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COB-2023-1543 ANALYTICAL SOLUTIONS FOR THE FOUR PARAMETER BURGERS MODEL USING THE GENERALIZED INTEGRATING FACTOR

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Abstract. *Analytical solutions for second order differential equations like the ones governing idealized viscoelastic constitutive models are only known for some specific forms of excitation. The extension of the analytical solutions to a broader set of excitations is of great interest for the data fit procedure, as well as for many other uses. This work proposed the application of the generalized integrating factor method, recently proposed by the authors, to analytically solve the differential equation of Burgers viscoelastic model for different forms of excitations (either strain or stress). Analytical expressions to compute the derivative of the complete solution with respect to material coefficients, used to fit experimental data, are also obtained. Two examples are provided to validate the analytical solutions associated to senoidal and to constant (relaxation) problems.*

Keywords: *Burgers model, analytical solution, generalized integrating factor, viscoelasticity modelling*

1. INTRODUCTION

Viscoelastic materials have both elastic and viscous behavior, such that the constitutive relations are rate dependent. They can be found in many practical applications, like rubbers, plastics, biological materials and metals subjected to high temperature, to name a few. Thus, a proper description of their constitutive behavior is of great relevance. Idealized viscoelastic constitutive models have been regularly used to simulate the mechanical response of these materials due their simplicity. Yet, their solutions allow the identification of important characteristics of the material, like transient creep and gradual recovery when the loading cases, and they conform very well to experimental data of a variety of engineering materials. These models are generally obtained by the combination of classic models such as Maxwell and Kevin-Voigt. Among these models the four parameters viscoelastic model of Burgers (Shames and Cozzarelli, 1992), is among the most used ones, and it is described by a second order ordinary differential equation (ODE) with constant coefficients. This model is a combination of the Maxwell and the Kelvin-Voigt model and there are different possible differential equations for the Burgers model, notably with the first and second time derivative of the strain field (Hu and Gutierrez, 2022), and with the strain field and its first time derivative (Malkin and Isayev, 2022). Both possibilities can be unified by considering all derivative terms for strain in a single constitutive relation of the form

$$a_0\sigma(t) + a_1\dot{\sigma}(t) + a_2\ddot{\sigma} = b_0\varepsilon(t) + b_1\dot{\varepsilon}(t) + b_2\ddot{\varepsilon}(t) \tag{1}$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are constants depending on the rheological characteristics of the model.

There are many mechanical tests used to obtain the material coefficients of such model, usually imposing controlled strain (relaxation) or controlled stress (creep). Therefore, we investigate the analytical solution for Eq. (1) for these two possibilities.

2. STRAIN EXCITATION

Assuming a known imposed strain $\tilde{\varepsilon}(t)$, it is possible to rewrite Eq. (1) as

$$a_0\sigma(t) + a_1\dot{\sigma}(t) + a_2\ddot{\sigma} = f(t) \tag{2}$$

where

$$f(t) = b_0\tilde{\varepsilon}(t) + b_1\dot{\tilde{\varepsilon}}(t) + b_2\ddot{\tilde{\varepsilon}}(t). \tag{3}$$

Splitting the constant coefficient a_1 as $a_1 = F_{11} + F_{12}$, where both are also constant, results in

$$a_0\sigma(t) + F_{11}\dot{\sigma}(t) + F_{12}\dot{\sigma}(t) + a_2\ddot{\sigma} = f(t) \tag{4}$$

and, finally, we multiply the previous equation by a generalized integrating factor $\mu_2(t)$

$$\mu_2(t)a_2\ddot{\sigma} + \mu_2(t)F_{11}\dot{\sigma}(t) + \mu_2(t)F_{12}\dot{\sigma}(t) + \mu_2(t)a_0\sigma(t) = \mu_2(t)f(t) \tag{5}$$

or

$$p_{22}\ddot{\sigma} + \dot{p}_{22}\dot{\sigma}(t) + p_{21}\dot{\sigma}(t) + \dot{p}_{21}\sigma(t) = \mu_2(t)f(t). \tag{6}$$

Thus, it is possible to write

$$\dot{p}_{22}(t) = \mu_2(t)F_{11} = \dot{\big(\mu_2(t)a_2\big)} = \dot{\mu}_2(t)a_2 \tag{7}$$

such that

$$\frac{\dot{\mu}_2(t)}{\mu_2(t)} = \frac{F_{11}}{a_2}. \tag{8}$$

In a similar fashion,

$$\dot{p}_{21}(t) = \mu_2(t)a_0 = \dot{\big(\mu_2(t)F_{12}\big)} = \dot{\mu}_2(t)F_{12} \tag{9}$$

such that

$$\frac{\dot{\mu}_2(t)}{\mu_2(t)} = \frac{a_0}{F_{12}}. \tag{10}$$

Thus, from equation Eqs. (8) and (10) and using the fact that $a_1 = F_{11} + F_{12}$ results in

$$\frac{F_{11}}{a_2} = \frac{a_0}{a_1 - F_{11}} \tag{11}$$

or

$$F_{11}^2 - a_1F_{11} + a_0a_2 = 0 \tag{12}$$

which has two roots

$$F_{11} = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2}. \tag{13}$$

Consequently, using Eq. (8), the integrating factor is given by

$$\mu_2(t) = \exp\left(\int \frac{F_{11}}{a_2}\,\mathrm{d}t\right) = \exp\left(\frac{F_{11}}{a_2}t\right). \tag{14}$$

