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A COMPARISON BETWEEN SIMILARITY SOLUTION AND EXPERIMENTAL DATA OF INCOMPRESSIBLE PLANAR MIXING LAYER

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Abstract. Similarity solutions of the boundary layer equations for a plane mixed layer flowing from left to right are traditionally solved by assuming that the separating streamline emanating from the horizontal divider plate remains horizontal downstream of it. This effectively makes the vertical velocity profile unidirectional, flowing from the slowest to the fastest layer. However, experimental and direct numerical simulation (DNS) data contradict these findings. It turns out that entrainment occurs, where the fluid flows from both layers towards the center line of the mixing layer. As a consequence, this center line moves towards the slower flow more and more as the streams flow downstream. The present work tries to correct this problem by modifying the center line boundary condition used in similarity solution and then comparing with the experimental data present in the references bibliographies.

Keywords: Mixing layers, Blasius Solution, Shooting method, Experimental data

1. INTRODUCTION

During the last decades the phenomenon of the mixing layer, which configures a free shear flow has been the source of extensive experimental and numerical research for engineering application (Teixeira and Alves, 2012). In this way, the mixing layer has several applications such as the dissemination of underwater waste in oceans and rivers, control of the pattern of temperature factors in gas turbine engines, mixing of reagents in a combustion chamber and dispersion of combustion from chimneys to the atmosphere (Teixeira and Alves, 2012; Silva, 2022; Shchepakina *et al.*, 2023). However, the focus this work is planar mixing layer, where is formed when two streams with different speeds meet downstream of a splitter plate (Teixeira and Alves, 2011). As shown in Fig. 1, where parallel flows with velocities U_1 (upper) and U_2 (lower) interacting with each other and take the x -axis as the longitudinal direction and y is the normal direction in physical space. The origin is taken as the point at which the two fluids must first come into contact.

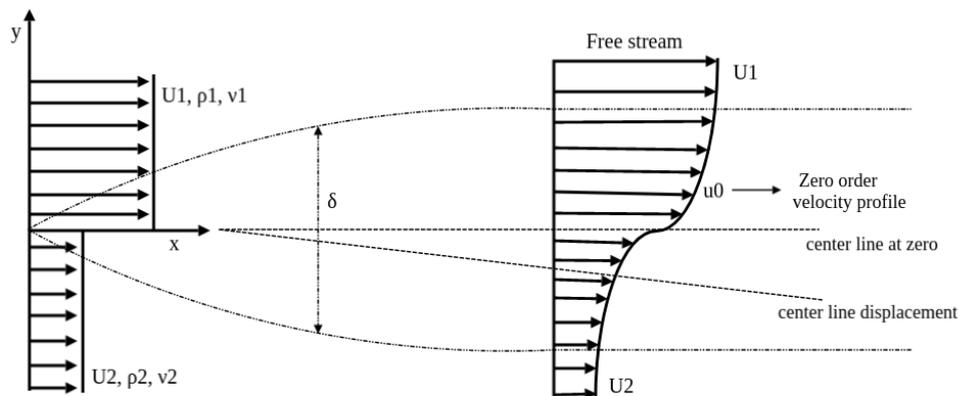


Figure 1. Schematic drawing of the flow to mixing layer phenomenon.

Lessen (1949), Lock (1951), Chapman (1943) and Crane (1957) began analyzing the mixed layer on uniform, laminar and parallel stream. However, a difference between these works is the classification of flow for incompressible and compressive cases, this information changes the analysis of the problem completely (Haukea and Hughesb, 1998). In this way, Lock (1951) described the incompressible case planar mixing through the boundary layer equation's similarity solutions, where thoroughly examined for the stream function with no pressure gradient, which results in the Blasius equation. These similarity solutions are accurate in describing the flow behavior of the planar mixed layer (Diyesperov, 1986). However, a good description only occurs when it is in a region far enough away from the splitter plate for the similarity theory to be valid, but not far enough away to leave laminar flow and transition to turbulent flow (Salemi, 2007).

Blasius equation for the planar mixing layer is mathematically classified as an ordinary differential equation (ODE) of third order non-linear (Schlichting and Gersten, 2017). This presupposes that in order to address this phenomenon, three boundary conditions are required. Nevertheless, there are only two evident boundary conditions are that the longitudinal component of the velocity at the upper and lower edges of the boundary layer, where match the velocity of the uniform stream (Klemp and Acrivos, 1972). But, there is a lack of the third boundary condition present at the interface when both flows are subsonic, which transforms the problem into infinite solutions (Ting, 1959). This implies that the interface between the streams show a center line that directs the position of the flow on space, where this center line is unknown, can be displaced up or down on the physical domain (Casarella and Choo, 1968).

