol{R}^T$ and $\boldsymbol{I}_3 - \boldsymbol{R}$.

4 EIGENVECTORS OF A ROTATION MATRIX

The eigenvector corresponding to the unit eigenvalue of $\boldsymbol{R} \in SO\,(3)$ gives the direction of the axis, but there are another two eigenvectors which are, in general, complex conjugates of each other associated with the complex conjugate eigenvalues, $\lambda_{2,\,3} = \cos\theta \pm i\,\sin\theta$. Denoting those eigenvectors by $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, one can interpret them with the help of two real vectors defined by

$$\boldsymbol{e}_1 = \frac{1}{2}\,(\boldsymbol{v}_2 + \boldsymbol{v}_3)$$
$$\boldsymbol{e}_2 = \frac{i}{2}\,(\boldsymbol{v}_2 - \boldsymbol{v}_3)$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthogonal to each other and both are orthogonal to $\boldsymbol{v}$, the axis. Therefore, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ span a plane through the origin orthogonal to the axis, see Fig. 3.

The action of $\boldsymbol{R}$ on $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is given by

$$\begin{aligned}
R\,\boldsymbol{e}_1 &= \frac{1}{2}\,R\,(\boldsymbol{v}_2 + \boldsymbol{v}_3) = \frac{1}{2}\,(\lambda_2\,\boldsymbol{v}_2 + \lambda_3\,\boldsymbol{v}_3) = \frac{1}{2}\,(\cos\theta + i\,\sin\theta)\,\boldsymbol{v}_2 + \frac{1}{2}\,(\cos\theta - i\,\sin\theta)\,\boldsymbol{v}_3 \\
&= \cos\theta\,\frac{1}{2}\,(\boldsymbol{v}_2 + \boldsymbol{v}_3) + \sin\theta\,\frac{i}{2}\,(\boldsymbol{v}_2 - \boldsymbol{v}_3) = \cos\theta\,\boldsymbol{e}_1 + \sin\theta\,\boldsymbol{e}_2 \\
R\,\boldsymbol{e}_2 &= \frac{i}{2}\,R\,(\boldsymbol{v}_2 - \boldsymbol{v}_3) = \frac{i}{2}\,(\lambda_2\,\boldsymbol{v}_2 - \lambda_3\,\boldsymbol{v}_3) = \frac{1}{2}\,(i\,\cos\theta - \sin\theta)\,\boldsymbol{v}_2 - \frac{1}{2}\,(i\,\cos\theta + \sin\theta)\,\boldsymbol{v}_3 \\
&= \cos\theta\,\frac{i}{2}\,(\boldsymbol{v}_2 - \boldsymbol{v}_3) - \sin\theta\,\frac{1}{2}\,(\boldsymbol{v}_2 + \boldsymbol{v}_3) = \cos\theta\,\boldsymbol{e}_2 - \sin\theta\,\boldsymbol{e}_1
\end{aligned}$$

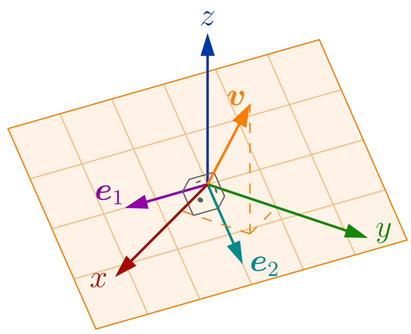


Figure 3: Rotation plane orthogonal to $\boldsymbol{v}$ spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

and this implies that the action of $\boldsymbol{R}$ is, in fact, a rotation by an angle θ about an axis through the origin and in $\boldsymbol{v}$ direction.

5 ANTISYMMETRIC MATRIX ADJOINT TO A VECTOR

A real vector $\boldsymbol{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T$ from $\mathbb{R}^3$ can be represented by a 3×3 antisymmetric matrix (that is, a matrix which satisfies $\boldsymbol{A}^T = -\boldsymbol{A}$, also called a skew symmetric matrix), see (Room, 1952; Selig, 2005),

$$\boldsymbol{V} = \mathrm{ad}\left(\boldsymbol{v}\right) = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \tag{6}$$

and this matrix can be used in the place of vector $\boldsymbol{v}$ in the cross product since

$$\boldsymbol{V}\,\boldsymbol{u} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} v_y\,u_z - v_z\,u_y \\ v_z\,u_x - v_x\,u_z \\ v_x\,u_y - v_y\,u_x \end{bmatrix} = \boldsymbol{v} \times \boldsymbol{u}.$$

The square of $\boldsymbol{V}$ is

$$\begin{aligned} \boldsymbol{V}^2 &= \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \\ &= \begin{bmatrix} -{v_y}^2 - {v_z}^2 & v_x\,v_y & v_x\,v_z \\ v_x\,v_y & -{v_x}^2 - {v_z}^2 & v_y\,v_z \\ v_x\,v_z & v_y\,v_z & -{v_x}^2 - {v_y}^2 \end{bmatrix} \\ &= \begin{bmatrix} {v_x}^2 & v_x\,v_y & v_x\,v_z \\ v_x\,v_y & {v_y}^2 & v_y\,v_z \\ v_x\,v_z & v_y\,v_z & {v_z}^2 \end{bmatrix} - \begin{bmatrix} {v_x}^2 + {v_y}^2 + {v_z}^2 & 0 & 0 \\ 0 & {v_x}^2 + {v_y}^2 + {v_z}^2 & 0 \\ 0 & 0 & {v_x}^2 + {v_y}^2 + {v_z}^2 \end{bmatrix} \\ &= \boldsymbol{v}\,\boldsymbol{v}^T - \boldsymbol{v}^T\,\boldsymbol{v}\,\boldsymbol{I}_3 \end{aligned}$$