Now using this integrating factor in Eq. (5) and integrating with respect to t results in

$$a_2\dot{\sigma}(t) + F_{12}\sigma(t) = \frac{1}{\mu_2(t)}\int \mu_2(t)f(t)\,\mathrm{d}t + \frac{C_1}{\mu_2(t)}, \tag{15}$$

which is a first order differential equation and C_1 is an integration constant. Thus, the $\mu_2(t)$ integrating factor and the subsequent integration acted as an order reduction scheme. Equation (15) is, then, suitable for solution using traditional Leibniz integrating factor $\mu_1(t)$, (Boyce and Diprima, 2001),

$$\underbrace{\mu_1(t)a_2}_{p_{11}}\dot{\sigma}(t) + \underbrace{\mu_1(t)F_{12}}_{\dot{p}_{11}}\sigma(t) = \frac{\mu_1(t)}{\mu_2(t)}\int \mu_2(t)f(t)\,\mathrm{d}t + \mu_1(t)\frac{C_1}{\mu_2(t)}, \tag{16}$$

thus,

$$\dot{\big(p_{11}(t)\big)} = \dot{\mu}_1(t)a_2 = \mu_1(t)F_{12} \implies \frac{\dot{\mu}_1(t)}{\mu_1(t)} = \frac{F_{12}}{a_2} \implies \mu_1(t) = \exp\left(\frac{F_{12}}{a_2}t\right). \tag{17}$$

Applying this integrating factor to Eq. (16) yields

$$\sigma(t) = \frac{1}{\mu_1(t)a_2}\int \frac{\mu_1(t)}{\mu_2(t)}\int \mu_2(t)f(t)\,\mathrm{d}t\,\mathrm{d}t + \frac{C_1}{\mu_1(t)a_2}\int \frac{\mu_1(t)}{\mu_2(t)}\,\mathrm{d}t + \frac{C_2}{\mu_{1(t)}a_2}, \tag{18}$$

where C_2 is also an integration constant. This equation can have the integrating factors replaced by their previously calculated analytical expressions, resulting in

$$\sigma(t) = \frac{1}{a_2}\exp\left(\frac{F_{11}-a_1}{a_2}t\right)\int \exp\left(\frac{a_1-2F_{11}}{a_2}t\right)\int \exp\left(\frac{F_{11}}{a_2}t\right)f(t)\,\mathrm{d}t\,\mathrm{d}t$$
$$+\frac{C_1}{a_2}\exp\left(\frac{F_{11}-a_1}{a_2}t\right)\int \exp\left(\frac{a_1-2F_{11}}{a_2}t\right)\mathrm{d}t + \frac{C_2}{a_2}\exp\left(\frac{F_{11}-a_1}{a_2}t\right). \tag{19}$$

Evaluating the integrals that do not depend on $f(t)$ results in

$$\sigma(t) = \frac{1}{a_2}\exp\left(\frac{F_{11}-a_1}{a_2}t\right)\int \exp\left(\frac{a_1-2F_{11}}{a_2}t\right)\int \exp\left(\frac{F_{11}}{a_2}t\right)f(t)\,\mathrm{d}t\,\mathrm{d}t$$
$$+\frac{C_1}{a_2}\exp\left(\frac{F_{11}-a_1}{a_2}t\right)\frac{a_2}{a_1-2F_{11}}\exp\left(\frac{a_1-2F_{11}}{a_2}t\right) + \frac{C_2}{a_2}\exp\left(\frac{F_{11}-a_1}{a_2}t\right) \tag{20}$$

and as both C_1 and C_2 are generic constants, Eq. (20) can be simplified to

$$\sigma(t) = \underbrace{\frac{1}{a_2}\exp\left(\frac{F_{11}-a_1}{a_2}t\right)\int \exp\left(\frac{a_1-2F_{11}}{a_2}t\right)\int \exp\left(\frac{F_{11}}{a_2}t\right)f(t)\,\mathrm{d}t\,\mathrm{d}t}_{\sigma_p}$$
$$+C_1\exp\left(\frac{-F_{11}}{a_2}t\right) + C_2\exp\left(\frac{F_{11}-a_1}{a_2}t\right). \tag{21}$$