Lock (1951) and Ting (1959) were the first to use the asymptotic expansions method to study the absence of a third boundary condition and find a unique solution, considering the normal velocity of the problem to be zero $v(0) = 0$, i.e., there is no displacement of the center line. The contrary is seen in numerical and experimental analysis, where this null condition is false on the common interface for the situations that are most similar to the real ones (Huang and Ho, 1990; Olsen and Dutton, 2003). Therefore, to investigate the effects of this third boundary conditions for the mixing layer similarity solution, a numerical solution approach is applied.

One of the most well-known techniques for mixing layer is shooting method (Sandham, 1989; White, 1991; Lu and Lele, 1994). This method solves the boundary value problem (BVP) by reducing it to an initial value problem (IVP) (Ahsan and Farrukh, 2013). However, Osborne (1969) shown two issues with the technique, including the difficulty of finding a good estimative for the inical conditions, which causes divergence in the iterative method of solving nonlinear algebraic equations, and ill-conditioned mathematical problems.

Our interest in this study is show a correction approach of the third boundary condition to determine the position of center line displacement through the similarity solution for plane mixing layer. In this way, was adopt also a experimental data from Huang and Ho (1990) to validate the difference between the theory and experimental analysis with velocity profiles. This investigation is the beginning of a proposal for future optimizations of parameters for the CFD simulation of the mixture problem, to make the simulation more efficient and accurate.

2. METHODOLOGY

2.1 Governing Equations

The Prandtl boundary layer equations for two-dimensional steady laminar flow of incompressible fluid and no gradient pressure over mixing layer phenomen are given by Eq.(1) and (2).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (2)$$

Where, u and v are velocity coordinates on the (x, y) axes and a ν is the kinematic viscosity. Then, similar transformations search for reduce the Eq.(1) and (2) into a single ODE called Blasius equation, Eq.(3). To carry out these transformations implies reducing the number of independent variables, in this case x and y , to just one dimensionless variable η , known as the stretching factor ($\eta = (U_1/\nu x)^{1/2}$).

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0 \quad (3)$$

The boundary conditions in the upper part when $u = U_1$ for $\eta = +\infty$ and for the lower condition $u = U_2$ for $\eta = -\infty$, referring to Eq.(4) and(5). The question arises about the lack of the third condition present on interface. Lock (1951) assumed the interface center line like a constant, that way streamline ($\psi=0$) and

imposing the velocity $v = 0$ when $y = 0$, this condition is satisfied forming a boundary condition with null displacement center line at $\eta = 0$, Eq.(6).

$$\eta \rightarrow +\infty : f'(\eta) = 1 \quad (4)$$

$$\eta \rightarrow -\infty : f'(\eta) = \frac{U_2}{U_1} = \lambda \quad (5)$$

$$\eta \rightarrow 0 : f(\eta = 0) = 0 \quad (6)$$

The other requirements at the interface to be satisfied are that the velocity and the normal and transverse components of the stress and the tangential stress is $\mu(\partial u/\partial y)$ must be continuous. Forming the complete boundary conditions over the interface described by Eq.(7) and (8).

$$f'_1(0) = f'_2(0) \quad (7)$$

$$f''_1(0) = f''_2(0) \quad (8)$$

where, the subscripts 1 and 2 indicate the upper and lower of the interface between the respective fluids.

2.2 Center line displacement for similarity solution

This study suggests an adapted boundary condition in the interface region so that a controlled displacement of the center line for mixing layer. That way, to change the condition imposed by the literature in Eq.(6), when $\eta = 0$, assumes a constant to change the position of the center line which becomes $\eta = C$, Eq.(9). Therefore, redefining and satisfying the similar solutions, consider that the velocity component $v(x, y)$ must be equal to zero ($v = 0$), which presents a boundary condition described in Eq.(10).

$$C = y\sqrt{\frac{U_\infty}{2\nu x}} \quad (9)$$

$$\eta \rightarrow C : f(\eta = C) = Cf'(C) \quad (10)$$

3. NUMERICS

For all numerical methods in this paper, FORTRAN 90 algorithms are used, and Mathematica 12 is used to depict the data so that the mixing layer phenomenon may be seen.

