

COB-2023-0466

FRACTURE NUCLEATION AND PROPAGATION BASED ON DISCRETE TOPOLOGICAL DERIVATIVE

William Souza Garcia

Marcel Duarte da Silva Xavier

Departamento de Engenharia Mecânica, Universidade Federal Fluminense - PGMEC/UFF - Rua Passo da Pátria 156, 24210-240, Niterói - RJ, Brasil

william_garcia@id.uff.br, marcelxavier@id.uff.br

Abstract. *Scientific investigation of fracture mechanics problems, such as crack/damage nucleation and propagation in brittle and ductile materials, remains of large interest in several branches of engineering. The understanding of this phenomena allows, for instance, to extend components useful life and predict failures before its occurrence. Computational modelling has become an indispensable tool, in particular to fracture analysis, since relatively few practical problems have closed-form analytical solutions. In the past years, the topological derivative method has accumulated a large scientific contribution and demonstrated to be able to capture the whole nucleation and propagation damaging process, including important features like kinking and bifurcations. The present work aims to introduce the discrete topological derivative method to fracture mechanics problems, and take benefits from the advantages of discrete analysis to present a simplified approach. For this purpose, the numerical results related to the minimization of a shape functional, given by the sum of the total potential energy and the dissipation term of the system, with respect to the distribution of the healthy and damaged phases, under an irreversibility constraint, after properly validated, are compared to experimental and numerical predictions references established in the literature. The purpose is to analyze the differences between the estimative given by discrete and traditional approaches. Therefore, this study aims to achieve consistent results using an extremely simple and quite efficient algorithm, enabling the approach to non-linear problems.*

Keywords: *Discrete Topological Derivative, Fracture Mechanics, Crack Propagation.*

1. INTRODUCTION

The Topological Derivative measures the sensitivity of a given shape functional with respect to the introduction of an infinitesimal perturbation in the domain such as hole insertions, inclusions, source terms or cracks. This concept was rigorously introduced by Sokołowski and Żochowski (1999) and since then, this tool has proven to be extremely useful in various physics and engineering problems such as topological optimization Allaire *et al.* (2005), image processing Larrabide *et al.* (2008), and in particular in the context of fracture mechanics Xavier *et al.* (2017).

The objective of this work, involves a simplification of the sensitivity analysis of the shape functional through the discrete approach, which consequently requires the preliminary discretization of the damage model of Francfort and Marigo (1993) applied to the plane elasticity context, in order to study the nucleation and propagation of fractures in this scenario. For this purpose, initially the topological derivative for the associated discrete shape functional is presented, with respect to nucleation of a small triangular inclusion applying the scheme of material interpolation scheme of Huang and Xie (2009). Then, the associated sensitivity is used to propose a simple numerical scheme to determine the nucleation and propagation of fractures in the domain. In other words, the topological derivative is used as a descent direction to minimize the Francfort-Marigo functional, indicating, at each iteration, the regions that will be damaged. In order to validate the proposed methodology, the experiment of Bittencourt *et al.* (1996) is presented, comparing the results obtained with the continuous and discrete approach.

The paper is organized as follows. In Section 2 the mechanical model is revisited. The associated topological derivative is presented in Section 3. The resulting algorithm is detailed in Section 4. Finally, the numerical result obtained is presented in Section 5, ending with some conclusions in Section 6.

2. MECHANICAL MODEL

The main idea of the damage model of Francfort and Marigo (1993) consists of introducing an elastic body composed of two different materials, represented by the parameter $\rho_0 \ll 1$. The change from the original material to the damage one will occur only if the elastic energy released by this transition overcome a material-dependent threshold. In other words, the occurrence of damage is determined by the relation

$$\frac{1}{2} \mathbb{C} \varepsilon \cdot \varepsilon - \frac{1}{2} \rho_0 \mathbb{C} \varepsilon \cdot \varepsilon > \kappa, \quad (1)$$

where \mathbb{C} is the fourth-order elasticity tensor, ε is the second order strain tensor and κ is a material property that represents the damage toughness.

Two conditions are expected for this model. Firstly, the healthy material should be more stiffer than the damaged material, i.e.,

$$(1 - \rho_0) \mathbb{C} \varepsilon \cdot \varepsilon > 0 \quad \forall \varepsilon, \quad (2)$$

to characterize the stiffness loss associated with the damage. Secondly, the damage ($\mathbb{C} \rightarrow \rho_0 \mathbb{C}$) is permanent, i.e., the material is unable to return to its original state ($\rho_0 \mathbb{C} \rightarrow \mathbb{C}$). Thus, irreversibility imposes a constraint on the evolution of the phenomenon.