Using $f(t)$ defined by Eq. (3), one can integrate the particular solution, σ_p, by parts. The inner convolution is evaluated as

$$IC(t) = \int \exp\left(\frac{F_{11}}{a_2}t\right)\big(b_0\varepsilon(t) + b_1\dot{\varepsilon}(t) + b_2\ddot{\varepsilon}(t)\big)\,\mathrm{d}t = b_0\int \exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t + b_1\exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)$$
$$-b_1\frac{F_{11}}{a_2}\int \exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t + b_2\exp\left(\frac{F_{11}}{a_2}t\right)\dot{\varepsilon}(t) - b_2\frac{F_{11}}{a_2}\int \exp\left(\frac{F_{11}}{a_2}t\right)\dot{\varepsilon}(t)\,\mathrm{d}t$$
$$= b_1\exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t) + \left[b_0 - b_1\frac{F_{11}}{a_2}\right]\int \exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t + b_2\exp\left(\frac{F_{11}}{a_2}t\right)\dot{\varepsilon}(t) - b_2\frac{F_{11}}{a_2}\exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)$$
$$+b_2\frac{F_{11}^2}{a_2^2}\int \exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t$$
$$= b_2\exp\left(\frac{F_{11}}{a_2}t\right)\dot{\varepsilon}(t) + \left[b_1 - b_2\frac{F_{11}}{a_2}\right]\exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t) + \left[b_0 - b_1\frac{F_{11}}{a_2} + b_2\frac{F_{11}^2}{a_2^2}\right]\int \exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t, \tag{22}$$

while the outer convolution results in

$$
\begin{aligned}
OC(t) = \int \exp\left(\frac{a_1 - 2F_{11}}{a_2}t\right) IC(t)\,\mathrm{d}t = \int b_2 \exp\left(\frac{a_1 - F_{11}}{a_2}t\right)\dot{\varepsilon}(t) + \left[b_1 - b_2\frac{F_{11}}{a_2}\right]\exp\left(\frac{a_1 - F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t + \\
+ \left[b_0 - b_1\frac{F_{11}}{a_2} + b_2\frac{F_{11}^2}{a_2^2}\right]\int \exp\left(\frac{a_1 - 2F_{11}}{a_2}t\right)\int \exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t\,\mathrm{d}t \\
= b_2 \exp\left(\frac{a_1 - F_{11}}{a_2}t\right)\varepsilon(t) - b_2\frac{a_1 - F_{11}}{a_2}\int \exp\left(\frac{a_1 - F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t + \left[b_1 - b_2\frac{F_{11}}{a_2}\right]\int \exp\left(\frac{a_1 - F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t \\
+ \left[b_0 - b_1\frac{F_{11}}{a_2} + b_2\frac{F_{11}^2}{a_2^2}\right]\int \exp\left(\frac{a_1 - 2F_{11}}{a_2}t\right)\int \exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t\,\mathrm{d}t.
\end{aligned}
\tag{23}
$$

Thus, using the previous three equations, $\sigma_p(t)$ can be simplified to

$$
\begin{aligned}
\sigma_p(t) = \frac{b_2}{a_2}\varepsilon(t) + \frac{1}{a_2}\left[b_1 - b_2\frac{a_1}{a_2}\right]\exp\left(\frac{F_{11} - a_1}{a_2}t\right)\int \exp\left(\frac{a_1 - F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t \\
+\frac{1}{a_2}\left[b_0 - b_1\frac{F_{11}}{a_2} + b_2\frac{F_{11}^2}{a_2^2}\right]\exp\left(\frac{F_{11} - a_1}{a_2}t\right)\int \exp\left(\frac{a_1 - 2F_{11}}{a_2}t\right)\int \exp\left(\frac{F_{11}}{a_2}t\right)\varepsilon(t)\,\mathrm{d}t\,\mathrm{d}t,
\end{aligned}
\tag{24}
$$

which is a closed form that depends only on $\varepsilon(t)$, since its derivatives were transferred to the exponential function of the convolution and, consequently, no continuity assumption is necessary.

The integration constants must be calculated using an initial value problem, *i.e.*, $\sigma(0)$ and $\dot{\sigma}(0)$ must be known. Substituting these conditions in Eq. (21) results in

$$\sigma(0) = \sigma_p(0) + C_1 + C_2 \tag{25}$$

and

$$\dot{\sigma}(0) = \dot{\sigma}_p(0) - C_1\frac{F_{11}}{a_2} + C_2\frac{F_{11} - a_1}{a_2} \tag{26}$$

such that

$$C_1 = \frac{(F_{11} - a_1)\big(\sigma_p(0) - \sigma(0)\big) + a_2\big(\dot{\sigma}(0) - \dot{\sigma}_p(0)\big)}{a_1 - 2F_{11}} \tag{27}$$

and

$$C_2 = \frac{F_{11}\big(\sigma_p(0) - \sigma(0)\big) + a_2\big(\dot{\sigma}_p(0) - \dot{\sigma}(0)\big)}{a_1 - 2F_{11}}. \tag{28}$$

3. STRESS EXCITATION

Since all coefficients in Eq. (1) are constant and both stress and strain have the same number of coefficients and the same derivative orders, all equations above can be analogously obtained for strain in terms of stress. Assuming the excitation as

$$f(t) = a_0\tilde{\sigma}(t) + a_1\dot{\tilde{\sigma}}(t) + a_2\ddot{\tilde{\sigma}}(t). \tag{29}$$