3.1 Shooting method for Blasius equation

The numerical method implemented to solve the Blasius problem is the shooting method. This method transforms Eq.(3) into a first-order reduced ODE's system, Eq.(11). Afterwards, the formation of the system of equations uses a spatial discretization with the application of the 4th Order Runge-Kutta method to carry out the solution march, however to adopt this method there is the need for the initial conditions to be satisfied for both regions of the flow. For this, it was necessary to divide the domain into two parts and march from the center of the mixture layer towards the ends to the borders in $+\infty$ and $-\infty$.

$$\begin{cases} \phi'_1 = \phi_2 \\ \phi'_2 = \phi_3 \\ \phi'_3 = -\frac{1}{2}\phi_1\phi_3 \end{cases} \quad (11)$$

Where, the terms f , f' , f'' and f''' are respectively represented by ϕ_1 , ϕ_2 , ϕ_3 and ϕ'_3 . The initial conditions Eq.(12) is used when the center line is not displacement showed in section 2.1. Soon, two guesses of march $a1$ and $a2$ are necessary, which are constants of the interaction.

$$\begin{cases} \phi'_1(\eta = 0) = 0 \\ \phi'_2(\eta = 0) = a1 \\ \phi'_3(\eta = 0) = a2 \end{cases} \quad (12)$$

Nonetheless, the numerical solution to verify the displacement of the center line, applies the same method of shooting with the system of ordinary equations, Eq.(11), for this displacement to occur, must change the initial conditions where $\eta = C$, introducing the new initial conditions of march Eq.(13).

$$\begin{cases} \phi'_1(\eta = C) = C a1 \\ \phi'_2(\eta = C) = a1 \\ \phi'_3(\eta = C) = a2 \end{cases} \quad (13)$$

4. RESULTS AND VALIDATION

4.1 Experimental results from the literature

The experimental data was used to compare with the Blasius solution described by the similarity method, verify the behavior of the center line displacement through the velocity profiles and show a novel approach to the boundary conditions to determine the correct direction of the center line displacement between the experiment and the adapted similar solution. Currently, there are many well-known works to show the experimental results of the mixing layer phenomenon. Thus, this work uses the experimental results of Huang and Ho (1990) conducted in an open-circuit wind tunnel. The tunnel experiment was split into two streams that were separated by a plate and later combined to produce a mixing section. As a result, the velocity profiles measured at the trailing edge of the splitter plate represented both the wake and the mixing layer, but the wake vanished near $x = 0.8$. Browand and Latigo (1979) and Huang (1985) published papers that provide detailed documentation of the wind tunnel and flow properties used by Huang and Ho (1990).

4.2 Validation of the incompressible planar mixing layer

The planar mixing layer's validation to similarity solution is done through the numerical solution when the center line displacement is zero ($\eta = 0$) and compare with the data provided by Lock (1951). For this validation, derivatives of the dependent variable $f(\eta)$ are used with a speed ratio of $\lambda = 0$.