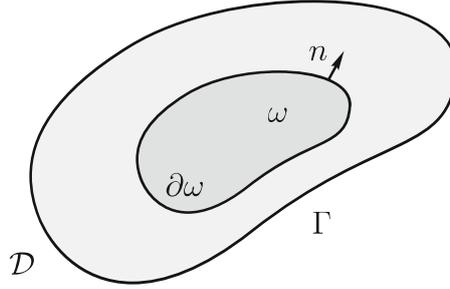


Figure 1 – Geometric domain \mathcal{D} decomposed into $\omega \subset \mathcal{D}$ and $\mathcal{D} \setminus \omega$

More precisely, let us consider an open and bounded geometrical domain $\mathcal{D} \subset \mathbb{R}^2$, with Lipschitz boundary $\Gamma := \partial \mathcal{D}$, and the subdomains $\omega \subset \mathcal{D}$ and $\mathcal{D} \setminus \omega$ composed of different physical properties, as shown in Fig. 1. The interaction between ω e $\mathcal{D} \setminus \omega$ is governed by a transmission condition acting on the interface $\partial \omega$, see Novotny and Sokolowski (2020). Francfort and Marigo proposed a functional that should be minimized at each time instant t_i , whose arguments are the displacement field u_i and the damage distribution $\rho : \mathcal{D} \rightarrow \{1, \rho_0\}$ defined as

$$\rho(x) := \begin{cases} 1, & \text{se } x \in \mathcal{D} \setminus \omega, \\ \rho_0, & \text{se } x \in \omega. \end{cases} \quad (3)$$

Since $\rho_0 \ll 1$, $\mathcal{D} \setminus \omega$ and ω are used to represent the healthy and damaged parts of the elastic body, respectively, as shown in Fig. 2. That is, if $\rho(x) = 1$ one recovers the healthy material \mathbb{C} , otherwise, if $\rho(x) = \rho_0$ one obtains the damaged material $\rho_0 \mathbb{C}$.

The Francfort-Marigo functional $\Psi_\omega(u_i)$ is defined as the sum of the total potential energy and an energy dissipation term, namely

$$\Psi_\omega(u_i) = \mathcal{J}(u_i) + \kappa |\omega| \quad (4)$$

where $|\omega|$ is the Lebesgue measure of ω and $\mathcal{J}(u_i)$ is the total potential energy defined as

$$\mathcal{J}(u_i) = \frac{1}{2} \int_{\mathcal{D}} \sigma(u_i) \cdot \varepsilon(u_i) - \int_{\Gamma_N} \bar{q} \cdot u \quad (5)$$

where the vectorial function u is solution of the following variational problem:

$$\begin{cases} \text{Find } u \in \mathcal{U}, \text{ such that:} \\ \int_{\mathcal{D}} \sigma(u) \cdot \varepsilon(\eta) = \int_{\Gamma_N} \bar{q} \cdot \eta, \quad \forall \eta \in \mathcal{V} \end{cases} \quad (6)$$

Some terms in the above equation require explanation. The stress tensor $\sigma(\varphi)$ is defined as

$$\sigma(\varphi) = \rho \mathbb{C} \varepsilon(\varphi), \quad (7)$$

while the strain tensor $\varepsilon(\varphi)$ is given by the symmetric part of the gradient of φ , namely

$$\varepsilon(\varphi) = \frac{1}{2} (\nabla \varphi + (\nabla \varphi)^T). \quad (8)$$

We restrict ourselves to isotropic material, so that the elasticity tensor \mathbb{C} can be represented as

$$\mathbb{C} = 2\mu \mathbb{I} + \lambda (\mathbf{I} \otimes \mathbf{I}) \quad (9)$$

where \mathbb{I} and \mathbb{II} are the second and fourth identity tensors, respectively, and μ and λ are the Lamé's coefficient, both considered to be constant in all domain. Particularly, in the case of the plain stress state assumption, we have

$$\mu = \frac{E}{2(1 + \nu)} \quad \text{and} \quad \lambda = \frac{\nu E}{1 - \nu^2} \quad (10)$$

while in the case of plain strain state,

$$\mu = \frac{E}{2(1 + \nu)} \quad \text{and} \quad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad (11)$$

where E is the Young modulus and ν the Poisson's coefficient.

Finally, the displacement field is solution to the following boundary value problem: Find u_i , such that

$$\begin{cases} \operatorname{div} \sigma(u_i) = 0 & \text{in } \mathcal{D}, \\ \sigma(u_i) = \rho \mathbb{C} \varepsilon(u_i), \\ u_i = g_i & \text{in } \Gamma_D, \\ \sigma(u_i) n = \bar{q} & \text{in } \Gamma_N. \end{cases} \quad (12)$$

where $g_i = g_{i-1} + \Delta g_i$ is used to denote a prescribed displacement on the boundary $\Gamma_D \subset \Gamma$ depending on the time instant t_i and the increment Δg_i . Thus, the total applied displacement g computed as the sum

$$g = g_0 + \sum_{i=1}^N \Delta g_i \quad (13)$$

where N is the total number of increments. Finally, $\Gamma_0 \subset \Gamma$ is used to denote a traction free boundary. Therefore, $\Gamma = \Gamma_D \cup \Gamma_0$, such that $\Gamma_D \cap \Gamma_0 = \emptyset$.

