27th ABCM International Congress of Mechanical Engineering
December 04-08, 2023, Florianópolis-SC, Brazil

COB-2023-0777 ON THE USE OF THE GENERALIZED INTEGRATING FACTOR FOR SOLVING COUPLED SYSTEMS OF ORDINARY SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. *Systems of coupled linear second order Ordinary Differential Equations, ODEs, appear in fields like vibration, electric circuitry, applied mathematics and physics. Analytically solving these differential equations, especially for non-trivial excitation functions, is paramount for problems that depend on solution accuracy. Nonetheless, there is no general analytic solution procedure for non-homogeneous linear ordinary second order differential equations. This work proposes an extension of the Leibniz integrating factor to account for ODEs of higher orders. The approach consists in splitting the coefficients of intermediate derivatives to allow for systematic use of the original concept of integrating factors. Analytical solutions are obtained through a double convolution, avoiding the need of proposing a candidate solution. The method is then particularized for second order ODEs with constant coefficients. Closed-form particular solutions due to Heaviside multiplied by polynomials and Dirac's delta excitations are obtained by using the proposed approach. Unlike existing methods as uncoupled eigen decomposition and Fourier series, the proposed approach can be used to derive analytical solutions by means of exponential maps due to the use of the integrating factor. It is shown that proportional damping renders many efficient particularization to the analytical solutions. Two test cases are addressed and compared to numerical approximations. It is shown that the proposed methodology is accurate, provides the homogeneous and the particular solutions in separate, and is not sensitive to time discretization like numerical methods.*

Keywords: *Integrating factor, constant coefficients second order differential equations, vibration differential equations, electric circuitry differential equations*

1. INTRODUCTION

Systems of coupled second order differential equations (ODEs) are extremely important in Engineering problems such as mechanical vibrations (Kelly, 2000; Rao, 2017; Hughes, 2000), electric circuitry (Irwin and Nelms, 2006; Agarwal and Lang, 2005), control (Zabczyk, 2009), and many other real-world problems (Boyce and Diprima, 2001; Kreyszig *et al.*, 2011). These systems can be represented in matrix notation as

$$\mathbf{M}(t)\ddot{\mathbf{y}}(t) + \mathbf{C}(t)\dot{\mathbf{y}}(t) + \mathbf{K}(t)\mathbf{y}(t) = \mathbf{f}(t), \quad (1)$$

where t is the independent variable, $\mathbf{M}(t)$, $\mathbf{C}(t)$ and $\mathbf{K}(t)$ are matrix coefficients dependent, $\mathbf{y}$ is the vector of dependent variables and $\dot{\mathbf{y}}$ and $\ddot{\mathbf{y}}$ are its derivatives with respect to t.

Many methods have been developed to address these ODEs and they can be broadly classified in analytical, state variables and numerical methods. Analytical methods include undetermined coefficients, variation of parameters, Fourier and Laplace transforms (Abell and Braselton, 2023; Wordu *et al.*, 2022; Boyce and Diprima, 2001; Kreyszig *et al.*, 2011), state variables can be used to lower the order of the differential equation to a first order problem with higher dimensionionality (Chahande and Arora, 1994), and numerical methods are the most used family of techniques. A review of these methods is out of the scope of this work and some examples can be found in (Bathe and Baig, 2005; Hilber *et al.*, 1977; Wilson *et al.*, 1972; Houbolt, 1950; Hulbert and Chung, 1996; Newmark, 1959; Song *et al.*, 2023).

Each method presents an intrinsic disadvantage. Undetermined coefficients and variation of parameters need a previous knowledge of a candidate solution and the homogeneous solution, respectively, (Boyce and Diprima, 2001). Integral transforms need the costly and cumbersome operation of the inverse transform (Abell and Braselton, 2023) and state variables double the dimensionality of the problem, that can be already large (Chahande and Arora, 1994). Numerical methods can be computationally expensive and inaccurate (Song *et al.*, 2023). Therefore, this work proposes a method to systematically find analytical solutions to systems of coupled ODEs, that is a generalization of the Leibniz integrating factor.

It is shown that the Leibniz integrating factor is a particular case of the broader generalized integrating factor and that the integrating factor itself depends on the particular solution of a Riccati nonlinear ODE. The Riccati equation is shown to have a straightforward particular solution in the case of systems of coupled ODEs with constant matrix coefficients, which are used in vibration analysis, electric circuitry and control, to name a few. This particular solution is a constant matrix, that is found by directly solving a quadratic matrix equation. Under the mild assumption that matrices $\mathbf{K}$ and $\mathbf{C}$ commute, the quadratic matrix equation has a closed-form solution (Fogaça and Cardoso, 2023).