where $\boldsymbol{I}_3$ is the 3×3 identity matrix and $\boldsymbol{v}^T\,\boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{v} = {v_x}^2 + {v_y}^2 + {v_z}^2$. Note that $\boldsymbol{V}^2$ is symmetric, thus $\left(\boldsymbol{V}^2\right)^T = \boldsymbol{V}^2$. Also, $\boldsymbol{V}\,\left(\boldsymbol{V}\,\boldsymbol{u}\right) = \boldsymbol{v} \times \left(\boldsymbol{v} \times \boldsymbol{u}\right) = \left(\boldsymbol{v}\,\boldsymbol{v}^T - \boldsymbol{v}^T\,\boldsymbol{v}\,\boldsymbol{I}_3\right)\,\boldsymbol{u}$ because of the triple product expansion, or Lagrange's formula[2], see (Itō, 1993), given by $\boldsymbol{a} \times \left(\boldsymbol{b} \times \boldsymbol{c}\right) = \boldsymbol{b}\,\left(\boldsymbol{a} \cdot \boldsymbol{c}\right) - \boldsymbol{c}\,\left(\boldsymbol{a} \cdot \boldsymbol{b}\right)$. For a unit vector, $\boldsymbol{V}^2$ simplifies to $\boldsymbol{V}^2 = \boldsymbol{v}\,\boldsymbol{v}^T - \boldsymbol{I}_3$. The rotation matrix $\boldsymbol{R} \in SO\left(3\right)$ can be written as

$$\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{v}}\left(\theta\right) = \boldsymbol{I}_3 + \sin\theta\,\boldsymbol{V} + \left(1 - \cos\theta\right)\,\boldsymbol{V}^2 \tag{7}$$

which is clearly a polynomial function of $\boldsymbol{V}$. Moreover, $\boldsymbol{V}^2$ has an interesting geometrical interpretation : for a unit vector $\boldsymbol{v}$, $-\boldsymbol{V}^2\,\boldsymbol{u}$ is the component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{v}$ as shown in Fig. 4.

The cube of $\boldsymbol{V}$ is

$$\begin{aligned} \boldsymbol{V}^3 &= \begin{bmatrix} 0 & \left({v_x}^2 + {v_y}^2 + {v_z}^2\right)\,v_z & -\left({v_x}^2 + {v_y}^2 + {v_z}^2\right)\,v_y \\ -\left({v_x}^2 + {v_y}^2 + {v_z}^2\right)\,v_z & 0 & \left({v_x}^2 + {v_y}^2 + {v_z}^2\right)\,v_x \\ \left({v_x}^2 + {v_y}^2 + {v_z}^2\right)\,v_y & -\left({v_x}^2 + {v_y}^2 + {v_z}^2\right)\,v_x & 0 \end{bmatrix} \\ &= -\boldsymbol{v}^T\,\boldsymbol{v}\,\boldsymbol{V} \end{aligned}$$

[2]Joseph-Louis Lagrange, 1736 – 1813 (born Giuseppe Luigi Lagrangia or Giuseppe Ludovico De la Grange Tournier).

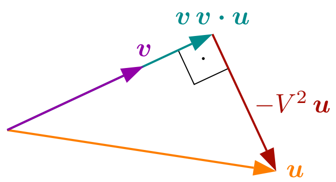


Figure 4: The $\boldsymbol{u}$ components parallel $(\boldsymbol{v}\,\boldsymbol{v}\cdot\boldsymbol{u})$ and orthogonal $\left(-\boldsymbol{V}^2\,\boldsymbol{u}\right)$ to $\boldsymbol{v}$

and, again, for a unit vector, it simplifies to $\boldsymbol{V}^3 = -\boldsymbol{V}$. Therefore, any polynomial of $\boldsymbol{V}$ can be reduced to the second order at most.

6 CHASLES' THEOREM

To prove Chasles's theorem we just need to show that any rigid transformation can be written in the form given by Eqs. (1) and (2). Essentially this just means finding the pitch and axis of the transformation. That is, knowing $\boldsymbol{G}$ sub-matrices $\boldsymbol{R}$ and $\boldsymbol{t}$ one must determine h, θ, $\boldsymbol{v}$ and $\boldsymbol{c}$. This is covered in many textbooks on Robotics, see, *e.g.*, references (Selig, 1992, 2005), but here a new equation for locating the axis is provided.

Since the trace of a matrix is equal to the sum of its eigenvalues (Strang, 2016) and the eigenvalues of the rotation sub-matrix of $\boldsymbol{G}$ are known, see Section 3, the rotation angle can be obtained by solving

$$\mathrm{Tr}\,(\boldsymbol{R}) = \mathrm{Tr}\,(\boldsymbol{G}) - 1 = 1 + 2\,\cos\theta \tag{8}$$

for θ. Using $\arccos$ would limit the solutions to the interval $0 \le \theta \le \pi\,\mathrm{rad}$, thus $\sin\theta \ge 0$. A second solution is possible, but it simply implies a reverse in axis direction.

The vector $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{R}$ and its transpose since $\boldsymbol{R}^T$ is a rotation in the opposite direction. Therefore

$$\left(\boldsymbol{R} - \boldsymbol{R}^T\right)\,\boldsymbol{v} = \boldsymbol{0}$$

and since the difference above is an antisymmetric matrix, its adjoint vector, see Section 5, is proportional to $\boldsymbol{v}$. It can be shown that the proportionality constant is $2\,\sin\theta$ using, *e.g.*, Eg. (7). So, for $\theta \ne 0$ and $\pm\pi\,\mathrm{rad}$

$$\boldsymbol{V} = \frac{1}{2\,\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^T\right) \tag{9}$$

and $\boldsymbol{v}$ is written taking into account Eq. (6). For $\theta = 0$ there is no rotation, namely, the rigid motion is a pure translation and the transitional vector is found in the last column of $\boldsymbol{G}$. If $\theta = \pm\pi\,\mathrm{rad}$, from Eg. (7) we have that

$$\boldsymbol{R}_{\boldsymbol{v}}\,(\pm\pi) = \boldsymbol{I}_3 + 2\,\boldsymbol{V}^2 = 2\,\boldsymbol{v}\,\boldsymbol{v}^T - \boldsymbol{I}_3$$

and one component of $\boldsymbol{v}$ is computed using that maximum diagonal element of $\boldsymbol{R}$; the other two are determined using the remaining elements of the respective row or column.