Changing a for b, Eq. (13) becomes

$$F_{11} = \frac{b_1 \pm \sqrt{b_1^2 - 4b_0b_2}}{2}; \tag{30}$$

while the complete strain solution is given by

$$
\begin{aligned}
\varepsilon(t) = \underbrace{\frac{1}{b_2}\exp\left(\frac{F_{11} - b_1}{b_2}t\right)\int \exp\left(\frac{b_1 - 2F_{11}}{b_2}t\right)\int \exp\left(\frac{F_{11}}{b_2}t\right) f(t)\,\mathrm{d}t\,\mathrm{d}t}_{\varepsilon_p} \\
+C_1\exp\left(\frac{-F_{11}}{b_2}t\right) + C_2\exp\left(\frac{F_{11} - b_1}{b_2}t\right);
\end{aligned}
\tag{31}
$$

and the particular solution to strain is found to be

$$\varepsilon_p(t) = \frac{a_2}{b_2}\sigma(t) + \frac{1}{b_2}\left[a_1 - a_2\frac{b_1}{b_2}\right]\exp\left(\frac{F_{11}-b_1}{b_2}t\right)\int \exp\left(\frac{b_1-F_{11}}{b_2}t\right)\sigma(t)\,\mathrm{d}t$$

$$+\frac{1}{b_2}\left[a_0 - a_1\frac{F_{11}}{b_2} + a_2\frac{F_{11}^2}{b_2^2}\right]\exp\left(\frac{F_{11}-b_1}{b_2}t\right)\int \exp\left(\frac{b_1-2F_{11}}{b_2}t\right)\int \exp\left(\frac{F_{11}}{b_2}t\right)\sigma(t)\,\mathrm{d}t\,\mathrm{d}t. \tag{32}$$

Finally, the integration constants are

$$C_1 = \frac{(F_{11}-b_1)\left(\varepsilon_p(0)-\varepsilon(0)\right) + b_2\left(\dot\varepsilon(0)-\dot\varepsilon_p(0)\right)}{b_1-2F_{11}} \tag{33}$$

and

$$C_2 = \frac{F_{11}\left(\varepsilon_p(0)-\varepsilon(0)\right) + b_2\left(\dot\varepsilon_p(0)-\dot\varepsilon(0)\right)}{b_1-2F_{11}}. \tag{34}$$

4. MODEL FITTING AND SENSITIVITY ANALYSIS

The analytical solutions to the previous differential equations must be fitted to experimental data in order to adjust the coefficients a or b to match a given viscoelastic material. This task is usually performed by means of an optimization problem where the coefficients a or b are used as a set of design variables.

This curve fitting approach is shown in Tirella *et al.* (2014) and consist of a least squares problem, and as this type of problem is commonly solved using gradient-based optimization algorithms, sensitivity analysis is paramount (Schomaker *et al.*, 2007).

As both stress and strain solution share the same mathematical structure (compare Eq. (21) to Eq. (31) and Eq. (24) to Eq. (32)), the expressions for both solutions can be evaluated using a single expression. Therefore, we introduce the generic variables $\gamma(t)$ and $\kappa(t)$, which can mean stress or strain, interchangeably. The coefficients a and b can also be represented in a similar manner by r and s, such that

$$\gamma(t) = C_1\exp\left(\frac{-F_{11}}{r_2}t\right) + C_2\exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \frac{s_2}{r_2}\kappa(t) + \frac{1}{r_2}\left[s_1 - s_2\frac{r_1}{r_2}\right]\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int \exp\left(\frac{r_1-F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t$$

$$+\frac{1}{r_2}\left[s_0 - s_1\frac{F_{11}}{r_2} + s_2\frac{F_{11}^2}{r_2^2}\right]\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int \exp\left(\frac{r_1-2F_{11}}{r_2}t\right)\int \exp\left(\frac{F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t\,\mathrm{d}t, \tag{35}$$

with the integration constants written with the generic coefficients r and s:

$$C_1 = \frac{(F_{11}-r_1)\left(\gamma_p(0)-\gamma(0)\right) + r_2\left(\dot\gamma(0)-\dot\gamma_p(0)\right)}{r_1-2F_{11}} \tag{36}$$

and

$$C_2 = \frac{F_{11}\left(\gamma_p(0)-\gamma(0)\right) + r_2\left(\dot\gamma_p(0)-\dot\gamma(0)\right)}{r_1-2F_{11}}. \tag{37}$$

Finally, the partition of the coefficient of the term with the first time derivative, F_{11}, is

$$F_{11} = \frac{r_1 \pm \sqrt{r_1^2 - 4r_0r_2}}{2}. \tag{38}$$

One of the advantages of computing the generic response $\gamma(t)$ given by Eq. (35) is the direct evaluation of the sensitivities of $\gamma(t)$ with respect to the coefficients a or b of the Eq. (1).