Consequently, analytical solutions for ODEs with constant matrix coefficients are provided. These solutions are built using exponential maps, which are known to provide accurate solutions (Song *et al.*, 2023), and extensive research have been made in the past two decades to considerably reduce its computational cost (Botchev, 2013; Vo and Sidje, 2017; Gaudreault *et al.*, 2018; Sastre *et al.*, 2019; Aprahamian and Higham, 2014).

Hence, this work shows the mathematical background and exemplifies the use and advantage of the generalized integrating factor for the homogeneous solution of systems of coupled ODEs with constant matrix coefficients. Two widely used cases of excitation functions - Dirac's delta and Heaviside are investigated in details. These solutions are compared to the reference Newmark-beta numerical method by means of two examples.

2. INTEGRATING FACTOR

Let a system of n ordinary linear first order differential equations

$$\mathbf{C}(t)\dot{\mathbf{y}}(t) + \mathbf{K}(t)\mathbf{y}(t) = \mathbf{f}(t), \tag{2}$$

where $\mathbf{C}$ and $\mathbf{K}$ are $n \times n$ matrix coefficients dependent upon time, t, $\mathbf{y}$ and $\dot{\mathbf{y}}$ are the dependent variable and its first time derivative and $\mathbf{f}$ is an excitation function vector, also function of time. In the well-known Leibniz integrating factor, (Boyce and Diprima, 2001; Kreyszig *et al.*, 2011), the intuition to solve Eq. (2) is to multiply the differential equation (DE) by a function $\boldsymbol{\mu}_1(t)$, the integrating factor, transforming the differential equation into a fully integrable DE,

$$\underbrace{\boldsymbol{\mu}_1(t)\mathbf{C}(t)}_{\mathbf{P}(t)}\dot{\mathbf{y}}(t) + \underbrace{\boldsymbol{\mu}_1(t)\mathbf{K}(t)}_{\dot{\mathbf{P}}(t)}\mathbf{y}(t) = \boldsymbol{\mu}_1(t)\mathbf{f}(t). \tag{3}$$

using the derivative product rule,

$$\int_{t_0}^{t} \underbrace{\boldsymbol{\mu}_1\mathbf{C}}_{\mathbf{P}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_1\mathbf{K}}_{\dot{\mathbf{P}}}\mathbf{y}\,\mathrm{d}t = \boldsymbol{\mu}_1\mathbf{C}\mathbf{y} = \int \boldsymbol{\mu}_1\mathbf{f}\,\mathrm{d}t + \mathbf{C}_1 \implies \mathbf{y} = \boldsymbol{\mu}_1^{-1}\mathbf{C}^{-1}\int \boldsymbol{\mu}_1\mathbf{f}\,\mathrm{d}t + \boldsymbol{\mu}_1^{-1}\mathbf{C}^{-1}\mathbf{C}_1, \tag{4}$$

where $\mathbf{C}_1$ is an integration constant and the time dependence is dropped for easy of notation. The integrating factor $\boldsymbol{\mu}_1$ is the solution to the following DE,

$$\dot{\boldsymbol{\mu}}_1\mathbf{C} = \boldsymbol{\mu}_1\left[\mathbf{K} - \mathbf{C}\right] \implies \boldsymbol{\mu}_1 = \exp\left(\int \left[\mathbf{K} - \mathbf{C}\right]\mathbf{C}^{-1}\,\mathrm{d}t\right). \tag{5}$$

The same procedure can be applied to a second order ODE,

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}. \tag{6}$$

Assuming that the coefficient matrix $\mathbf{M}$ is invertible for all t, it is possible to normalize the previous equation. Also, we split the coefficient of the first time derivative as

$$\underbrace{\boldsymbol{\mu}_2\mathbf{I}}_{\mathbf{P}_{2,1}}\ddot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,1}}_{\dot{\mathbf{P}}_{2,1}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,2}}_{\mathbf{P}_{2,2}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\bar{\mathbf{K}}}_{\dot{\mathbf{P}}_{2,2}} = \boldsymbol{\mu}_2\bar{\mathbf{f}}, \tag{7}$$

where $\mathbf{F}_{2,1,1} + \mathbf{F}_{2,1,2} = \bar{\mathbf{C}}$, $\bar{\mathbf{C}} = \mathbf{M}^{-1}\mathbf{C}$, $\bar{\mathbf{K}} = \mathbf{M}^{-1}\mathbf{K}$ and $\bar{\mathbf{f}} = \mathbf{M}^{-1}\mathbf{f}$. Using the procedure detailed in Fogaça and Cardoso (2023) and similar intuition with the $\mathbf{P}_{i,j}$, $\mathbf{F}_{2,1,1}$, and consequently $\mathbf{F}_{2,1,2}$, is given as the solution to a Riccati ODE,

$$\mathbf{F}_{2,1,1}^2 = \dot{\bar{\mathbf{C}}} + \mathbf{F}_{2,1,1}\bar{\mathbf{C}} - \bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,1}, \tag{8}$$

while the integrating factor is given by

$$\boldsymbol{\mu}_2 = \exp\left(\int \mathbf{F}_{2,1,1}\,\mathrm{d}t\right). \tag{9}$$