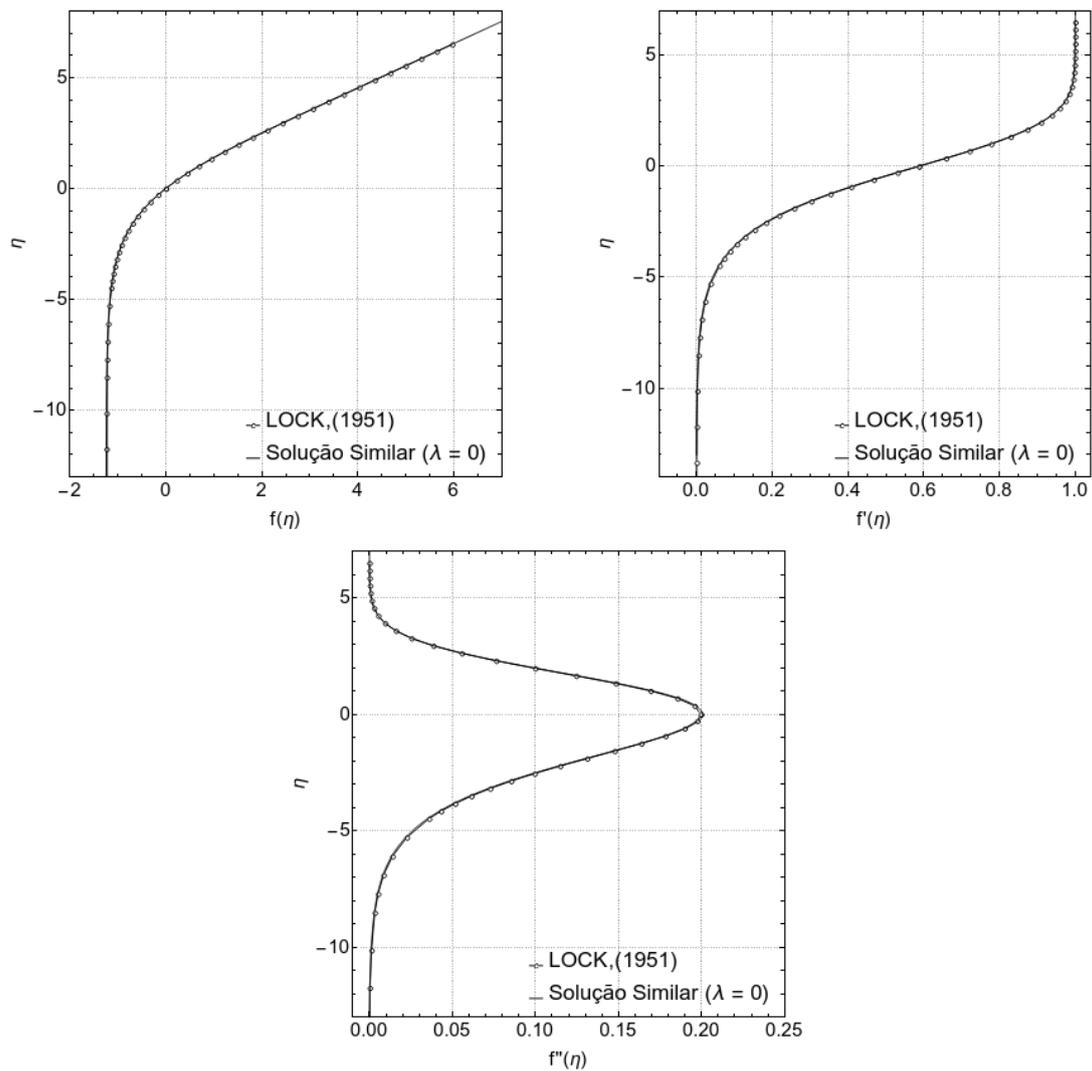


Figure 2. Validation between the similarity solution and data from Lock (1951) for the values of $f(\eta)$, $f'(\eta)$, and $f''(\eta)$ for $\lambda = 0$.

4.3 Velocity distributions for the incompressible mixing without displacement of center line

Based on initial conditions, Eq.(12), this study examined how velocity distributions behave for the incompressible mixing layer. For this, it was determined that $\lambda = 1/2$ for various positions of x (0.1, 0.5, 1.0) resulted in the velocity coordinate profiles for $u(x, y)$ and $v(x, y)$, Fig.(3).