The scalar multiplication of both sides of Eq. (2) by $\boldsymbol{v}$ leads to

$$\boldsymbol{v}\cdot\boldsymbol{t} = \theta\,h\,\boldsymbol{v}\cdot\boldsymbol{v} = \theta\,h \tag{10}$$

because $\boldsymbol{v}\cdot\boldsymbol{v} = 1$ and $\boldsymbol{v}\cdot(\boldsymbol{I}_3 - \boldsymbol{R})\,\boldsymbol{c} = 0$ since $\boldsymbol{v}$ belongs to the null space of $\boldsymbol{I}_3 - \boldsymbol{R}$, see Eq. (5). Once θ and $\boldsymbol{v}$ have already been obtained, h can be isolated in Eq. (10) leaving only $\boldsymbol{c}$ to be determined.

Reminding that the scalar product can be represented by a matrix product using transposition, Eq. (2) can be rearranged replacing $\theta\,h$ from Eq. (10) as

$$(\boldsymbol{I}_3 - \boldsymbol{R})\,\boldsymbol{c} = \boldsymbol{t} - \boldsymbol{v}\cdot\boldsymbol{t}\,\boldsymbol{v} = \left(\boldsymbol{I}_3 - \boldsymbol{v}\,\boldsymbol{v}^T\right)\,\boldsymbol{t} = -\boldsymbol{V}^2\,\boldsymbol{t}. \tag{11}$$

Equation (11) can be solved for $\boldsymbol{c}$, but this is not straightforward because the rank of $\boldsymbol{I}_3 - \boldsymbol{R}$ is two, see Section 3, meaning that, despite the fact that Eq. 11 represents three consistent scalar equations, only two of them are linearly independent. The reason is that any point on the axis can be used to compute $\boldsymbol{t}$ by Eq. (2), no particular point is better or worse than the others. When a particular point $\boldsymbol{c}$ is substituted in Eq. (2), some pieces of information are lost and the original point cannot be recovered from $\boldsymbol{t}$. To surmount this problem, it is customary to impose an extra constraint, *e.g.*, setting a particular value for one coordinate of $\boldsymbol{c}$ or solving the equations for two coordinates depending on the third one (Selig, 1992, 2005). Herein a different constraint is proposed, and this leads to a closed form equation for computing $\boldsymbol{c}$.

7 THE POINT CLOSEST TO THE ORIGIN ON AN AXIS

Denoting the point on the axis closer to the origin by $\boldsymbol{c}_0$, one can use the vector equation of a line or pure geometrical intuition to establish the $\boldsymbol{c}_0$ and $\boldsymbol{v}$ are orthogonal, so $\boldsymbol{v} \cdot \boldsymbol{c}_0 = 0$ or, using matrix notation,

$$\boldsymbol{v}^T \boldsymbol{c}_0 = 0. \tag{12}$$

What makes Eq. (12) a very interesting constraint is the fact that $\boldsymbol{v}^T$ is in the left null space of $\boldsymbol{I}_3 - \boldsymbol{R}$ and, therefore, it is linearly independent of the rows of $\boldsymbol{I}_3 - \boldsymbol{R}$. Thus, appending Eq. (12) to Eq. (11) yields

$$\begin{bmatrix} \boldsymbol{I}_3 - \boldsymbol{R} \\ \boldsymbol{v}^T \end{bmatrix} \boldsymbol{c}_0 = \begin{bmatrix} -\boldsymbol{V}^2 \boldsymbol{t} \\ 0 \end{bmatrix} \tag{13}$$

where the 4×3 matrix on left-hand side has full rank which allows for a single solution for $\boldsymbol{c}_0$.

For a $m \times n$ matrix $\boldsymbol{A}$, the Moore[3]-Penrose[4] pseudoinverse given by

$$[A^\dagger]_{n\times m} = \left([A^T]_{n\times m}\, [A]_{m\times n}\right)^{-1}_{n\times n} [A^T]_{n\times m}$$

exists and is unique if $\boldsymbol{A}$ is taken from the left-hand of Eq. (13) and it can be used as a regular inverse to multiply the right-hand side of Eq. (13) to obtain $\boldsymbol{c}_0$.

To simplify the computation, $\boldsymbol{R}$ is replaced by the polynomial function of $\boldsymbol{V}$ given by Eq. (7). Therefore, using the properties of Section 5, the matrices needed can be written as

$$\begin{aligned} \begin{bmatrix} \boldsymbol{I}_3 - \boldsymbol{R} \\ \boldsymbol{v}^T \end{bmatrix} &= \begin{bmatrix} -\sin\theta\, \boldsymbol{V} - (1-\cos\theta)\ \boldsymbol{V}^2 \\ \boldsymbol{v}^T \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{I}_3 - \boldsymbol{R} \\ \boldsymbol{v}^T \end{bmatrix}^T &= \begin{bmatrix} -\sin\theta\, \boldsymbol{V}^T - (1-\cos\theta)\ \boldsymbol{V}^{2\,T} & \boldsymbol{v} \end{bmatrix} \\ &= \begin{bmatrix} \sin\theta\, \boldsymbol{V} - (1-\cos\theta)\ \boldsymbol{V}^2 & \boldsymbol{v} \end{bmatrix} \end{aligned}$$

and their product yields

$$\begin{aligned} \begin{bmatrix} \boldsymbol{I}_3 - \boldsymbol{R} \\ \boldsymbol{v}^T \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I}_3 - \boldsymbol{R} \\ \boldsymbol{v}^T \end{bmatrix} &= \begin{bmatrix} \sin\theta\, \boldsymbol{V} - (1-\cos\theta)\ \boldsymbol{V}^2 & \boldsymbol{v} \end{bmatrix} \begin{bmatrix} -\sin\theta\, \boldsymbol{V} - (1-\cos\theta)\ \boldsymbol{V}^2 \\ \boldsymbol{v}^T \end{bmatrix} \\ &= -\sin^2\theta\, \boldsymbol{V}^2 + (1-\cos\theta)^2\ \boldsymbol{V}^4 + \boldsymbol{v}\,\boldsymbol{v}^T \\ &= \boldsymbol{I}_3 - (1 - 2\,\cos\theta)\ \boldsymbol{V}^2 \end{aligned} \tag{14}$$

which is a polynomial in $\boldsymbol{V}$ and, therefore, the inverse of this matrix is also a polynomial in $\boldsymbol{V}$ with the form $a\,\boldsymbol{I}_3 + b\,\boldsymbol{V} + c\,\boldsymbol{V}^2$, where a, b and c are real coefficients to be determined. Moreover, because $\boldsymbol{V}^3 = -\boldsymbol{V}$, no term of degree greater than two should appear in the inverse. The coefficients a, b and c are determined bearing in mind that the product of a matrix by its inverse is the identity matrix, thus