Thus, Equation (7) is reduced to a first order ODE,

$$\boldsymbol{\mu}_2\mathbf{I}\dot{\mathbf{y}} + \boldsymbol{\mu}_2\mathbf{F}_{2,1,2}\mathbf{y} = \int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \mathbf{C}_2, \tag{10}$$

where $\mathbf{C}_2$ is an integration constant. One observes that Eq. (10) is in the form of Eq. (2) and, hence, can be solved using Eq. (4). One also observes that the Leibniz integrating factor is a particular case of the proposed generalized integrating factor.

3. PARTICULARIZING FOR CONSTANT COEFFICIENTS

Coupled systems of ODEs with constant matrix coefficients are used in many engineering problems, notably in electric circuitry and vibration. Therefore, this section particularizes the equations of Section 2to the constant coefficients case.

As $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ are constant matrices, it is fair to assume that $\mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,2}$ are also constant. In many damping models the damping matrix, $\mathbf{C}$, commutes with the stiffness matrix, $\mathbf{K}$, like in proportional damping, and this commutativity remains to $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$. Using these assumptions and some Linear Algebra, Higham (2008); Higham and Kim (2001), Eq. (8) has the following algebraic solution (Fogaça and Cardoso, 2023),

$$\mathbf{F}_{2,1,1} = \frac{1}{2}\bar{\mathbf{C}} + \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}. \tag{11}$$

Fogaça and Cardoso (2023) also shown that the assumptions made to Eq. (11) imply that $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}}$ also commute. Therefore, the integrating factors can be reduced and combined to give the following solution to Eq. (6),

$$\mathbf{y} = \underbrace{\exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\mathbf{F}_{2,1,1}t\right)\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t}_{\mathbf{y_p}} + \underbrace{\exp\left(-\mathbf{F}_{2,1,1}t\right)\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t\right)\mathbf{C}_1}_{\mathbf{y_h}}, \tag{12}$$

in which, $\mathbf{y_p}$ and $\mathbf{y_h}$ are the particular solution (due to excitation) and the homogeneous solution, respectively.

As $\mathbf{C}_1$ and $\mathbf{C}_2$ are any constant vector, the complete solution can be written as

$$\mathbf{y} = \mathbf{y_h} + \mathbf{y_p} = \exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t\right)\mathbf{C}_1 + \mathbf{y_p}, \tag{13}$$

and its derivative with respect to time is given by

$$\dot{\mathbf{y}} = -\mathbf{F}_{2,1,1}\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2 + \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t\right)\mathbf{C}_1 + \dot{\mathbf{y}}_\mathbf{p}. \tag{14}$$

Considering initial conditions at time $t = 0$ yields

$$\mathbf{y}(0) = \exp(\mathbf{0})\,\mathbf{C}_2 + \exp(\mathbf{0})\,\mathbf{C}_1 + \mathbf{y_p}(0), \tag{15}$$

and

$$\dot{\mathbf{y}}(0) = -\mathbf{F}_{2,1,1}\exp(\mathbf{0})\,\mathbf{C}_2 + \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\exp(\mathbf{0})\,\mathbf{C}_1 + \dot{\mathbf{y}}_\mathbf{p}(0), \tag{16}$$

which can be summarized in a linear system when initial conditions $\mathbf{y}(0) = \mathbf{u}_0$ and $\dot{\mathbf{y}}(0) = \mathbf{v}_0$ are imposed,

$$\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] & -\mathbf{F}_{2,1,1} \end{bmatrix} \begin{Bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_0 - \mathbf{y_p}(0) \\ \mathbf{v}_0 - \dot{\mathbf{y}}_\mathbf{p}(0) \end{Bmatrix}; \tag{17}$$

whose solution is

$$\mathbf{C}_2 = \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \Big(\mathbf{v}_0 - \dot{\mathbf{y}}_\mathbf{p}\left(0\right) - \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\left(\mathbf{u}_0 - \mathbf{y}_\mathbf{p}\left(0\right)\right)\Big), \tag{18}$$

and

$$\mathbf{C}_1 = \mathbf{u}_0 - \mathbf{y}_\mathbf{p}\left(0\right) - \mathbf{C}_2. \tag{19}$$