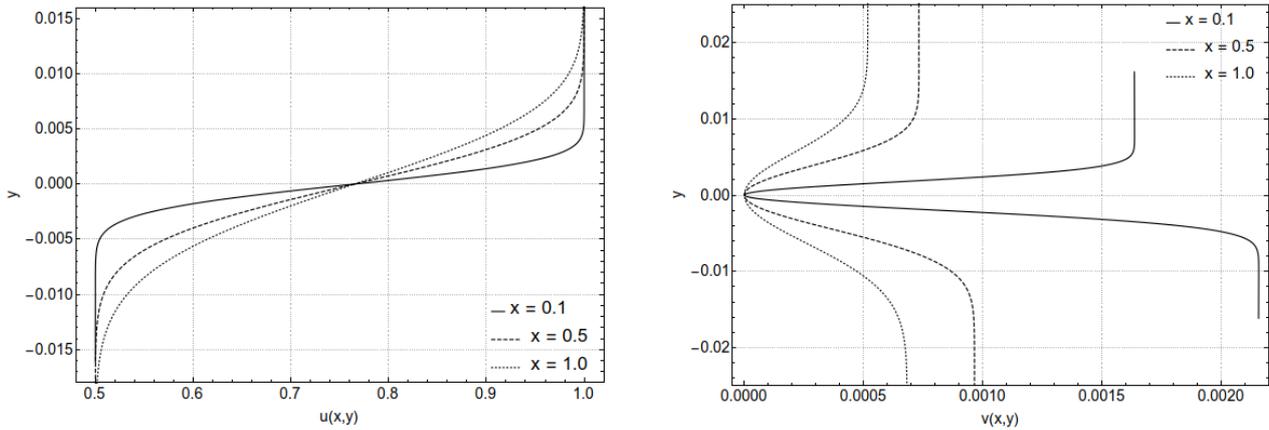


Figure 3. Velocity profile of the coordinates: longitudinal velocity $u(x, y)$ and transverse velocity profile $v(x, y)$ in relation to the y axis to three different positions x .

These velocity coordinate profiles are based on $u(x, y)$, which provides information about the effects of shear thickness, and $v(x, y)$, where displays the direction of the mixing flow's inlet and outflow. Thus, Figure 3 shows the longitudinal velocity u at three points on the x -axis, where the thickness shear layer increases with the grow position of x , smoothing the discontinuity introduced at $x=0$ and transverse velocity v for $y = 0$ is always positive regardless of the position of x . Therefore, no part of the fluid in the upper region can be accelerated by contact with the lower stream, and no part of the fluid in the lower region can be decelerated. As a result, the acceleration is always zero at the point of contact between the two fluid regions, which is the streamline through the origin, and the profile has an inflexion point at that location.

4.4 Velocity distributions for the incompressible mixing with displacement of center line

In order to find a similar solution with the displacement of the center line, the initial condition of the interface Eq.(14) was applied, which adopts analyzes referring to the positive ($C > 0$) and negative ($C < 0$) values of the constant for the velocity profiles of u and v , respectively, as shown in Figs. 4, 5 and 6 for various x -axis positions.

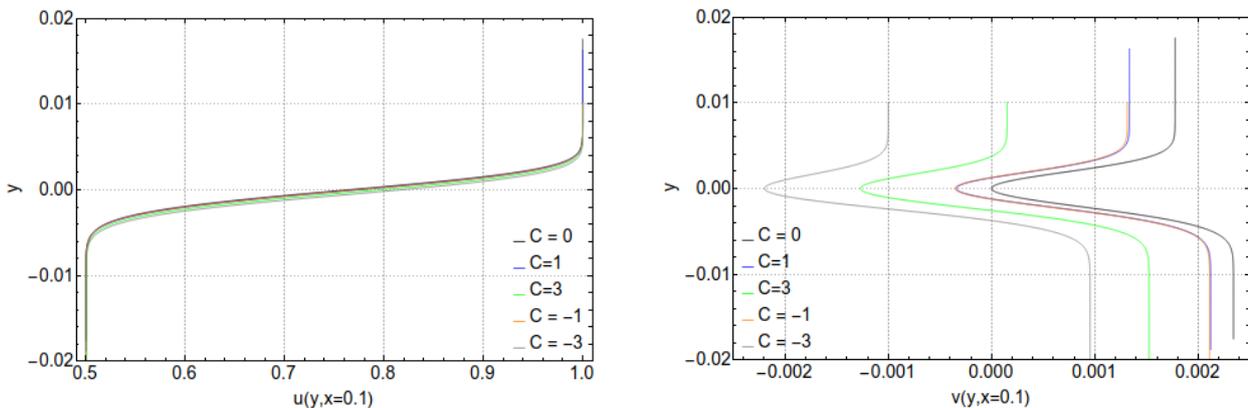


Figure 4. Profiles of longitudinal u and transversal u velocity for various constants C with the position $x = 0.1$.