The FEM (Finite Elements Method) formulation of the equilibrium equation associated with the problem in question can be written as:

$$KU = F \quad (14)$$

where $K = K^\top = \rho^p K_0$ is the global stiffness matrix of the system and establishes the correspondence between forces

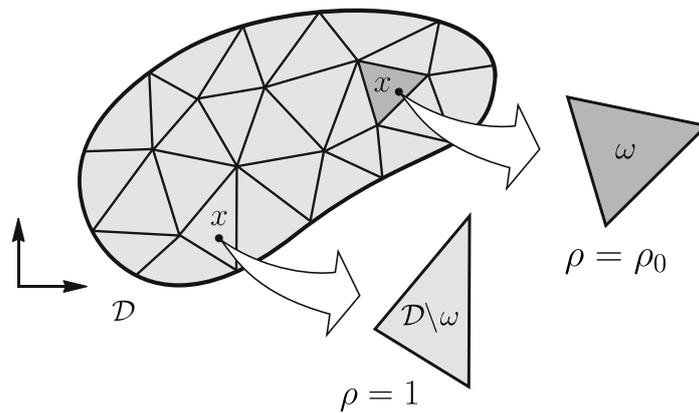


Figure 2 – Intermediated domain.

and nodal displacements in the element, K_0 is the nominal stiffness matrix, with ρ given by eq. (3), F , the force vector and U , the discrete solution of the problem. Now, we have all elements to state the Francfort-Marigo damage model, which consists in minimizing the functional $\Psi_\omega(u_i)$, for each time increment t_i , with respect to the set $\omega \subset \mathcal{D}$. That is

$$\text{Minimize}_{\omega \subset \mathcal{D}} \Psi_\omega(u_i) := \mathcal{J}(u_i) + \kappa |\omega|, \quad \text{subjected to (12)} \quad (15)$$

This model is purely energetic in the sense that damage evolution is based just on the energy density distribution. As a direct consequence, it is not able to distinguish the difference between traction and compression stress states and thus not suited to describe the crack closure phenomenon.

Another important feature of the model concerns the characterization of a critical load. In problems without singularities, critical load is the one that allows local strain-energy density to achieve a critical value. In problems with stress singularities, however, the strain energy density rises locally to unbounded values and consequently above any finite threshold. Nevertheless, experiments like those of Griffith indicate the existence of a critical nonzero load even in the presence of such singularities, which reveals a limitation on the straightforward application of the Francfort-Marigo model in these cases. An existing remedy in the literature proposes a modification in the (discrete) numerical scheme of the model by introducing a new material property κ_s used in conjunction with a scaling factor associated with a mesh size measure Allaire *et al.* (2011). Here, we replace κ by a modified energy release parameter κ_δ (see (1)) defined by the ratio

$$\kappa = \kappa_\delta := \frac{\kappa_s}{\delta} \quad (16)$$

where δ is a scaling factor associated with the width of the initial damage. From the physical point of view, when δ becomes smaller, the parameter κ_δ increases in a similar way as the energy density, so that the critical load converges to a finite nonzero value. This strategy has shown to be effective in problems of crack propagation where the fracture is represented by a damaged region of small width δ , since letting $\delta \rightarrow 0$ forces the damage region to be crack-like. In the original Bourdin *et al.* (2008) work, the crack was approximated by a smeared region by Ambrosio and Tortorelli (1992) functional, whereas in our approach the contrary is done: a damage converges to a crack. In the anti-plane case theoretical results in this respect were derived by Maso and Iurlano (2013). Note the use of κ_s is explicitly taken into account in these approximations.

3. SENSIBILITY ANALYSIS

Thus, the idea is to remove an element K^e and replace it by γK^e , with $K^e = \rho^p K_0^e$, where K_0^e is the nominal stiffness matrix of e -th finite element, p is the penalization factor (in general, $p = 3$, according to Huang and Xie (2009)) and $\gamma = \gamma(x)$ is the contrast in material properties defined as

$$\gamma(x) := \begin{cases} \rho_0, & \text{if } x \in \mathcal{D} \setminus \omega \\ \rho_0^{-1}, & \text{if } x \in \omega \end{cases} \quad (17)$$