4. EXCITATION FUNCTIONS

For many applications, finding just the homogeneous solution is sufficient, like for Modal Analysis (Ewins, 2000; Kelly, 2006). Nonetheless, some problems require the non-homogeneous solution, *e.g.*, forced vibrations and active circuits. Especially, particular solutions due to excitation functions modeled using continuous functions, like polynomials and periodic ones, and discontinuous functions, like Dirac's delta and Heaviside step. Some other applications, however, need both solutions, which are important for transient analysis, Fitzpatrick (2018); Oral *et al.* (2022).

Hence, this section derives analytical expressions to two cases of discontinuous excitation - Dirac's delta and Heaviside step. These solutions can be linearly summed to the homogeneous solution found in Eq. (12) for transient problems. The analytical solutions to these excitation functions are important not only to simulation problems, but also in optimization, since analytical expressions for the solution field facilitate the sensitivity analysis of many optimization problems, as for example, with integral measures Jog (2002); Zhu *et al.* (2009) and for topology optimization Long *et al.* (2021); Wang *et al.* (2023); Zhang *et al.* (2022); Silva *et al.* (2019, 2020).

4.1 Dirac's delta

Dirac's delta distribution is widely used to model impact in mechanical vibrations (Kelly, 2000, 2006; Boyce and Diprima, 2001). Impact is a recurrent excitation case in a large set of mechanical systems in real world problems, being considered in both simulation as well as in optimization. In most cases, numerical solutions are obtained using cumbersome parameterizations of the Dirac's delta with limited accuracy, like in Eftekhari (2015), and requires a very fine time discretization. Therefore, is is important to derive analytical solutions to this class of excitation.

Let the normalized excitation vector for Dirac's delta excitation be defined as

$$\bar{\mathbf{f}} = \sum_{k=0}^{n_k} c_{1k}\delta\left(t - t_{1k}\right)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + \sum_{k=0}^{n_k} c_{nk}\delta\left(t - t_{nk}\right)\mathbf{M}^{-1}\mathbf{e}_n = \sum_{j=1}^{n}\sum_{k=0}^{n_k} c_{jk}\delta\left(t - t_{jk}\right)\mathbf{v}_j, \tag{20}$$

where $\delta(t - t_{nk})$ is the Dirac's delta distribution at time t_{nk}, n_k is the number of deltas, c_{jk} are constant coefficients, $\mathbf{e}_j$ are the canonical basis vector at each direction j. The particular solution can be obtained by considering Eq. (20) and Eq. (12) such that

$$\mathbf{y}_\mathbf{p} = \sum_{j=1}^{n}\left(\sum_{k=0}^{n_k} c_{jk}\mathcal{H}\left(t - t_{jk}\right)\left[\exp\Big(\mathbf{F}_{2,1,1}\left(t_{jk} - t\right)\Big) - \exp\Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\left(t_{jk} - t\right)\Big)\right]\right)\left[\mathbf{C} - 2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{e}_j, \tag{21}$$

where $\mathcal{H}$ is the Heaviside step function. It is worth noticing that the solution given by Eq. (21) is rather simple and that the linear system $\left[\mathbf{C} - 2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{e}_j$ must be evaluated only once for each loaded DOF j, independently of the number of impulses in this DOF and independently of the time span. Also, there is no time step, since we are not using a time discretization procedure. Computational cost of the exponential maps alone will be further discussed in the Appendix.

4.2 Heaviside step function

Heaviside step functions are particularly important since they can be used to turn on and off other expressions. These unitary step functions can be multiplied by different functions and these combinations are then used to model more complex behavior and to parameterize other functions.

Thus, they can model forces suddenly appearing and vanishing in mechanical systems as well as switches, both for electric current and fluid flow (Boyce and Diprima, 2001; Irwin and Nelms, 2006; Agarwal and Lang, 2005). Let the normalized excitation vector be defined as

$$\bar{\mathbf{f}} = \sum_{k=0}^{n_k} c_{1k}f_{1k}\mathcal{H}\left(t - t_{1k}\right)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + \sum_{k=0}^{n_k} c_{nk}f_{2k}\mathcal{H}\left(t - t_{nk}\right)\mathbf{M}^{-1}\mathbf{e}_n = \sum_{j=1}^{n}\sum_{k=0}^{n_k} c_{jk}f_{jk}\mathcal{H}\left(t - t_{jk}\right)\mathbf{v}_j. \tag{22}$$

where f_{jk} is any function of time multiplying the Heaviside step function at the j-th DOF and from k-th time interval onward. Considering Eq. (22) and Eq. (12) one finds the following particular solution

$$\mathbf{y_p} = \exp\Big(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\Big) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \mathcal{H}\left(t - t_{jk}\right) \int_{t_{jk}}^{t} \exp\Big(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\Big) \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) f_{jk}(t)\, \mathrm{d}t\, \mathrm{d}t\, \mathbf{v}_j. \tag{23}$$