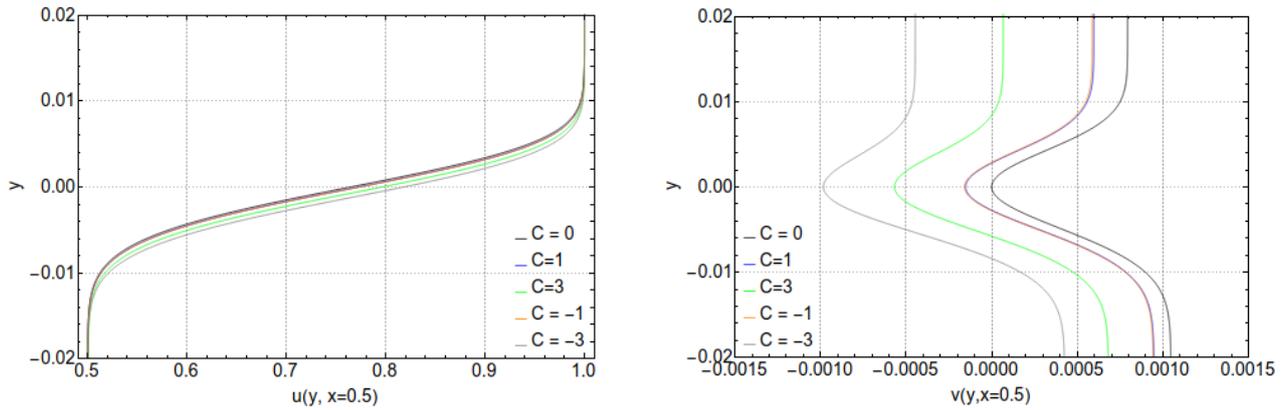


Figure 5. Profiles of longitudinal u and transversal u velocity for various constants C with the position $x = 0.5$.

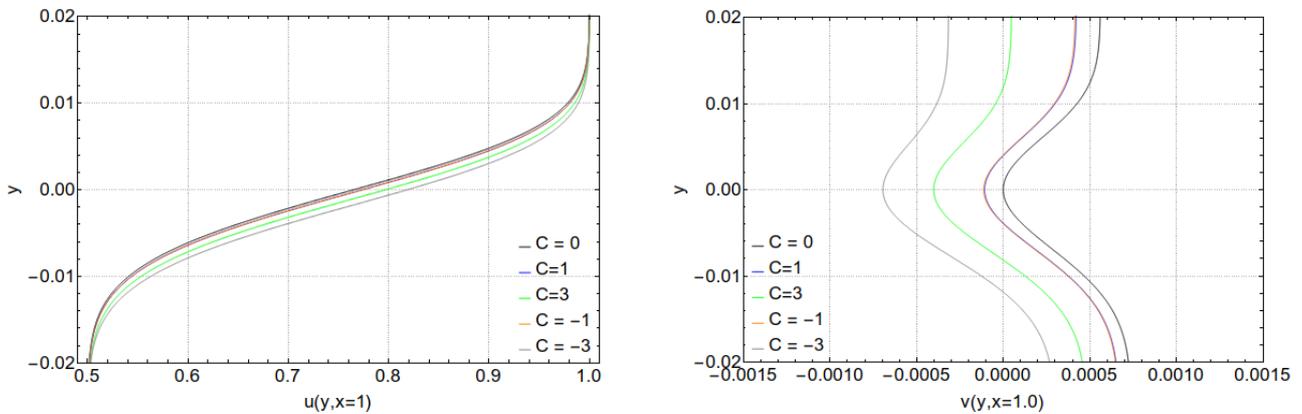


Figure 6. Profiles of longitudinal u and transversal u velocity for various constants C with the position $x = 1.0$.

The similarity approach's implementation of constants is shown in Figs. (4), (5), and (6). For the longitudinal velocity u with several constant presents a similar characteristic to the thickness of the shear layer in the various positions of $x(0.1, 0.5, 1)$. Nonetheless, the transverse velocity v shows different behaviors when changed the constants value, which the case $C=0$ is returned to the result presented in Section 4.2, where the flow is always positive with fluid entering at upper and lower part of the mixing layer. Although the positive constants $C = 1$ and $C = 3$ exhibit a direction change in the middle with a fluid inlet at the upper part, this analysis gives the false impression regarding the idea of the mixing phenomena. However, for negative constants, when $C = -1$ presents behavior similar to $C = 1$, but when $C = -3$ is positive only in the lower part and negative in the upper part, i.e., it shows that the direction of flow is altered, proving the displacement of the center line to the negative side of y .

4.5 Comparison of experimental data and Blasius solution for mixing layer

This section investigates the comparison between the velocity profile with experimental data presented in Section 4.1, the Blasius solution for $\eta = 0$ described by Lock, and the new proposal for changing the boundary condition in the Blasius similar solution. However, the longitudinal velocity profile u is not relevant to analyzing the center line displacement or comparing the solutions. Therefore, this comparison used only the analysis of the transverse velocity profile v for the three cases discussed, Fig. 7.