Nonetheless, differentiation under the integral sign must be carried out. According to Protter and Morrey (2012), whenever the limits of integration do not change with the integrand, the differentiation can be carried out directly under the integral sign, which is also known as Leibniz rule. The exponential functions inside the convolutions are continuous everywhere as are their derivatives, thus, if the excitation field function is also continuous, so is the whole integrand and the Leibniz rule can be used. Hence, assuming that $\kappa(t)$ is continuous, the derivative of a field (stress or strain) with respect to a generic design variable p is obtained in closed form as

$$\begin{aligned}\frac{d\gamma(t)}{dp} = {} & \frac{dC_1}{dp}\exp\left(\frac{-F_{11}}{r_2}t\right) - C_1\frac{d}{dp}\left[\frac{F_{11}}{r_2}\right]\exp\left(\frac{-F_{11}}{r_2}t\right) + \frac{dC_2}{dp}\exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \\ & C_2\frac{d}{dp}\left[\frac{F_{11}-r_1}{r_2}\right]\exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \frac{d}{dp}\left[\frac{s_2}{r_2}\right]\kappa(t) + \\ & \frac{d}{dp}\left[\frac{1}{r_2}\left(s_1 - s_2\frac{r_1}{r_2}\right)\right]\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t + \\ & \frac{d}{dp}\left[\frac{F_{11}-r_1}{r_2}\right]\frac{1}{r_2}\left(s_1 - s_2\frac{r_1}{r_2}\right)\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t + \\ & \frac{d}{dp}\left[\frac{r_1-F_{11}}{r_2}\right]\frac{1}{r_2}\left(s_1 - s_2\frac{r_1}{r_2}\right)\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t + \\ & \frac{d}{dp}\left[\left(\frac{1}{r_2}\left(s_0 - s_1\frac{F_{11}}{r_2} + s_2\frac{F_{11}^2}{r_2^2}\right)\right)\right]\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-2F_{11}}{r_2}t\right)\int\exp\left(\frac{F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t\,\mathrm{d}t + \\ & \frac{d}{dp}\left[\frac{F_{11}-r_1}{r_2}\right]\left(\frac{1}{r_2}\left(s_0 - s_1\frac{F_{11}}{r_2} + s_2\frac{F_{11}^2}{r_2^2}\right)\right)\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-2F_{11}}{r_2}t\right)\int\exp\left(\frac{F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t\,\mathrm{d}t + \\ & \frac{d}{dp}\left[\frac{r_1-2F_{11}}{r_2}\right]\left(\frac{1}{r_2}\left(s_0 - s_1\frac{F_{11}}{r_2} + s_2\frac{F_{11}^2}{r_2^2}\right)\right)\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-2F_{11}}{r_2}t\right)\int\exp\left(\frac{F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t\,\mathrm{d}t + \\ & \frac{d}{dp}\left[\frac{F_{11}}{r_2}\right]\left(\frac{1}{r_2}\left(s_0 - s_1\frac{F_{11}}{r_2} + s_2\frac{F_{11}^2}{r_2^2}\right)\right)\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-2F_{11}}{r_2}t\right)\int\exp\left(\frac{F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t\,\mathrm{d}t, \end{aligned} \quad (39)$$

which, by using the linearity property of the differentiation operator, can be simplified to

$$\begin{aligned}\frac{d\gamma(t)}{dp} = {} & \left(\frac{dC_1}{dp} - C_1\frac{d}{dp}\left[\frac{F_{11}}{r_2}\right]\right)\exp\left(\frac{-F_{11}}{r_2}t\right) + \left(\frac{dC_2}{dp} + C_2\frac{d}{dp}\left[\frac{F_{11}-r_1}{r_2}\right]\right)\exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \\ & \frac{d}{dp}\left[\frac{s_2}{r_2}\right]\kappa(t) + \frac{d}{dp}\left[\frac{1}{r_2}\left(s_1 - s_2\frac{r_1}{r_2}\right)\right]\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t + \\ & \frac{d}{dp}\left[\left(\frac{1}{r_2}\left(s_0 - s_1\frac{F_{11}}{r_2} + s_2\frac{F_{11}^2}{r_2^2}\right)\right)\right]\exp\left(\frac{F_{11}-r_1}{r_2}t\right)\int\exp\left(\frac{r_1-2F_{11}}{r_2}t\right)\int\exp\left(\frac{F_{11}}{r_2}t\right)\kappa(t)\,\mathrm{d}t\,\mathrm{d}t. \end{aligned} \quad (40)$$

5. PERIODIC EXCITATION

Periodic excitations are important for dynamic testing of viscoelastic materials, as for example, in the measurement of dynamic moduli (Zhang *et al.*, 2022). Nonetheless, periodic excitation is also important for the design of dampers, like in seismic applications and other attenuation devices (Karavasilis *et al.*, 2011; Zhu *et al.*, 2011). Hence, lets exploit the idea for Eq.(35) of representing either stress either strain as a generic field variable $\gamma(t)$ by defining the excitation $\kappa(t)$ as a periodic function in complex form

$$\kappa(t) = \sum_{j=1}^{n} c_j \exp\left(\beta_j t + \phi_j\right), \quad \beta_j, \phi_j \in \mathbb{C}; \quad (41)$$

where $\beta_j = i\omega_j$ is a complex angular frequency $\phi_j = i\varphi_j$ is a complex phase. Using Eq. (35), the complete solution is found to be:

$$\begin{aligned}\gamma(t) = {} & C_1\exp\left(\frac{-F_{11}}{r_2}t\right) + C_2\exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \sum_{j=1}^{n} c_j\exp\left(\beta_j t + \phi_j\right)\left(\frac{s_2}{r_2} + \left(\frac{1}{r_1 - F_{11} + r_2\beta_j}\right)\left[s_1 - s_2\frac{r_1}{r_2}\right]\right. \\ & \left. + \left(\frac{r_2}{\left(r_1 - F_{11} + r_2\beta_j\right)\left(F_{11} + r_2\beta_j\right)}\right)\left[s_0 - s_1\frac{F_{11}}{r_2} + s_2\frac{F_{11}^2}{r_2^2}\right]\right), \end{aligned} \quad (42)$$

and, using Eq. (40), the derivative of the complete solution due to periodic excitation is

$$\frac{d\gamma(t)}{dp} = \left(\frac{dC_1}{dp} - C_1 \frac{d}{dp}\left[\frac{F_{11}}{r_2}\right]\right) \exp\left(\frac{-F_{11}}{r_2}t\right) + \left(\frac{dC_2}{dp} + C_2 \frac{d}{dp}\left[\frac{F_{11}-r_1}{r_2}\right]\right) \exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \sum_{j=1}^{n} c_j \exp\left(\beta_j t + \phi_j\right) \left(\frac{d}{dp}\left[\frac{s_2}{r_2}\right] + \frac{d}{dp}\left[\frac{1}{r_2}\left(s_1 - s_2\frac{r_1}{r_2}\right)\right] \frac{r_2}{r_2\beta_j + r_1 - F_{11}} + \frac{d}{dp}\left[\frac{1}{r_2}\left(s_0 - s_1\frac{F_{11}}{r_2} + s_2\frac{F_{11}^2}{r_2^2}\right)\right] \frac{r_2^2}{\left(r_2\beta_j + F_{11}\right)\left(r_2\beta_j + r_1 - F_{11}\right)}\right). \tag{43}$$

6. CONSTANT EXCITATION

Constant excitation functions are fundamental for creep and relaxation tests, where stress is kept constant throughout the test and the strain is measured, Cespi *et al.* (2007); Martins *et al.* (2015), or vice-versa, Shi *et al.* (2020); Martins *et al.* (2015). Using Eq. (35), the complete solution to the constant excitation field, $\frac{d\kappa}{dt} = 0$, is analytically obtained

$$\gamma(t) = C_1 \exp\left(\frac{-F_{11}}{r_2}t\right) + C_2 \exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \frac{s_0 r_2 \kappa}{F_{11}\left(r_1 - F_{11}\right)}. \tag{44}$$

However, using Eq. (11), one can simplify the denominator of the particular solution to

$$\gamma(t) = C_1 \exp\left(\frac{-F_{11}}{r_2}t\right) + C_2 \exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \frac{s_0}{r_0}\kappa, \tag{45}$$

avoiding the calculation of any complex quantity in the particular solution. Previous solution is an example of exponential series found in many relaxation response of amorphous polymers.

Furthermore, using Eq.(40), the derivative of the complete solution due to constant excitation is given by

$$\frac{d\gamma(t)}{dp} = \left(\frac{dC_1}{dp} - C_1 \frac{d}{dp}\left[\frac{F_{11}}{r_2}\right]\right) \exp\left(\frac{-F_{11}}{r_2}t\right) + \left(\frac{dC_2}{dp} + C_2 \frac{d}{dp}\left[\frac{F_{11}-r_1}{r_2}\right]\right) \exp\left(\frac{F_{11}-r_1}{r_2}t\right) + \kappa \frac{d}{dp}\left[\frac{s_0}{r_0}\right]. \tag{46}$$

7. EXAMPLES

This section provides examples of solution of the Burgers model using the Kelvin representation. According to Mackiewicz and Szydło (2019), the constitutive differential equation is given by

$$\frac{\eta_1\eta_2}{E_1E_2}\ddot{\sigma} + \left(\frac{\eta_1}{E_1} + \frac{\eta_1}{E_2} + \frac{\eta_2}{E_2}\right)\dot{\sigma} + \sigma = \eta_1\dot{\varepsilon} + \frac{\eta_1\eta_2}{E_2}\ddot{\varepsilon}, \tag{47}$$

and the properties used are: $E_1 = 742\ MPa$; $\eta_1 = 148.891\ GPa/s$; $E_2 = 1483\ MPa$ and $\eta_2 = 34.914\ GPa/s$, experimentally obtained for asphalt mixtures (Mackiewicz and Szydło, 2019).

Two examples are studied: the response due to a periodic (sinusoidal) excitation and the response due to a constant strain field (relaxation test). Analytical solutions obtained with the proposed approach are compared to a well established 4th order Runge-Kutta numerical method with adaptive time step (Tsitouras, 2011).

7.1 Dynamic test

Let stress be the periodic excitation field, $\kappa = \sigma(t) = 60sin\left(0.02\pi t\right)\ [kPa]$, as shown in Figure 1 (bottom). The solution given by the proposed approach, Eq. (42), is shown in Fig 1 (top image, solid blue line) and the numerical solution is also shown in the top image as a dotted dark line. As it can be seen, both solutions overlap.

Figure 2 shows the homogeneous and the particular solutions obtained by using the proposed approach. It is important to stress that the generalized integrating factor can be used to independently evaluate both solutions, unlike the numerical approach.