This solution is, however, dependent of the double convolutions of the functions f_{jk}. Therefore, to illustrate the implementation and further details, these functions should be particularized for a given family of functions.

4.2.1 Particularization for quadratic polynomials

Polynomials are functions with wide-spread use in Engineering, like for example the ramp function in circuitry analysis (Irwin and Nelms, 2006; Agarwal and Lang, 2005). Polynomials are also often used to approximate other functions (Arora, 2017; Haftka and Gürdal, 1992). In the following, we study the case of a second order polynomial.

Let the excitation function be defined as

$$\bar{\mathbf{f}} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} \Big(c_{jk0} + c_{jk1} t + c_{jk2} t^2\Big) \mathcal{H}\left(t - t_{jk}\right) \mathbf{v}_j, \tag{24}$$

where c_{jk0}, c_{jk1} and c_{jk2} are constant coefficients. Applying Eq. (24) to Eq. (23) and with further simplifications, the particular solution is

$$\begin{aligned}
\mathbf{y_p}^{(2)} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} \mathcal{H}\left(t - t_{jk}\right) \Bigg\{ & c_{jk2} t^2 \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \mathbf{F}_{2,1,1}^{-1} + t \Bigg(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \Big(c_{jk1} \mathbf{F}_{2,1,1}^{-1} - 2 c_{jk2} \mathbf{F}_{2,1,1}^{-2} \Big) \\
& - 2 c_{jk2} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2} \mathbf{F}_{2,1,1}^{-1} \Bigg) + 2 c_{jk2} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-3} \mathbf{F}_{2,1,1}^{-1} - \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2} \Big(c_{jk1} \mathbf{F}_{2,1,1}^{-1} - 2 c_{jk2} \mathbf{F}_{2,1,1}^{-2} \Big) \\
& + \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \Big(2 c_{jk2} \mathbf{F}_{2,1,1}^{-3} - c_{jk1} \mathbf{F}_{2,1,1}^{-2} + c_{jk0} \mathbf{F}_{2,1,1}^{-1} \Big) + \exp\Big(\mathbf{F}_{2,1,1} \left(t_{jk} - t\right) \Big) \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \\
& \Bigg(- \Big(c_{jk2} t_{jk}^2 + c_{jk1} t_{jk} + c_{jk0} \Big) \mathbf{F}_{2,1,1}^{-1} + \left(2 c_{jk2} t_{jk} + c_{jk1}\right) \mathbf{F}_{2,1,1}^{-2} - 2 c_{jk2} \mathbf{F}_{2,1,1}^{-3} \Bigg) + \Bigg(\exp\Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] \left(t_{jk} - t\right) \Big) \\
& \Bigg(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \Bigg(\mathbf{F}_{2,1,1}^{-1} \Big(- c_{jk2} t_{jk}^2 - c_{jk1} t_{jk} - c_{jk0} \Big) + \mathbf{F}_{2,1,1}^{-2} \left(2 c_{jk2} t_{jk} + c_{jk1}\right) - 2 c_{jk2} \mathbf{F}_{2,1,1}^{-3} \Bigg) \\
& + \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2} \Big(\mathbf{F}_{2,1,1}^{-1} \left(2 c_{jk2} t_{jk} + c_{jk1}\right) - 2 c_{jk2} \mathbf{F}_{2,1,1}^{-2} \Big) - 2 c_{jk2} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-3} \mathbf{F}_{2,1,1}^{-1} - \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} (\\
& - \mathbf{F}_{2,1,1}^{-1} \Big(c_{jk2} t_{jk}^2 + c_{jk1} t_{jk} + c_{jk0} \Big) + \mathbf{F}_{2,1,1}^{-2} \left(2 c_{jk2} t_{jk} + c_{jk1}\right) - 2 c_{jk2} \mathbf{F}_{2,1,1}^{-3} \Big) \Bigg) \Bigg) \Bigg\} \mathbf{M}^{-1} \mathbf{e}_j.
\end{aligned} \tag{25}$$

5. RESULTS

A three DOF system described by

$$\mathbf{M} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 600 & -400 & 0 \\ -400 & 1000 & -300 \\ 0 & -300 & 700 \end{pmatrix} \tag{26}$$

and $\mathbf{C} = \beta \mathbf{K}$ is used to investigate the analytical solutions obtained in this manuscript. The analytical solutions are compared to the well known Newmark-beta method (Newmark, 1959). The comparisons are done both visually, through the pictured graphs, and using the Root Mean Squared Error (RMSE), (James *et al.*, 2013). The implementation of the proposed approach is available in the public repository *https://github.com/CodeLenz/Giffndof.jl*.