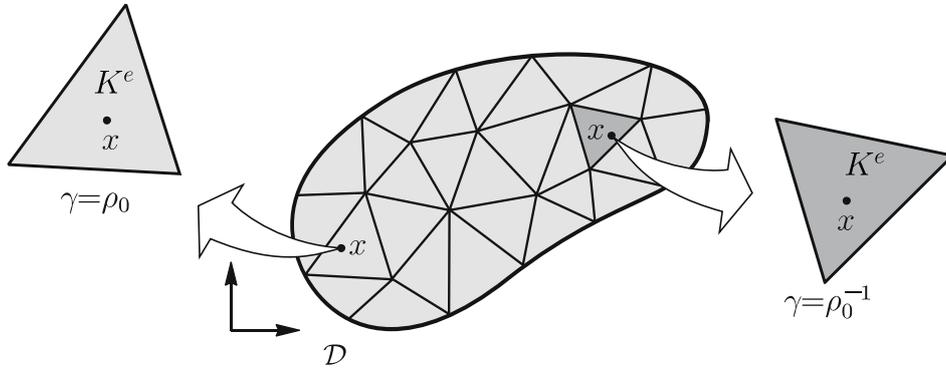


Figure 3 – Geometric domain contrast in material properties.

with $0 < \gamma < \infty$. The perturbed form of the stiffness matrix K can be evaluated as

$$\begin{aligned} K_\rho &= K - \rho^p K_0^e + \gamma \rho^p K_0^e \\ &= K - (1 - \gamma) \rho^p K_0^e \end{aligned} \quad (18)$$

Therefore, the solution associated with the perturbed problem is given by the equation

$$K_\rho U_\rho = F \quad (19)$$

such that $U_\rho = U + \tilde{U}$ can be written, where \tilde{U} is the portion of U_ρ concerning the disturbance.

In order to assess the magnitude of the \tilde{U} portion, we compare the Eqs. (14), (18) e (19):

$$\begin{aligned} KU &= K_\rho U_\rho \\ KU &= KU_\rho - (1 - \gamma) \rho^p K_0^e U_\rho \\ K\tilde{U} &= (1 - \gamma) \rho^p K_0^e U_\rho. \end{aligned} \quad (20)$$

Taking the inner product of Eq. (20) with respect to the \tilde{U} ,

$$K\tilde{U} \cdot \tilde{U} = (1 - \gamma)p\rho^{p-1}K_0^e U_\rho^e \cdot \tilde{U}. \quad (21)$$

Provided that the matrix K is positive-definite, from the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \|\tilde{U}\|^2 &\leq C_1 \|K_0^e U_\rho^e\| \|\tilde{U}\| \\ &\leq C_2 |\omega^e| \|\tilde{U}\| \\ \Rightarrow \|\tilde{U}\| &= O(|\omega^e|), \end{aligned} \quad (22)$$

where C_1 e C_2 are positive real constants. That is, the magnitude of $\|\tilde{U}\|$ has the order of the volume of the disturbed element $|\omega^e|$.

The sensitivity associated with the perturbed stiffness matrix K_ρ with respect to the damage distribution ρ is defined as follows

$$\begin{aligned} \dot{K}_\rho &:= \lim_{\Delta\rho \rightarrow 0} \frac{K_{\rho+\Delta\rho} - K_\rho}{\Delta\rho} \\ &= -(1 - \gamma)p\rho^{p-1}K_0^e \end{aligned} \quad (23)$$

Therefore, the sensitivity of Eq. (19) is given by

$$\begin{aligned} K_\rho \dot{U}_\rho &= -\dot{K}_\rho U_\rho \\ K_\rho \dot{U}_\rho &= (1 - \gamma)p\rho^{p-1}K_0^e U_\rho^e \\ \dot{U}_\rho &= (1 - \gamma)p\rho^{p-1}K_\rho^{-1}K_0^e U_\rho^e \end{aligned} \quad (24)$$

The introduction of the contrast γ induces the method of domain representation by level-set function, as proposed by Amstutz and Andrä (2006). This simple strategy allows the BESO sensitivity to guide the level-set function towards a minimum local of the optimization problem based on the associated optimality criterion.

In order to simplify future analyses we rewrite Eqs. (23) and (24) as follows:

$$\dot{K} = -(1 - \gamma)p\rho^{p-1}K_0^e \quad \text{e} \quad \dot{U} = (1 - \gamma)p\rho^{p-1}K^{-1}K_0^e U^e \quad (25)$$

3.1 Discrete sensitivity in plane elasticity

The discrete shape functional of Francfort-Marigo with respect to the nucleation of a small inclusion in plane elasticity is given by

$$\Psi = \Psi_E + \Psi_G \quad (26)$$

where Ψ_E give by

$$\Psi_E = \frac{1}{2}KU \cdot U - F \cdot U \quad (27)$$

is the total potential energy of the elastic system and Ψ_G is give by

$$\Psi_G = \kappa|\omega^e| \quad (28)$$

is the Griffith energy dissipation term.