$$\begin{aligned} \boldsymbol{I}_3 &= \left(\boldsymbol{I}_3 - (1 - 2\,\cos\theta)\ \boldsymbol{V}^2\right)\ \left(a\,\boldsymbol{I}_3 + b\,\boldsymbol{V} + c\,\boldsymbol{V}^2\right) \\ &= a\,\boldsymbol{I}_3 + 2\,b\ (1-\cos\theta)\ \boldsymbol{V} + (c - (1 - 2\,\cos\theta)\ (a-c))\ \boldsymbol{V}^2 \end{aligned}$$

and comparing the coefficients of $\boldsymbol{V}$ on both sides,

$$\begin{aligned} a &= 1 \\ 2\,b\ (1-\cos\theta) &= 0 \\ c - (1 - 2\,\cos\theta)\ (a-c) &= 0. \end{aligned}$$

Assuming that we do have a proper rotation, so $\theta \neq 0$, b must be zero. From the first and third equations,

$$c = \frac{1}{2}\,\frac{1 - 2\,\cos\theta}{1-\cos\theta}$$

[3]Eliakim Hastings Moore, 1862 – 1932.
[4]Roger Penrose, 1931.

and , consequently,

$$\left(\begin{bmatrix}\boldsymbol{I}_3-\boldsymbol{R}\\ \boldsymbol{v}^T\end{bmatrix}^T \begin{bmatrix}\boldsymbol{I}_3-\boldsymbol{R}\\ \boldsymbol{v}^T\end{bmatrix}\right)^{-1} = \boldsymbol{I}_3 + \frac{1}{2}\,\frac{1-2\,\cos\theta}{1-\cos\theta}\,\boldsymbol{V}^2$$

and the pseudoinverse is

$$\begin{aligned}\begin{bmatrix}\boldsymbol{I}_3-\boldsymbol{R}\\ \boldsymbol{v}^T\end{bmatrix}^{\dagger} &= \left(\begin{bmatrix}\boldsymbol{I}_3-\boldsymbol{R}\\ \boldsymbol{v}^T\end{bmatrix}^T \begin{bmatrix}\boldsymbol{I}_3-\boldsymbol{R}\\ \boldsymbol{v}^T\end{bmatrix}\right)^{-1} \begin{bmatrix}\boldsymbol{I}_3-\boldsymbol{R}\\ \boldsymbol{v}^T\end{bmatrix}^T\\ &= \left(\boldsymbol{I}_3 + \frac{1}{2}\,\frac{1-2\,\cos\theta}{1-\cos\theta}\,\boldsymbol{V}^2\right)\begin{bmatrix}\sin\theta\,\boldsymbol{V}-(1-\cos\theta)\,\boldsymbol{V}^2 & \boldsymbol{v}\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{2}\,\left(\cot\left(\frac{\theta}{2}\right)\,\boldsymbol{V}-\boldsymbol{V}^2\right) & \boldsymbol{v}\end{bmatrix}.\end{aligned}$$

Using the relations $\boldsymbol{V}\,\boldsymbol{v} = \boldsymbol{v}\times\boldsymbol{v} = \boldsymbol{0}$, $\boldsymbol{V}^3 = -\boldsymbol{V}$ and $\boldsymbol{V}^4 = -\boldsymbol{V}^2$, we eventually have,

$$\begin{aligned}\boldsymbol{c}_0 &= \begin{bmatrix}\boldsymbol{I}_3-\boldsymbol{R}\\ \boldsymbol{v}^T\end{bmatrix}^{\dagger}\begin{bmatrix}-\boldsymbol{V}^2\,\boldsymbol{t}\\ 0\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{2}\,\left(\cot\left(\frac{\theta}{2}\right)\,\boldsymbol{V}-\boldsymbol{V}^2\right) & \boldsymbol{v}\end{bmatrix}\begin{bmatrix}-\boldsymbol{V}^2\,\boldsymbol{t}\\ 0\end{bmatrix}\\ &= \frac{1}{2}\left(\cot\left(\frac{\theta}{2}\right)\,\boldsymbol{V}-\boldsymbol{V}^2\right)\boldsymbol{t}\\ &= \frac{1}{2}\left(\boldsymbol{t}-(\boldsymbol{v}\cdot\boldsymbol{t})\,\boldsymbol{v}+\cot\left(\frac{\theta}{2}\right)\,\boldsymbol{v}\times\boldsymbol{t}\right).\end{aligned} \tag{15}$$

8 GEOMETRIC INTERPRETATION

Observe that $\boldsymbol{c}_0$, in Eq. (15), has two orthogonal components:

$$\begin{aligned}\boldsymbol{r}_1 &= -\frac{1}{2}\,\boldsymbol{V}^2\,\boldsymbol{t} = \frac{1}{2}\,(\boldsymbol{t}-(\boldsymbol{v}\cdot\boldsymbol{t})\,\boldsymbol{v})\\ \boldsymbol{r}_2 &= \frac{1}{2}\,\cot\left(\frac{\theta}{2}\right)\,\boldsymbol{V}\,\boldsymbol{t} = \frac{1}{2}\,\cot\left(\frac{\theta}{2}\right)\,\boldsymbol{v}\times\boldsymbol{t}\end{aligned}$$