5.1 Example 1

The fist example is the complete response due to an unitary impulse applied to the second DOF at $t = 1$s. A damping coefficient $\beta = 1 \times 10^{-2}$ is used. The analytical response for each DOF is shown as solid colored lines in Fig. 1 and the numerical solution obtained with $\Delta t = 1 \times 10^{-3}$s is shown as dotted lines. The fine time step is needed to properly represent the response due to the unitary impulse when using the numerical approach.

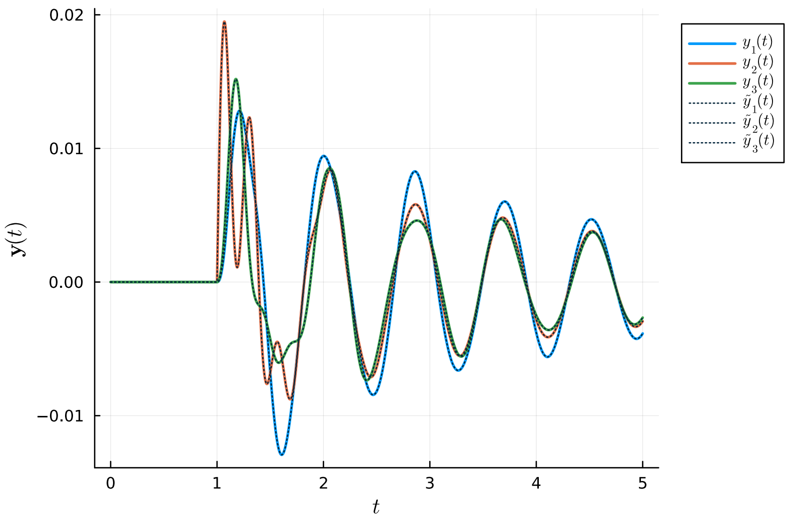

Figure 1. Complete solution due to an unitary impulse applied to the second DOF at $t = 1$. Solid lines depict the analytical solutions obtained by using the proposed approach and the dotted lines the solution obtained with the Newmark-beta method. The RMSE error of the response due to the Newmark-beta method using the response due to the Generalized Integrating Factor as reference is 4.9432×10^{-6}.

To exemplify the advantage of using an analytical approach we evaluate the same problem but with a very small damping coefficient of $\beta = 1 \times 10^{-6}$. Figure 2 shows the analytical displacement of the third DOF as a solid blue line and the numerical solution obtained with $\Delta t = 1 \times 10^{-2}$s as a dotted line. It is clear that the numerical solution deviates from the analytical solution as the time increases.

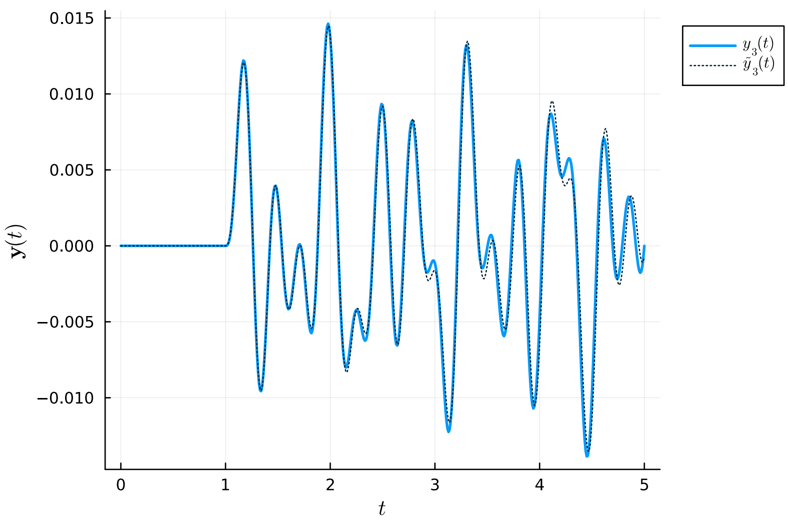

Figure 2. Complete solution due to an unitary impulse applied to the second DOF at $t = 1$. Solid line depict the analytical solutions for the third DOF obtained by using the proposed approach and the dotted line the solution obtained with the Newmark-beta method and $\Delta t = 1 \times 10^{-2}$. The RMSE error of the response due to the Newmark-beta method using the response due to the Generalized Integrating Factor as reference is 1.9027×10^{-3}.