Calculating the above equation with respect to the considered disturbance, one finds

$$\begin{aligned} \dot{\Psi} &= \frac{1}{2} \left[(\dot{K}U + K\dot{U}) \cdot U + KU \cdot \dot{U} \right] - F \cdot \dot{U} \\ &= \frac{1}{2} (\dot{K}U \cdot U + \underbrace{K\dot{U} \cdot U}_{\dot{U} \cdot K^\top U, \text{ where } K^\top = K} + KU \cdot \dot{U}) - F \cdot \dot{U} \\ &= \frac{1}{2} (\dot{K}U \cdot U + KU \cdot \dot{U} + KU \cdot \dot{U}) - F \cdot \dot{U} \\ &= \frac{1}{2} (\dot{K}U \cdot U + 2KU \cdot \dot{U}) - F \cdot \dot{U} \\ &= \frac{1}{2} \dot{K}U \cdot U + \underbrace{KU \cdot \dot{U}}_F - F \cdot \dot{U} \\ &= \frac{1}{2} \dot{K}U \cdot U + (F - F) \cdot \dot{U} \\ &= \frac{1}{2} \dot{K}U \cdot U \end{aligned} \quad (29)$$

and using the expression obtained for \dot{K} , provided by Eq. (25)

$$\dot{\Psi} = -\frac{1}{2}(1 - \gamma)p\rho^{p-1}K_0^e U^e \cdot U^e \quad (30)$$

By the definition of the contrast in Eq. (17) and the material characterization equation in (3), we finally obtain

$$\dot{\Psi} \approx \begin{cases} -\frac{3}{2}(1 - \rho_0)K_0^e U^e \cdot U^e, & \text{in } \mathcal{D} \setminus \omega \\ +\frac{3}{2}(1 - \rho_0)\rho_0 K_0^e U^e \cdot U^e, & \text{in } \omega \end{cases} \quad (31)$$

3.2 Sensitivity of the energy dissipation term

Since the last term in Eq. (15) represents energy dissipation, the topological derivative associated with this term can be given by

$$\dot{\Psi} = \begin{cases} +\kappa|\omega^e|, & \text{em } \mathcal{D} \setminus \omega \\ -\kappa|\omega^e|, & \text{em } \omega \end{cases} \quad (32)$$

4. RESULTING ALGORITHM

The topological sensitivity analysis provides a first-order correction to the shape functional when an infinitesimal perturbation is introduced into the domain. Therefore, it is possible to decrease the value of the shape functional by nucleating infinitesimal inclusions in regions where the topological derivative is negative. Since due to practical reasons only finite size perturbations can be created, an algorithm based on introducing finite size inclusions in these regions is proposed. If the size of the inclusions are small enough, but at the same time large enough to be treated numerically, the Francfort-Marigo functional is expected to decrease. The size of the inclusion is associated with the region ω^* where the topological derivative field is negative, i.e.,

$$\omega^* := \{x \in \mathcal{D} : D_T \Psi_\omega(x) < 0\} \quad (33)$$

In principle ω^* should not be a connected subset, i.e. there should be a nucleation of damage ahead of the previous damage, but also in other parts of the body. In the former case, the nucleation of damage produces the evolution of the damage set, while in the latter it means nucleation of genuine damage. It should be emphasized that from a theoretical point of view, the topological derivative keeps away from the damage region and for an infinitesimal inclusion only. On the other hand, the topological derivative can be used as a steeper descent direction in the optimization process as in any method based on the gradient of the objective functional. However, for practical purposes, since the numerical method introduces a grid of finite size, we will consider nucleation of inclusions of finite sizes but small enough such that a decreasing of the Francfort-Marigo functional in each iteration is ensured. It should also be noted that the topological gradient can be used instead of the shape gradient (as done in Allaire *et al.* (2011)) to compute the evolution time of the damage region.

That said, it is possible to design the algorithm by nucleating only at the points where the topological derivative achieves its minimum, or at all points where it is negative, while an intermediate choice would be to calibrate the size of the inclusion to be nucleated according to the characteristic size of the previous damaged region. This choice will be provided by the model parameter $\beta \in (0, 1)$, with extreme choices given by $\beta = 0$ (minimum points only), and $\beta = 1$ (the entire negative region), respectively. To this aim, let us introduce the quantity

$$D_T \Psi_\omega^* := \min_{x \in \omega^*} D_T \Psi_\omega(x), \quad (34)$$

thus, it is possible to define the inclusion to be nucleated $\omega^\beta \subset \omega^*$ as follows

$$\omega^\beta := \{x \in \omega^* : D_T \Psi_\omega(x) \leq (1 - \beta)D_T \Psi_\omega^*\}, \quad (35)$$