where $(\boldsymbol{v}\cdot\boldsymbol{t})\,\boldsymbol{v} = \boldsymbol{v}\,\boldsymbol{v}^T\,\boldsymbol{t}$ is a vector parallel to $\boldsymbol{v}$ and its length is equal to the projection of $\boldsymbol{t}$ onto $\boldsymbol{v}$; note that $\boldsymbol{v}\,\boldsymbol{v}^T$ is known as projection matrix. Therefore, $2\,\boldsymbol{r}_1$ is the $\boldsymbol{t}$ component orthogonal to $\boldsymbol{v}$, $\boldsymbol{t}_{\perp\boldsymbol{v}}$, as shown in Fig. 5a. The $\boldsymbol{t}$ component parallel to $\boldsymbol{v}$ has no influence in the axis position. As a consequence, we can focus on the plane orthogonal to $\boldsymbol{v}$ seen in Fig. 5b. In this plane, the axis passes through a point indicated as a cross in Fig. 5b at the tip of the vector $\boldsymbol{c}_0$. It also lies on the line that bisects the $\boldsymbol{t}$ component orthogonal to $\boldsymbol{v}$, $\boldsymbol{t}_{\perp\boldsymbol{v}}$. The vector $\boldsymbol{r}_1$ is the component of $\boldsymbol{c}_0$ in the same direction of $\boldsymbol{t}_{\perp\boldsymbol{v}}$. The second component of $\boldsymbol{c}_0$, $\boldsymbol{r}_2$, is orthogonal to $\boldsymbol{v}$, $\boldsymbol{t}$ and also $\boldsymbol{r}_1$ because $\boldsymbol{r}_1$ is a linear combination of $\boldsymbol{v}$ and $\boldsymbol{t}$. The $\boldsymbol{t}$ component orthogonal to $\boldsymbol{v}$ is $\boldsymbol{t}_{\perp\boldsymbol{v}} = \boldsymbol{t}-(\boldsymbol{v}\cdot\boldsymbol{t})\,\boldsymbol{v} = \left(\boldsymbol{I}_3-\boldsymbol{v}\,\boldsymbol{v}^T\right)\,\boldsymbol{t} = -\boldsymbol{V}^2\,\boldsymbol{t} = -\boldsymbol{v}\times(\boldsymbol{v}\times\boldsymbol{t})$ so $\left|-\boldsymbol{V}^2\,\boldsymbol{t}\right| = |\boldsymbol{v}|\;|\boldsymbol{v}\times\boldsymbol{t}|$ since $\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{t}$ are perpendicular. Then since $\boldsymbol{v}$ is a unit vector $\left|-\boldsymbol{V}^2\,\boldsymbol{t}\right| = |\boldsymbol{v}\times\boldsymbol{t}| = |\boldsymbol{v}|\;|\boldsymbol{t}|\,\sin\varphi = |\boldsymbol{t}|\,\sin\varphi$ where φ is the angle between $\boldsymbol{v}$ and $\boldsymbol{t}$, see $\boldsymbol{t}_{\perp\boldsymbol{v}}$ in Fig. 5a. From the right triangle with right angle at M, see Fig. 5b, the ratio between $\boldsymbol{v}\times\boldsymbol{t}$ and $\boldsymbol{t}_{\perp\boldsymbol{v}}$ is given by the cotangent of half θ, therefore

$$\frac{|\boldsymbol{r}_2|}{|\boldsymbol{r}_1|} = \frac{\left|\cot\left(\frac{\theta}{2}\right)\,\boldsymbol{v}\times\boldsymbol{t}\right|}{|\boldsymbol{t}-(\boldsymbol{v}\cdot\boldsymbol{t})\,\boldsymbol{v}|} = \cot\left(\frac{\theta}{2}\right)\,\frac{|\boldsymbol{t}|\,\sin\varphi}{|\boldsymbol{t}|\,\sin\varphi} = \cot\left(\frac{\theta}{2}\right).$$

9 MOMENT OF THE SCREW AXIS

One can think of the moment of the screw axis as the unique feature that defines the axis's position in space. For any given point $\boldsymbol{c}$ on the axis, the moment is

$$\boldsymbol{m} = \boldsymbol{c}\times\boldsymbol{v}$$

where the unit vector $\boldsymbol{v}$, as usual, gives the axis direction. Any other point on the axis is given by $\boldsymbol{c}+\lambda\,\boldsymbol{v}$, where λ is an appropriated scalar, so it is clear that the moment is unique for an axis. This is important because it is not possible to

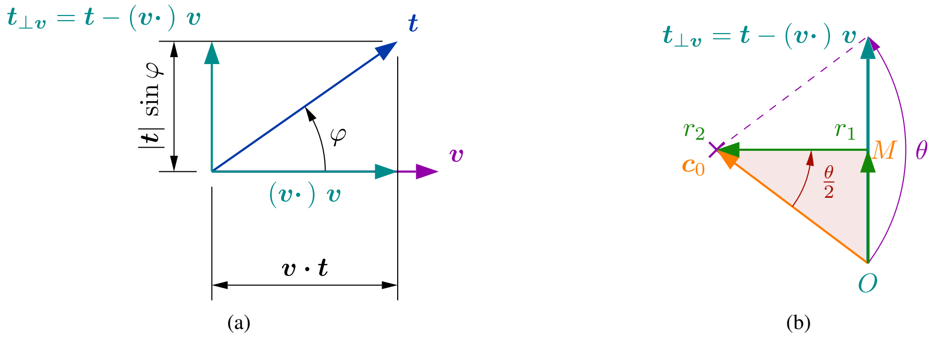


Figure 5: Geometric interpretation of Eq. (15): (a) plane spanned by $\boldsymbol{v}$ and $\boldsymbol{t}$ e (b) plane orthogonal to $\boldsymbol{v}$

solve Eq. (11) for $\boldsymbol{c}$ unless an additional constraint is imposed. However, replacing Eq. (7) into Eq. (11) bearing in mind that $\boldsymbol{V}\,\boldsymbol{c} = \boldsymbol{v} \times \boldsymbol{c} = -\boldsymbol{m}$ leads to

$$\begin{aligned} -\boldsymbol{V}^2\,\boldsymbol{t} &= \left(-\sin\theta\,\boldsymbol{V} - (1-\cos\theta)\;\boldsymbol{V}^2\right)\,\boldsymbol{c} \\ &= (\sin\theta\,\boldsymbol{I}_3 + (1-\cos\theta)\;\boldsymbol{V})\;\boldsymbol{m} \end{aligned}$$

and the inverse of the matrix on the right-hand side can be computed as in Section 7 or Appendix A yielding

$$(\sin\theta\,\boldsymbol{I}_3 + (1-\cos\theta)\;\boldsymbol{V})^{-1} = \frac{1}{\sin\theta}\,\boldsymbol{I}_3 - \frac{1}{2}\boldsymbol{V} + \frac{1}{2}\,\frac{\sin\theta}{1+\cos\theta}\,\boldsymbol{V}^2$$

which, after substitution and simplification, leads to

$$\begin{aligned} \boldsymbol{m} &= -\frac{1}{2}\left(\boldsymbol{V} + \cot\left(\frac{\theta}{2}\right)\,\boldsymbol{V}^2\right)\,\boldsymbol{t} \\ &= \frac{1}{2}\left(\boldsymbol{t}\times\boldsymbol{v} + \cot\left(\frac{\theta}{2}\right)\,(\boldsymbol{t} - (\boldsymbol{v}\cdot\boldsymbol{t})\;\boldsymbol{v})\right). \end{aligned} \tag{16}$$

Note that this allows us to compute the axis of the screw from the 4×4 transformation matrix so the Plücker coordinates of the screw axis is just

$$\hat{\$} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{m} \end{bmatrix}.$$

Note that it is not necessary to compute the pitch to get the Plücker coordinates of the screw axis.