5.2 Example 2

The same system used in the first example is now subjected to a more complicate excitation. Consider the excitation shown in Fig. 3 and described by

$$f(t) = \begin{cases} 0.0 & t < 1 \\ l(t) & 1 \le t < 3 \\ q(t) & 3 \le t < 5 \\ 5.0 & t \ge 5 \end{cases} \tag{27}$$

where $l(t) = 10(t-1)$ and $q(t) = -370 + 212.5t - 27.5t^2$. This load can also be written as

$$f(t) = l(t)\mathcal{H}(t-1) + \big(q(t) - l(t)\big)\,\mathcal{H}(t-3) + \big(5 - q(t)\big)\,\mathcal{H}(t-5) \tag{28}$$

In addition to load, we also consider a non-homogeneous initial condition in the second DOF, $y_2(0) = 0.01$. The

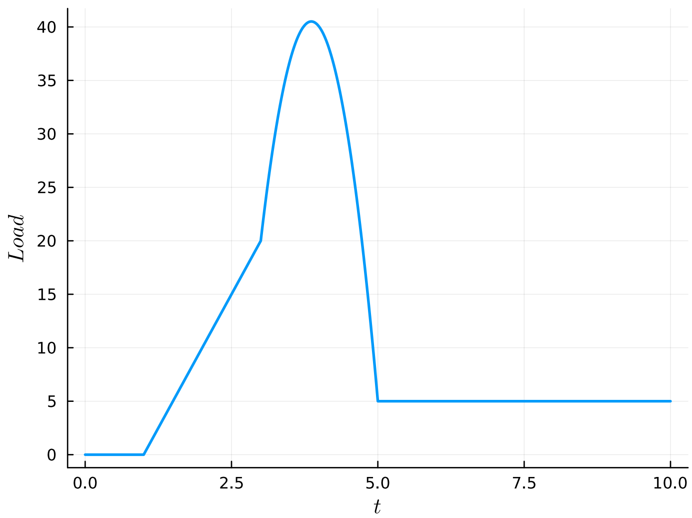

Figure 3. Load comprised of a linear varying load from $t = 1$ to $t = 3$, a quadratic varying load from $t = 3$ to $t = 5$ and a constant value for $t \ge 5$.

complete analytical response obtained by using the proposed approach is shown in Fig. 4.

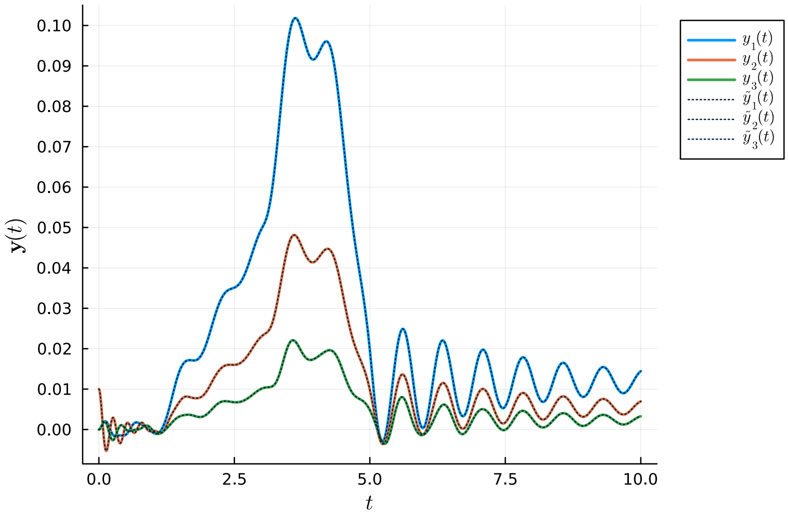

Figure 4. Complete solution due to the load depicted in Fig. 3 applied to the first DOF. Solid lines depict the analytical solutions for each DOF and dotted lines the solution obtained with the Newmark-beta method and $\Delta t = 1 \times 10^{-3}$. The RMSE error of the response due to the Newmark-beta method using the response due to the Generalized Integrating Factor as reference is 3.2405×10^{-7}.

As the analytical approach proposed in this work gives the homogeneous and the particular solutions in separate, it is also possible to visualize these responses, unlike when using a numerical method. Figure 5 shows the homogeneous (solid blue line), the particular (solid red line) and the complete (dotted line) analytical solutions for the second DOF due to the non-homogeneous initial condition and load given by Eq. (28).