where $\beta \in (0, 1)$ is chosen such that $|\omega^\beta| \approx \pi\delta^2/4$ (and $|\omega^\beta| \leq \pi\delta^2/4$), so that the size of the inclusion to be nucleated is here related to the width of the initial damage δ . Therefore, if the initial damage is crack-like (δ small), β will be taken as small as to satisfy $|\omega^\beta| \leq \pi\delta^2/4$. By this choice, a damage will evolve like a crack. As a matter of fact, the parameter β induces a threshold for the topological derivative $D_T \Psi_\omega(x)$ and the volume of the inclusion will only depend on δ , while its shape will depend on the contour lines (level-sets) of $D_T \Psi_\omega$. The experiment demonstrated in the next section shows that this strategy ensures a minimization of the Francfort-Marigo functional in each iteration, provided that the size of the nucleation to be nucleated ω^β is small enough.

The algorithm can be outlined as follows. Given the solution of the linear elasticity system (12), the associated topological derivative field (26) is evaluated. If the field is positive everywhere or $|\omega^*| < \pi\delta^2/4$, a perturbation of size $\pi\delta^2/4$ at any point of the domain is likely to increase the value of the functional. In this case, the algorithm will not propagate the damage, and it is possible to increase the load g_i further and run a new analysis. On the contrary, if the topological derivative field is negative in some undamaged region and the condition $|\omega^*| \geq \pi\delta^2/4$ is fulfilled, a damage ω^β will be nucleated inside ω^* , with $\beta : |\omega^\beta| \approx \pi\delta^2/4$ (and $|\omega^\beta| \leq \pi\delta^2/4$). Schematically, one can see the newly-damaged region as an half-disk of radius $\delta/2$ located at the tip of the pre-existing damage. Since the nucleation of a new damage ω^β modifies the problem, the solution to the elasticity system associated with the new topology have to be computed again. Finally, the new topological derivative field is evaluated and the process is repeated until the condition $|\omega^*| \geq \pi\delta^2/4$ is not fulfilled anymore for any load increment. The elasticity system is solved by the Finite Element Method. In order to improve the numerical results, the mesh at the crack tip is intensified in each iteration of the optimization process. The above procedure written in the form of pseudo-code is given in Algorithm 1.

Algorithm 1 – The damage evolution algorithm.

Input: $\mathcal{D}, \omega, \delta, N, g_0, \Delta g_i$
Output: The optimal topology ω^*

```

1  begin
2      for  $i = 1 : N$  do
3          solve elasticity system (12);
4          evaluate the topological derivative  $D_T\Psi_\omega$  according to (26);
5          compute the threshold  $\omega^*$  from (33);
6          while  $|\omega^*| \geq \pi\delta^2/4$  do
7              intensify the mesh at the crack tip;
8              solve elasticity system and evaluate  $D_T\Psi_\omega$ ;
9              compute the threshold  $\omega^*$  from (33);
10             compute the threshold  $\omega^\beta$  from (35);
11             nucleate new inclusion  $\omega^\beta$  inside  $\omega^*$ ;
12             update the damaged region:  $\omega \leftarrow \omega \cup \omega^\beta$ ;
13             solve elasticity system and evaluate  $D_T\Psi_\omega$ ;
14             compute the threshold  $\omega^*$  from (33);

```

5. BITTENCOURT'S EXPERIMENT

Some available experimental results used to test Algorithm 1 can be found in Ingraffea and Grigoriu (1990). The geometry of interest for these experiments is shown in Fig. 4 where all dimensions are given in inches. In particular, we highlight the three holes located between the load and initial crack. Thus, the scope is now the study of the influence of these holes on the crack trajectory.

The different cases treated by this geometry differ by the position of the crack with respect to the applied load, given on the one hand by the distance c , and on the other hand by the dimension of the initial crack length denoted as h , which are shown in Tab. 1. The additional parameters used to test the algorithm are shown in Tab. 2.

In the first case ($h = 1,5$ in and $c = 5,0$ in) the experimental trajectory does not reach the first hole, but it is immediately oriented toward the second one. The proposed algorithm to the Discrete Topological Derivative (DTD) was able to reproduce the experimental and numerical result presented to the Continuum Topological Derivative (CTD), as shown in Fig. 5a. In the second case ($h = 1,0$ in and $c = 6,0$ in) the experimental trajectory is oriented directly toward the second hole. Again, the proposed algorithm was able to reproduce the experimental and numerical results, as presented in Fig. 5b.

	c in	h in
Bittencourt 1	5.0	1.5
Bittencourt 2	6.0	1.0

Table 1 – Bittencourt's experiments. Position and length of the initial damage.