Once the moment is known, the point on the screw axis closest to the origin is given by

$$\boldsymbol{c}_0 = \boldsymbol{v}\times\boldsymbol{m}. \tag{17}$$

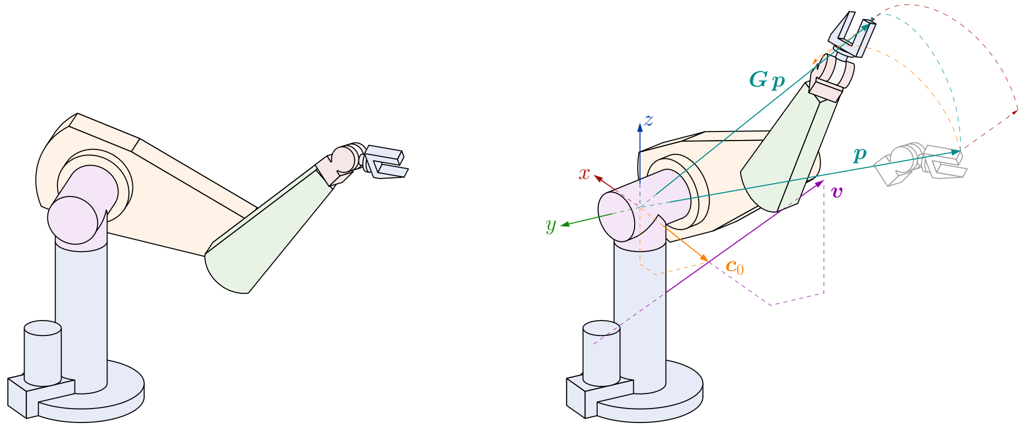
10 PLANAR MOTION

One important special case is the planar motion where the rotation is done about a point, the centre. The problem in planar motion is much simpler and doesn't require specialisation from the three-dimensional case because, in two dimensions, Eq. (2) becomes $(\boldsymbol{I}_2 - \boldsymbol{R})\;\boldsymbol{c} = \boldsymbol{t}$ where the matrix is never singular[5]. Here we provide a formula for the centre of rotation as an example of the use of Eq. (15).

To employ three-dimensional equations in the planar case one can select a particular coordinate plane, say the xy-plane, thus the axis direction is always perpendicular to that plane, therefore $\boldsymbol{v} = \hat{k}$. For $\boldsymbol{t} = \begin{bmatrix} t_x & t_y & 0 \end{bmatrix}^T$, we have

$$\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{t} &= 0 \\ \boldsymbol{v}\times\boldsymbol{t} &= \begin{bmatrix} -t_y \\ t_x \\ 0 \end{bmatrix}. \end{aligned}$$

[5] Unless the rotation angle is equal to any number of whole turns.



(a) (b)

Figure 6: PUMA: (a) initial configuration and (b) final configuration

Disregarding the last coordinate, we can write

$$\begin{bmatrix} -t_y \\ t_x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

and Eq. (15) becomes

$$\begin{aligned} \boldsymbol{c} &= \frac{1}{2}\left(\boldsymbol{I}_2 + \cot\left(\frac{\theta}{2}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} t_x \\ t_y \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -\cot\left(\frac{\theta}{2}\right) \\ \cot\left(\frac{\theta}{2}\right) & 1 \end{bmatrix}\boldsymbol{t} \end{aligned} \tag{18}$$

where the index of $\boldsymbol{c}_0$ were abolished because, in the plane, $\boldsymbol{c}$ denotes the unique centre of rotation. Equation (18) cannot be used if $\theta = 0$ or $2\,k\,\pi$, where $k \in \mathbb{Z}$, because any number of whole turns are indistinct from no rotation at all. In these cases, the motion is a pure translation in $\boldsymbol{t}$ direction. It is not hard to see that its inverse of $\boldsymbol{I}_2 - \boldsymbol{R}$ appears in Eq. (18).

11 NUMERICAL EXAMPLE

Figure 6a shows a PUMA robot in the initial configuration. Then the gripper is moved to the final configuration shown in Fig. 6b and the motion can be described by

$$\boldsymbol{G} = \begin{bmatrix} \frac{8-5\sqrt{2}}{18} & \frac{2-2\sqrt{2}}{9} & \frac{-8-7\sqrt{2}}{18} & \frac{-2-\sqrt{2}}{8} \\ \frac{2+4\sqrt{2}}{9} & \frac{1-4\sqrt{2}}{9} & \frac{-2+2\sqrt{2}}{9} & \frac{-1}{4} \\ \frac{-8-\sqrt{2}}{18} & \frac{-2-4\sqrt{2}}{9} & \frac{8-5\sqrt{2}}{18} & \frac{-\sqrt{2}}{8} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

and the $\cos\theta$ can be obtained from Eq. (8). Therefore, assuming that $0 \le \theta \le \pi$ rad, we have

$$\begin{aligned} \cos\theta &= -\frac{\sqrt{2}}{2} \\ \sin\theta &= \sqrt{1-\cos\theta} = \frac{\sqrt{2}}{2} \\ \cot\left(\frac{\theta}{2}\right) &= \frac{\sin\theta}{1-\cos\theta} = \sqrt{2}-1 \\ \theta &= \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\,\pi}{4}\,\text{rad.} \end{aligned}$$