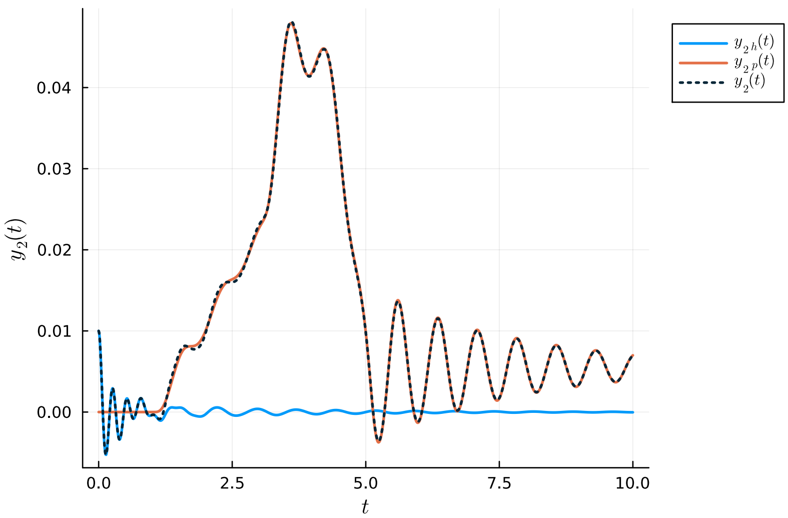

Figure 5. Homogeneous solution (solid blue line), particular solution (solid red line) and the complete solution (dotted line) for the second DOF.

6. CONCLUSION

The proposed generalized integrating factor has shown to be a effective method to tackle problems with systems of coupled ODEs, especially with constant matrix coefficients, like the ones arising in vibration and circuitry analysis. The method provides analytical solutions and is easy to implement, such that it can be used to address many fundamental problems in engineering. The generalized integrating factor gives the homogeneous and particular solutions in separate and independently of time discretization, unlike numerical methods. Finally, the complete response matched the numerically evaluated responses obtained with fine time discretization in all the examples.

ACKNOWLEDGEMENTS

Authors acknowledge CNPq (process number 303900/2020-2) and FAPESC (process numbers 2021TR843 and 2023TR000563 and scholarship CP48/2021). This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and process 88887.807649/2023-00. The authors also acknowledge Prof. Nicholas Higham for his kind replies to questions on exponential maps.

Appendix - EXPONENTIAL MAPS AND COMPUTATIONAL COST

Evaluating exponential maps is a computationally intensive task (Higham, 2008), especially if it has to be carried out along a time span. Despite the rapid advance in computational efficiency in the last two decades (Botchev and Knizhnerman, 2020; Vo and Sidje, 2017; Sastre *et al.*, 2019), a relation from (Aprahamian and Higham, 2014) can be used to evaluate an exponential map in different time steps if they are equally spaced, by calculating just the starting point and the recovering the last solution. The relation by (Aprahamian and Higham, 2014) is

$$\exp\left(\mathbf{A}\right)^{\alpha} = \exp\left(\alpha\mathbf{A}\right) \quad \forall\, \alpha \in \mathbb{Z}, \tag{29}$$

where $\mathbb{Z}$ is the set of integers.

Thus, assuming that the time span $t \in [t_i, t_f]$ is discretized in n_t intervals with length Δt, it is possible to write $t = k\Delta t$ such that the exponential map $\mathcal{E}(t)$ is

$$\mathcal{E}(t) = \exp\left(\mathbf{A}t\right) = \exp\left(k\Delta t\mathbf{A}\right) = \exp\left(\Delta t\mathbf{A}\right)^{k}, \tag{30}$$

where $k \in \mathbb{Z}$. Consequently, the exponential map in a discrete time span is evaluated through the following recursion

$$
\begin{aligned}
t_1: \quad \mathcal{E}(t_1) &= \exp(\Delta t \mathbf{A}) \\
t_2: \quad \mathcal{E}(t_2) &= \mathcal{E}(t_1)\exp(\Delta t \mathbf{A}) \\
t_3: \quad \mathcal{E}(t_3) &= \mathcal{E}(t_2)\exp(\Delta t \mathbf{A}) \\
&\vdots \\
t_k: \quad \mathcal{E}(t_k) &= \mathcal{E}(t_{k-1})\exp(\Delta t \mathbf{A}),
\end{aligned} \tag{31}
$$

such that the matrix $\exp(\Delta t \mathbf{A})$ has to be evaluated only once.

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