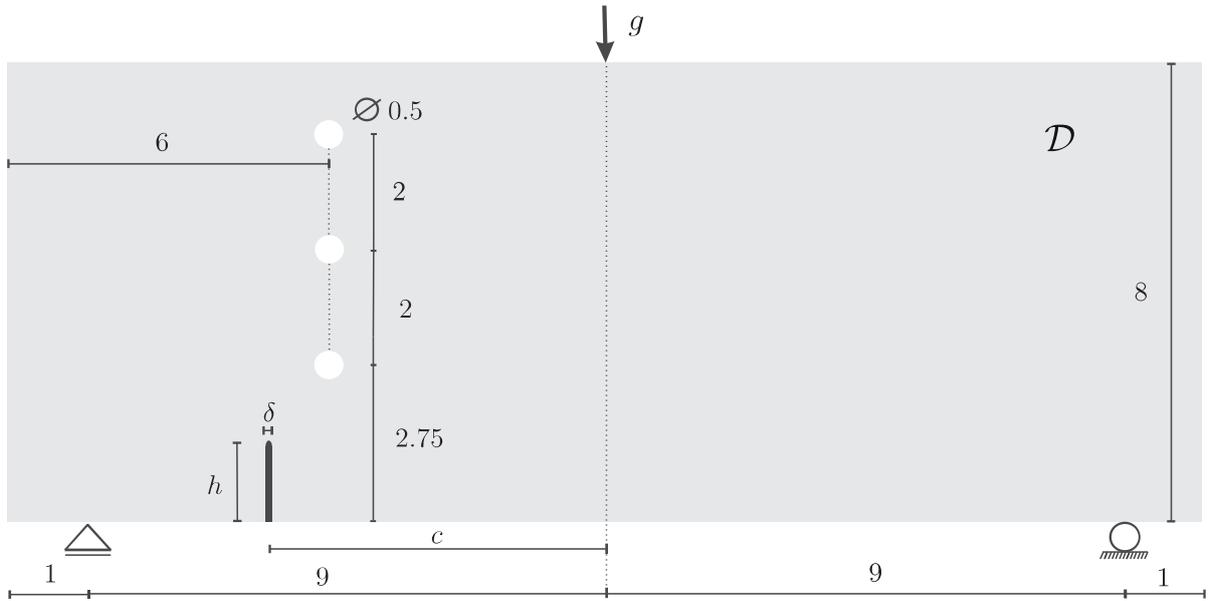


Figure 4 – Bittencourt’s experiment. Geometry and boundary conditions.

Parameter	Value	Parameter	Value
N	100	E	4.5×10^5 psi
δ	0.005 in	ρ_0	10^{-6}
l	$2\delta/3$ in	ν	0.35
g	0.20 in	κ_s	15 (in – lbf)/in

Table 2 – Bittencourt’s experiments. Parameters.

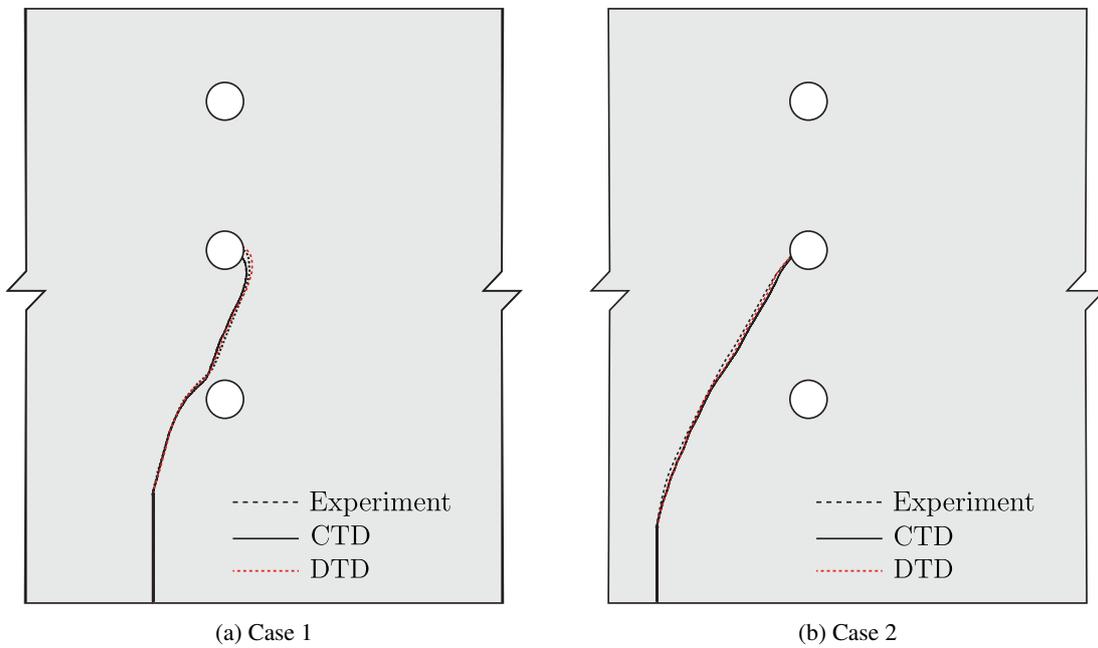


Figure 5 – Bittencourt’s experiment. Final results.