As a result, Fig. 7 for the experimental data shows that the faster layer pushes the fluid in the direction of the slower layer, causing the center line to shift, i.e., the region's fluid inflow behavior is always positive for the layer with the lowest velocity, while it is always negative for the layer with the highest velocity. However, the Blasius solution, in contrast to the experimental situations, has both positive flow directions with fluid entry at the top and bottom when $\eta = 0$ presenting a divergence in the flow and the center line from the experimental data. But with the new approach to the boundary conditions of the similar solution seen in the velocity profile, it was observed that the negative constants of the similar solution present a behavior in accordance with the experimental results.

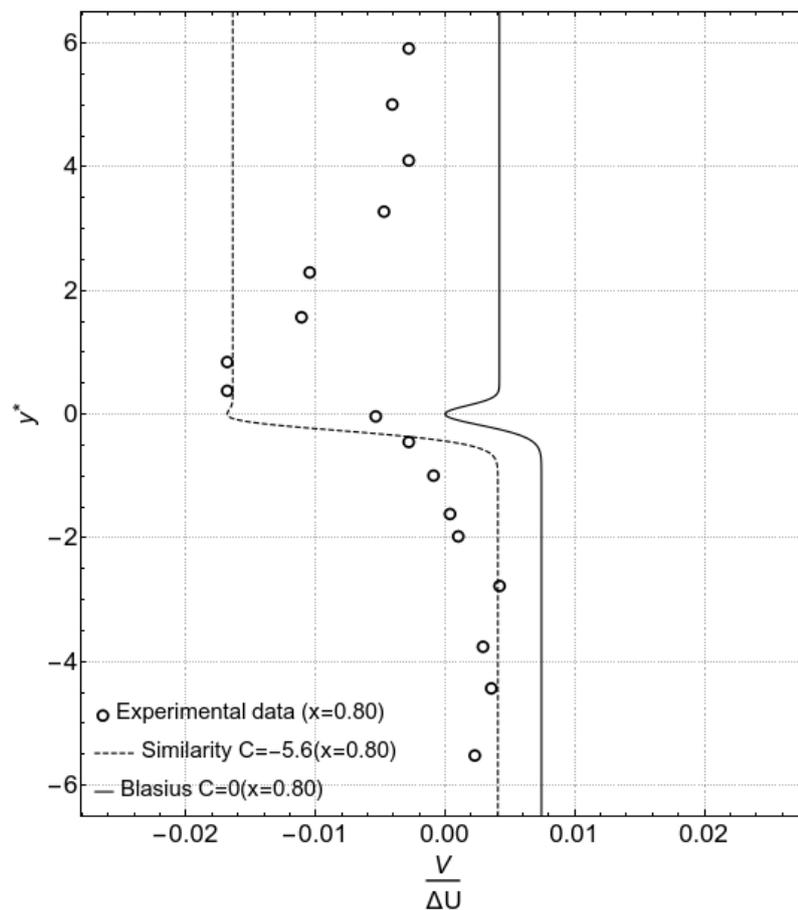


Figure 7. Comparison between the transversal velocity with the experimental data, the Blasius solution ($\eta = 0$) and the modified similarity solution.

5. CONCLUSION

The present paper shows a novel concept to find the position of the center line displacement of a incompressible planar mixed layer from a change in interface boundary conditions using a similarity solution. This change in the interface refers to the imposition of a constant that changes the position of the center line within the physical domain. However, the direction in which this displacement occurs is unknown, so experimental data was used to determine that there is a displacement that runs through the faster layer and pushes the center line towards the slower layer. Therefore, when comparing this new concept with the experimental case, it was concluded that the values of the negative constant findings of the similar solution come closer to the results of the true cases. Furthermore, this work makes it possible, with this new approach, to use inverse estimation problems with greater accuracy to search for the correct displacement of the center line and subsequently optimize boundary conditions and initial conditions in computational simulations of CFD on the phenomenon of mixing.