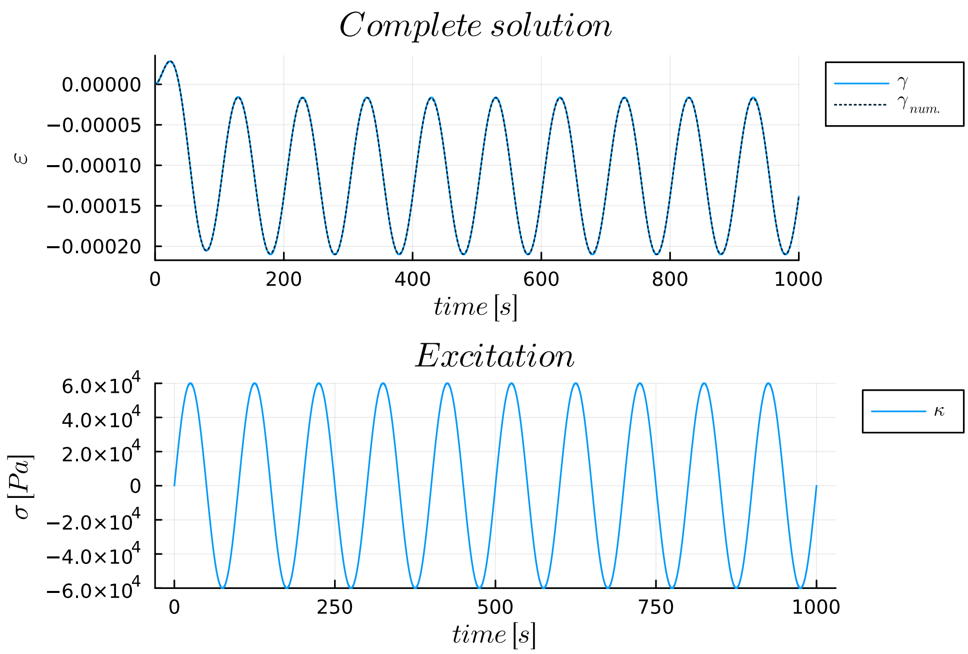

Figure 1. Sinusoidal stress excitation (bottom), analytical strain $\gamma(t)$ obtained by using the proposed approach and numerical solution$\gamma_{num}(t)$.

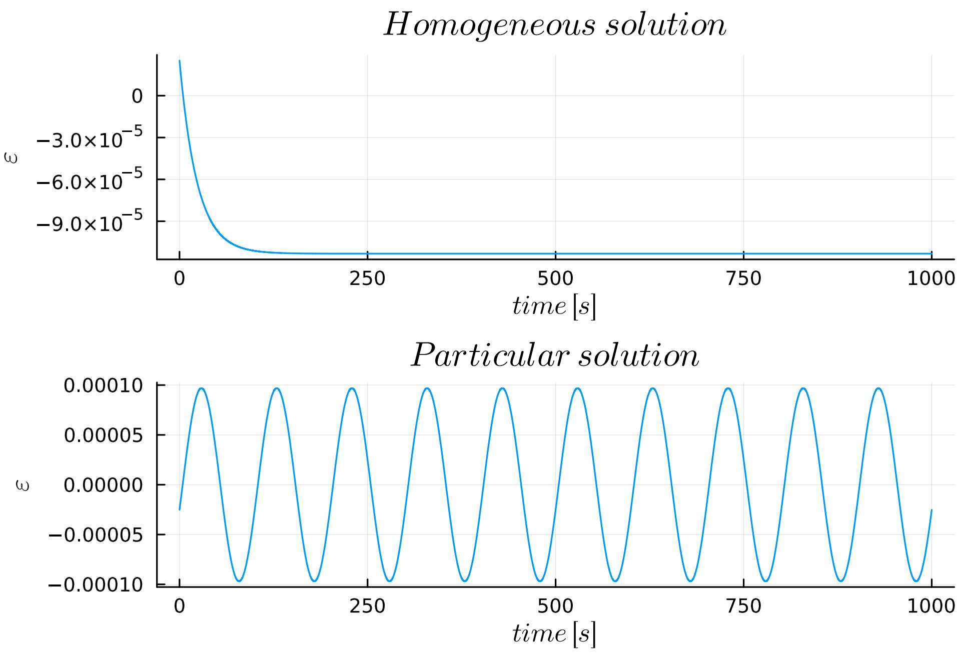

Figure 2. Homogeneous solution (top) and particular solution (bottom) associated to the complete analytical response $\gamma(t)$ shown in the top of Figure 1.

7.2 Relaxation test

Let strain be the constant excitation field $\kappa = \varepsilon(t) = 0.0001$, as shown in Figure 3 (bottom). The analytical result obtained by using the proposed approach is shown in the top of Fig. 3 as a solid blue line, and the solution obtained by using the numerical approach is shown as a dotted dark line. It can be seen that both solutions agree.

Figure 4 shows the homogeneous (top) and the particular solution (bottom) associated to $\gamma(t)$. Since coefficient of $\varepsilon(t)$ in the right hand side of Eq. (47), b_0, is null and the strain is constant, the excitation field is also null. Thus, the particular solution is trivially zero and the complete solution depends only on the homogeneous solution.