Note that adding or subtracting some full turns to/from θ results in a valid solution and the pitch would be adjusted accordingly when computed by Eq. (10). Moreover, the opposite of θ is also a solution, but in this case, the sine would change the sign too and, as a consequence, the direction of $\boldsymbol{v}$ would be reversed. Therefore, a general solution is

$$\theta = 2\,k\,\pi \pm \frac{3\,\pi}{4} \qquad \text{where } k \in \mathbb{Z}.$$

Now, Eq. (9) can be used to compute $\boldsymbol{V} = \mathrm{ad}\,(\boldsymbol{v})$ which allows us the assemble

$$\boldsymbol{v} = \frac{1}{3}\begin{bmatrix}-2\\-1\\2\end{bmatrix}.$$

The pitch is calculated by Eq. (10) applying $\boldsymbol{t}$ extracted from the last column of $\boldsymbol{G}$ (see Eq. (19)), θ and $\boldsymbol{v}$ as

$$h = \frac{1}{3\,\pi}\,\mathrm{m}/\mathrm{rad}.$$

The point closest to the origin can be determined straightforwardly or by first computing the moment of the screw axis by Eq. (16) as

$$\boldsymbol{m} = \frac{1}{12}\begin{bmatrix}-2\\2\\-1\end{bmatrix}$$

and, finally, using either Eq. (15) or Eq. (17)

$$\boldsymbol{c}_0 = \frac{1}{12}\begin{bmatrix}-1\\-2\\-2\end{bmatrix}.$$

Any point on the screw axis can be computed by

$$\boldsymbol{c} = \lambda\,\boldsymbol{v} + \boldsymbol{c}_0$$

where λ is a suitable real constant.

As expected, the obtained value can be used in Eqs. (7), (2) and (1) to assemble $\boldsymbol{G}$ of Eq. (19).

Visualizing the axis in Fig. 6b is a bit challenging. Therefore, dashed lines aligned with the coordinate axes are provided. To reach the point an the axis closest to the origin, departing from the origin, follow the orange dashed line in z, x and y direction order. Likewise, follow the magenta dashed line to reach the tip of $\boldsymbol{v}$ departing from the tip of $\boldsymbol{c}_0$, but now in x, y and z order.

The action of $\boldsymbol{G}$ on an arbitrary point $\boldsymbol{p}$ on the robot gripper[6] is shown in cyan in Fig. 6b. This displacement is lustrated by three paths: in cyan, the screw motion, in orange, first rotation then translation; and in red, first translation then rotation. Those are three out of an infinite number of possibilities. The actual robot cannot follow the last (red) path because it violates the workspace boundaries.

12 CONCLUSIONS

In this proof of Charles' theorem, we have assumed that rigid body displacements can always be represented by matrices. In the early 1800s, this was not the way that people thought of rigid body displacements. It was known that any rigid displacement (including reflections) could be generated by a maximum of four plane reflections. The proof of the theorem is then reduced to showing that the reflection planes can be arranged in orthogonal pairs. Reflection in a pair of orthogonal planes is equivalent to a half-turn (rotation by $\pi\,\mathrm{rad}$) about the intersection line of the two planes. A pair of half-turns is a finite screw displacement, the axis of the screw is the common perpendicular to the pair of lines, the rotation angle is twice the twist angle between the lines and the translation vector $\boldsymbol{t}$, is given by twice the separation between the lines along their common perpendicular. It is straightforward to see that any plane reflection and hence any pair of plane reflections can be written as matrices.

From this work, it is easy to see how the Plücker coordinates of the screw axis can be found. It is also simple to see how to find the screw invariants when the displacement is given in the adjoint (6×6) representation of the group of proper rigid body displacements.

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[6]Note that $\boldsymbol{p}$ might be written in extended coordinates as a 4×1 matrix $\begin{bmatrix}p_x & p_y & p_z & 1\end{bmatrix}^T$.

14 REFERENCES

Ceccarelli, M., 2000. "Screw axis defined by Giulio Mozzi in 1763 and early studies on helicoidal motion". *Mechanism and Machine Theory*, Vol. 35, No. 6, pp. 761–770. ISSN 0094-114X. doi:10.1016/S0094-114X(99)00046-4. URL http://www.sciencedirect.com/science/article/B6V46-3YRWJRR-1/2/a128b1fc54d0d3e9bd9fdd4c400e878e.

Chasles, M.F., 1830. "Note sur les propriétés générales du systéme de deux corpes semblables entre'eux, et placés d'une maniére quelconque dans l'espace; et sur le déplacement fini, ou infinitement petit, d'un corps solide libre". *Bulletin des Sciences Mathématiques, Astronomiques, Physiques et Chemiques*, Vol. 14, pp. 321–326. URL https://books.google.com/books?id=ERcAAAAAMAAJ&pg=PA321.

Garcia, S.R. and Horn, R.A., 2017. *A second course in linear algebraa second course in linear algebra*. Cambridge Mathematical Textbooks. Cambridge, Cambridge. ISBN 9781107103818. URL https://www.cambridge.org/9781107103818.

Gibson, C. and Hunt, K.H., 1990. "Geometry of screw systems–1: screws: genesis and geometry". *Mechanism and Machine Theory*, Vol. 25, No. 1, pp. 1–10. ISSN 0094-114X. doi:10.1016/0094-114X(90)90103-Q. URL http://www.sciencedirect.com/science/article/B6V46-482GT8Y-TP/2/286979301fe257660e6c44b6d49e9d4e.

Hunt, K.H., 1990. *Kinematic geometry of mechanisms*, Vol. 7 of *The Oxford engineering science series*. Clarendon, Oxford. ISBN ISBN-13: 9780198562337. Reprinted with corrections [from the 1978 edition].

Itō, K., ed., 1993. *Encyclopedic dictionary of mathematics*. MIT, Cambridge.

McCarthy, J.M. and Soh, G.S., 2011. *Geometric design of linkages*, Vol. 11 of *Interdisciplinary Applied Mathematics*. Springer, New York, 2nd edition. ISBN ISBN-13: 978-1-4419-7891-2 e-ISBN-13: 978-1-4419-7892-9. doi: 10.1007/978-1-4419-7892-9. URL http://link.springer.com/book/10.1007/978-1-4419-7892-9/page/1.