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7. REFERENCES

- Ahsan, M. and Farrukh, S., 2013. "A new type of shooting method for nonlinear boundary value problems". *Alexandria Engineering Journal*, Vol. 53, pp. 801–805.
- Browand, F.K. and Latigo, B.O., 1979. "Growth of the twodimensional mixing layer from a turbulent and

- nonturbulent boundary layer”. *Phys. Fluids*, Vol. 22, pp. 1011–1019.
- Casarella, M.J. and Choo, Y., 1968. “Displacement effects on the laminar mixing of two parallel streams”. *Aeronautical Quarterly* , Vol. 19, pp. 388 – 402.
- Chapman, D., 1943. *Laminar mixing of a compressible fluid*. Springer-Verlag Berlin Heidelberg. NACA-TN-1800.
- Crane, L.J., 1957. “The laminar and turbulent mixing of jets of compressible fluid. part ii, the mixing of two semi-infinite streams”. *Journal of Fluid Mechanics*, Vol. 3, p. 81.
- Diyesperov, V., 1986. “investigation of selfsimilar solutions describing flows in mixing layers”. *Pergamon Journals Ltd*, Vol. 50, pp. 303–312.
- Haukea, G. and Hughesb, T.J., 1998. “A comparative study of different sets of variables for solving compressible and incompressible flows”. *Comput. Methods Appl. Mech. Engrg.*, Vol. 153, p. 1–44.
- Huang, L.S., 1985. “Small scale transition in a two-dimensional mixing layer”. *PhD thesis, University of Southern California, Los Angeles, CA*.
- Huang, L.S. and Ho, C.M., 1990. “Small-scale transition in a plane mixing layer”. *J. Fluid Mech*, Vol. 210, pp. 475–500.
- Klemp, J.B. and Acrivos, A., 1972. “A note on the laminar mixing of two uniform parallel semi-infinite streams”. *J. Fluid Mech*, Vol. 55, pp. 25–30.
- Lessen, M., 1949. “On the stability of the free laminar boundary layer between parallel streams.” *NACA TN*, Vol. 50, pp. 303–312.
- Lock, R.C., 1951. “The velocity distribution in the laminar boundary layer between parallel streams”. *Mech. and Applied Math*, Vol. IV(1), p. 42–63.
- Lu, G. and Lele, S.K., 1994. “On the density ratio effect on the growth rate of a compressible mixing layer”. *Physics of Fluids*, Vol. 6, p. 1073–1075.
- Olsen, M.G. and Dutton, J.C., 2003. “Planar velocity measurements in a weakly compressible mixing layer”. *J. Fluid Mech.*, Vol. 486, p. 51–77.
- Osborne, M.R., 1969. “On shooting methods for boundary value problem”. *Journal of Mathematical Analysis and Applications*, Vol. 27, pp. 417–433.
- Salemi, L.C., 2007. *Análise de estabilidade linear de camada de mistura compressível binária*. Master’s thesis, Universidade Federal Fluminense, São José dos Campos.
- Sandham, N.D., 1989. *a numerical investigation of the compressible mixing layer*. Master’s thesis, University of Southampton, Southampton, England.
- Schlichting, H. and Gersten, K., 2017. *Boundary Layer Theory, Ninth Edition*. Springer-Verlag Berlin Heidelberg. Cap. 23, Pages 683-708.
- Shchepakina, E.A., Zubrilin, I.A., Kuznetsov, A.Y., Konstantin, Tsapenkov, D., Antonov, D.V., Strizhak, P.A., Yakushkin, D.V., Ulitichev, A.G., Dolinskiy, V.A. and Morales, M.H., 2023. “Physical and chemical features of hydrogen combustion and their influence on the characteristics of gas turbine combustion chambers”. *Appl. Sci.*, p. 13.
- Silva, M.C.N., 2022. *The influence of Kelvin-Helmholtz instability in a mixing layer: a simple model for study of fire safety between two parallel walls*. Ph.D. thesis, Universidade Federal do Pampa, Florianópolis, Brasil.
- Teixeira, R.S. and Alves, L.S.B., 2011. “self-excited compressible planar mixing layers at arbitrary mach numbers: numerical or physical phenomena”. *International Congress of Mechanical Engineering*, Vol. 21.
- Teixeira, R.S. and Alves, L.S.B., 2012. “Modelling far-field entrainment in compressible flows”. *International Journal of Computational Fluid Dynamics*, Vol. 26(1), p. 67–78.
- Ting, L., 1959. “on the mixing of two parallel streams”. *Journal of Mathematics and Physics*, Vol. 38, p. 153–165.
- White, F.M., 1991. *Viscous Fluid Flow*. McGraw Hill.

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