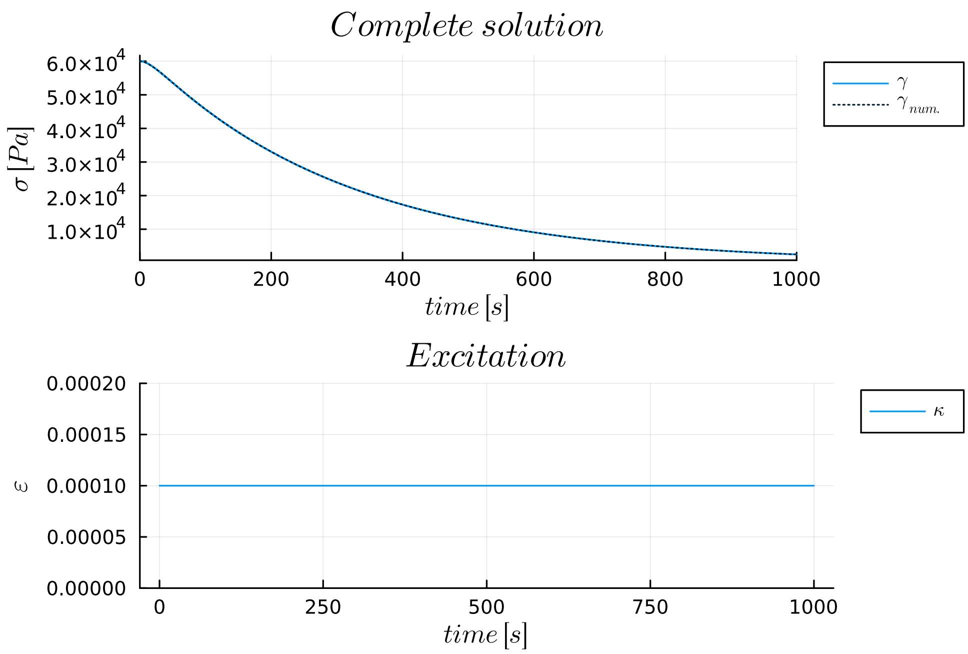

Figure 3. Constant strain excitation (bottom), stress field solution $\gamma(t)$ obtained by using the proposed approach (top) and the numerical solution $\gamma_{num}(t)$ (top).

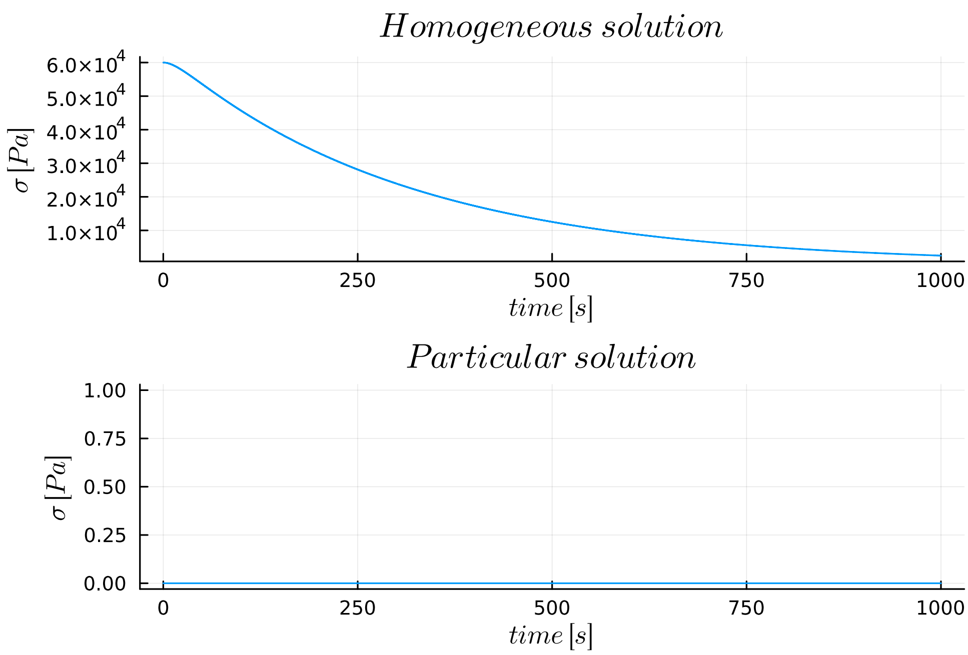

Figure 4. Comparison between the homogeneous and particular solutions of the stress field, when a constant strain excitation field is applied.

8. CONCLUSIONS

This work presented a family of analytical solutions for the Burgers 4-parameters viscoelastic constitutive model. Closed form solutions were obtained for both stress and strain excitation, allowing to analytically explore the application of this versatile model to a variety of cases without resorting to numerical methods. The derivative of these analytical responses with respect to material coefficients were also obtained and can be used to implement efficient data fit procedures. The results were validated by using a well known numerical method as reference. The general order reduction methodology introduced herein can be adapted for other types of second order differential equations and other material models in the future.

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