Room, T.G., 1952. "The composition of rotations in Euclidean three-space". *The American Mathematical Monthly*, Vol. 59, pp. 688–692. URL https://www.jstor.org/stable/2307548.

Selig, J.M., 1992. *Introductory robotics*. Prentice Hall, Hemel Hempstead. ISBN ISBN-13: 978-0-13-488875-0 ISBN-10: 0-13-488875-8.

Selig, J.M., 2005. *Geometric fundamentals of robotics*. Monographs in Computer Science. Springer, 2nd edition. ISBN ISBN-10: 0-387-20874-7 ISBN-13: 978-0-387-20874-9 e-ISBN-13: 978-0-387-27274-0. doi:10.1007/b138859. URL http://link.springer.com/book/10.1007/b138859/page/1.

Strang, G., 2016. *Introduction to linear algebra*. Wellesley - Cambridge, Wellesley. ISBN 978-0-9802327-7-6.

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A ANOTHER METHOD TO FIND THE INVERSE

Here, the inverse of the matrix on the right-hand side of Eq. (14) is computed by applying the Cayley[7]–Hamilton[8] theorem which states that *every square matrix over a commutative ring satisfies its own characteristic equation* (Garcia and Horn, 2017; Strang, 2016). Define the matrix $\boldsymbol{X}$ as

$$\boldsymbol{X} = \boldsymbol{I}_3 - (1 - 2\,\cos\theta)\ \boldsymbol{V}^2$$

where $\boldsymbol{V}^2 = \boldsymbol{v}\,\boldsymbol{v}^T - \boldsymbol{I}_3$ and $\boldsymbol{v}\,\boldsymbol{v}^T$ is the 3×3 projection matrix

$$\boldsymbol{v}\,\boldsymbol{v}^T = \begin{bmatrix} {v_x}^2 & v_x\,v_y & v_x\,v_z \\ v_x\,v_y & {v_y}^2 & v_y\,v_z \\ v_x\,v_z & v_y\,v_z & {v_z}^2 \end{bmatrix}.$$

The characteristic equation of $\boldsymbol{X}$ is obtained departing from the eigenvalues of $\boldsymbol{v}\,\boldsymbol{v}^T$ which are the values of λ that satisfies

$$\det\left(\boldsymbol{v}\,\boldsymbol{v}^T - \lambda\,\boldsymbol{I}_3\right) = \det\left(\begin{bmatrix} {v_x}^2 - \lambda & v_x\,v_y & v_x\,v_z \\ v_x\,v_y & {v_y}^2 - \lambda & v_y\,v_z \\ v_x\,v_z & v_y\,v_z & {v_z}^2 - \lambda \end{bmatrix}\right) = (1 - \lambda)\ \lambda^2 = 0$$

since ${v_x}^2 + {v_y}^2 + {v_z}^2 = 1$. So, the projection matrix has a single eigenvalue at one and a double eigenvalue at zero.

If a matrix is added to $k\,\boldsymbol{I}_n$, where k is a scalar, its eigenvalues are also added to k (Strang, 2016). Because $\boldsymbol{V}^2 = \boldsymbol{v}\,\boldsymbol{v}^T - \boldsymbol{I}_3$, the eigenvalues of $\boldsymbol{V}^2$ are zero (single) and minus one (double).

[7] Arthur Cayley, 1821 – 1895.

[8] William Rowan Hamilton, 1805 – 1865.

If a matrix is multiplied by a scalar k, its eigenvalues are also multiplied by k (Strang, 2016). Hence, the eigenvalues of $\boldsymbol{X}$ are those of $\boldsymbol{V}^2$ multiplied by $-(1-2\cos\theta)$ and added to one:

$$\begin{aligned}\lambda_1 &= -(1-2\cos\theta)\;0+1\boldsymbol{G} = \begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0}^T & 1\end{bmatrix} = 1\\ \lambda_{2,3} &= -(1-2\cos\theta)\;(-1)+1 = 2\;(1-\cos\theta)\,.\end{aligned}$$

Therefore, the characteristic equation of $\boldsymbol{X}$ is

$$(\lambda-1)\;(\lambda-2\;(1-\cos\theta))^2 = \lambda^3-(5-4\cos\theta)\;\lambda^2+4\;(2-\cos\theta)\;(1-\cos\theta)\;\lambda-4\;(1-\cos\theta)^2=0$$

and this equation must be satisfied by $\boldsymbol{X}$, so it becomes

$$\boldsymbol{X}^3-(5-4\cos\theta)\;\boldsymbol{X}^2+4\;(2-\cos\theta)\;(1-\cos\theta)\;\boldsymbol{X}-4\;(1-\cos\theta)^2\;\boldsymbol{I}_3=0.$$

Now, multiplying by $\boldsymbol{X}^{-1}$ and manipulating, we have

$$4\;(1-\cos\theta)^2\;\boldsymbol{X}^{-1}=\boldsymbol{X}^2-(5-4\cos\theta)\;\boldsymbol{X}+4\;(2-\cos\theta)\;(1-\cos\theta)\;\boldsymbol{I}_3.$$

Finally, replacing $\boldsymbol{X}$ by $\boldsymbol{I}_3-(1-2\cos\theta)\;\boldsymbol{V}^2$ on the right-hand side leads to

$$\begin{aligned}4\;(1-\cos\theta)^2\;\boldsymbol{X}^{-1} &= \left(\boldsymbol{I}_3-(1-2\cos\theta)\;\boldsymbol{V}^2\right)^2-\\ &\quad-(5-4\cos\theta)\;\left(\boldsymbol{I}_3-(1-2\cos\theta)\;\boldsymbol{V}^2\right)+\\ &\quad+4\;(2-\cos\theta)\;(1-\cos\theta)\;\boldsymbol{I}_3\\ &= 2\;(1-\cos\theta)\;(1-2\cos\theta)\;\boldsymbol{V}^2+4\;(1-\cos\theta)^2\;\boldsymbol{I}_3\\ \boldsymbol{X}^{-1} &= \boldsymbol{I}_3+\frac{1}{2}\,\frac{1-2\cos\theta}{1-\cos\theta}\,\boldsymbol{V}^2\end{aligned}$$

which is the desired inverse. Note that the identity $\boldsymbol{V}^4=-\boldsymbol{V}^2$ were used in the simplification.