6. CONCLUSIONS

In the present study, an algorithm for the Fracfort-Marigo damage model was proposed based only on the concept of discrete topological derivative, which allowed us to achieve overwhelming results. The designed algorithm incorporates a sequence of finite perturbations according to the contour lines of the topologically derived field, which can be seen as a

direct extension of the infinitesimal perturbation concept for numerical purposes.

This study validated a proposed discrete approach to solving fracture mechanics problems in simple geometries. The next steps include the application of this technique to non-linear and multiphysics problems, as well as exploring the sensitivity of the results to the indicated model parameters κ and β . Furthermore, the extension of the technique to the \mathbb{R}^3 domain and the evaluation of structures composed of more than one material are of practical-scientific interest and deserve additional investigation. The results of this study provide a solid foundation for future research in this area, and we believe that this simplified approach can be a useful tool for solving complex problems of engineering interest.

7. ACKNOWLEDGEMENTS

Thanks are addressed to CAPES (Brazilian Higher Education Staff Training Agency) for the master's scholarship to the first author.

8. REFERENCES

- Allaire, G., de Gournay, F., Jouve, F. and Toader, A.M., 2005. "Structural optimization using topological and shape sensitivity via a level set method". *Control and Cybernetics*, Vol. 34, No. 1, pp. 59–80.
- Allaire, G., Jouve, F. and Van Goethem, N., 2011. "Damage and fracture evolution in brittle materials by shape optimization methods". *Journal of Computational Physics*, Vol. 230, No. 12, pp. 5010–5044.
- Ambrosio, L. and Tortorelli, V., 1992. "On the approximation of free discontinuity problems". *Bollettino dell'Unione Matematica Italiana*, Vol. 6, pp. 105–123.
- Amstutz, S. and Andrä, H., 2006. "A new algorithm for topology optimization using a level-set method". *Journal of Computational Physics*, Vol. 216, No. 2, pp. 573–588.
- Bittencourt, T.N., Wawrzynek, P.A., Ingrassia, A.R. and Sousa, J.L., 1996. "Quasi-automatic simulation of crack propagation for 2d LEM problems". *Engineering Fracture Mechanics*, Vol. 55, No. 2, pp. 321–334.
- Bourdin, B., Francfort, G.A. and Marigo, J.J., 2008. "The variational approach to fracture". *Journal of Elasticity*, Vol. 91, No. 1-3, pp. 5–148.
- Francfort, G.A. and Marigo, J.J., 1993. "Stable damage evolution in a brittle continuous medium". *European Journal of Mechanics, A/Solids*, Vol. 12, No. 2, pp. 149–189.
- Huang, X. and Xie, Y.M., 2009. "Bi-directional evolutionary topology optimization of continuum structures with one or multiple materials". *Computational Mechanics*, Vol. 43, No. 3, pp. 393–401.
- Ingrassia, A.R. and Grigoriu, M., 1990. "Probabilistic fracture mechanics: A validation of predictive capability". Technical report, Cornell University, Ithaca, New York.
- Larrabide, I., Feijóo, R.A., Novotny, A.A. and Taroco, E., 2008. "Topological derivative: a tool for image processing". *Computers & Structures*, Vol. 86, No. 13–14, pp. 1386–1403.
- Maso, G.D. and Iurlano, F., 2013. "Fracture models as Gamma-limits of damage models". *Communications on Pure and Applied Mathematics*, Vol. 12, No. 4, pp. 1657–1686.
- Novotny, A.A. and Sokołowski, J., 2020. *An introduction to the topological derivative method*. Springer Briefs in Mathematics. Springer Nature Switzerland. doi:10.1007/978-3-030-36915-6.
- Sokołowski, J. and Żochowski, A., 1999. "On the topological derivative in shape optimization". *SIAM Journal on Control and Optimization*, Vol. 37, No. 4, pp. 1251–1272.
- Xavier, M., Fancello, E.A., Farias, J.M.C., Van Goethem, N. and Novotny, A.A., 2017. "Topological derivative-based fracture modelling in brittle materials: A phenomenological approach". *Engineering Fracture Mechanics*, Vol. 179, pp. 13–27.

9. RESPONSIBILITY NOTICE

The following text, properly adapted to the number of authors, must be included in the last section of the paper:
The author(s) is (are) solely responsible for the printed material included in this paper.