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UNCERTAINTY ANALYSIS OF THE BAND GAP FORMATION IN PERIODIC ROTORS

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Abstract. *Rotating machines are critical components of many industries, including power generation, transportation, and manufacturing. However, the vibration of these machines can lead to various issues, such as increased wear, reduced performance, and even catastrophic failure. To mitigate these problems, engineers need to design and optimize rotors that can operate efficiently and reliably under various operating conditions. One way to improve rotor performance is by taking advantage of band gaps. In rotating machines, structural periodicity can create band gaps (or modal spacing) on the frequency spectrum of the system. These band gaps regions are frequency zones where no resonance appears, with a large distance between two adjacent resonances, and they are characterized by a low vibrating response to excitation of the system. In this paper, we investigate the robustness of band gap formation in the presence of uncertainties by comparing results from a stochastic and a deterministic model. To achieve this, this study applies a probabilistic approach utilizing Monte Carlo theory and the Hoeffding–Sobol decomposition is evaluated both in a numerical investigation of a rotor with longitudinal periodicity. The results show that including uncertainties in modelling provides valuable insights into the robustness of band gap formation in rotating machines. The study sheds light on the potential benefits of probabilistic approaches for improving the predictability, efficiency, and reliability of rotating machines. The findings of this study can provide valuable insights for future research on the optimization and design of rotors with enhanced vibration performance.*

Keywords: *rotating machinery, wave propagation, modal analysis, uncertainty, band gaps.*

1. INTRODUCTION

Rotating machines play a vital role in numerous industries, encompassing power generation, transportation, and manufacturing. However, the inherent vibration exhibited by these machines can lead to various detrimental consequences, including accelerated wear, diminished performance, and even catastrophic failures. Consequently, engineers face the significant challenge of designing and optimizing rotors that can operate seamlessly under diverse operating conditions while mitigating the adverse effects of vibration. One promising avenue for enhancing rotor performance involves the concept of band gaps.

In the quest for improved rotor performance, one promising avenue of exploration lies on the use of the band gap phenomenon. In the context of rotating system, the structural periodicity in their design can give rise to band gaps, or modal spacing, within the frequency spectrum of the system (Richards and Pines, 2003; Alsaffar *et al.*, 2018). These band gaps are characterized by frequency zones wherein resonance does not occur, exhibiting a significant separation between adjacent resonances and a low vibration response to excitation (Deymier, 2013). Such characteristics are especially helpful in the designing of rotating machines, because critical speeds can be far from the operating range in a band gap zone. Therefore, it is interesting to design the structure to work at these regions (or to design the structure to present band gaps in the working conditions), since this has the potential to significantly enhance rotor performance, minimizing the deleterious effects of vibration and promoting efficient and reliable operation.

In practice, using structural periodicity in rotor systems, e.g., through periodic allocation of impellers along the rotor (Lamas and Nicoletti, 2022), affects the position and geometry of the vanes and volutes in the rotating machine, which affects its flow dynamics. Consequently, that will require a redesign of the machine's casing considering its efficiency and flow dynamics, and in rotating systems this is not always feasible, which leaves no room for big modifications. Thus, this paper aims to explore the effects of braking the longitudinal periodicity through slight variations in the periodic positioning of the impellers to assess the robustness of band gap formation. This is done to see if small variations, in order to fulfil design requirements, can be employed on the use of periodicity. To do that, this study applies a probabilistic approach utilizing Monte Carlo theory and the Hoeffding–Sobol decomposition (Sobol, 1993; Saltelli *et al.*, 2000) is evaluated both in a numerical investigation of a rotor with longitudinal periodicity, treating the impeller's positions as input random variables. The study sheds light on the potential benefits of probabilistic approaches for improving the predictability, efficiency, and reliability of rotating machines. The findings of this study can provide valuable insights for future research

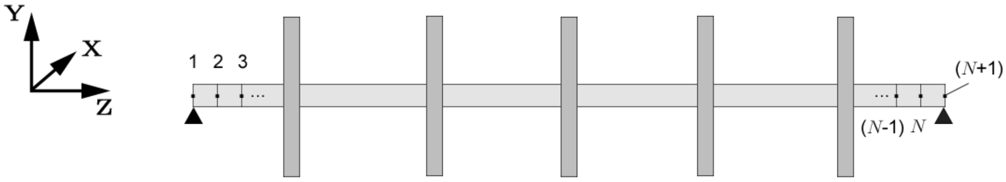

Figure 1: Finite element mesh representation of the periodic rotor with the bearings represented by black triangles.

on the optimization and design of rotors with enhanced vibration performance.

2. METHODOLOGY

2.1 Mathematical Modelling

Consider a rotor with five impellers periodically mounted along the shaft, as depicted in Fig. 1. To model the rotor are adopted finite elements based on Euler-Bernoulli beam theory, where the working elements (impellers) are considered rigid and represented by disks (Nelson and McVaugh, 1976). The same analysis can be made using finite elements based on the Timoshenko beam theory (Nelson, 1980). The shaft finite element has a constant radius and constant material properties, and it has two nodes, each one with four degrees of freedom: translation in lateral directions (x_i and y_i) and rotation around X and Y directions (β_i and γ_i). By adopting a mesh of N-connected elements to model the shaft, we obtain the system of ordinary differential equations:

$$\mathbf{M}\ddot{\mathbf{z}} - \Omega\,\mathbf{G}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{F} \tag{1}$$

where $\mathbf{M}$ is the inertia matrix, $\mathbf{G}$ is the gyroscopic matrix, $\mathbf{K}$ is stiffness matrices, Ω is the rotating speed of the rotor (in rad/s), $\mathbf{F}$ is the vector of external applied forces, and $\mathbf{z}$ is the vector of degrees of freedom:

$$\mathbf{z} = \{ \; x_1 \quad y_1 \quad \beta_1 \quad \gamma_1 \quad \cdots \quad x_i \quad y_i \quad \beta_i \quad \gamma_i \quad \cdots \quad x_{N+1} \quad y_{N+1} \quad \beta_{N+1} \quad \gamma_{N+1} \; \}^T \tag{2}$$

where N is the number of finite shaft elements in the model.

The elements that represent the impellers are modeled as rigid disks. Hence, they are represented in the model by their masses and moments of inertia, which are added to the inertia and gyroscopic matrices according to their position in the rotor, i.e. according to the node where the impeller is located in the model. The finite element matrices used in the model are found in Nelson and McVaugh (1976).

By adopting this model, one can find the natural frequencies of the rotor from the eigenvalues of the system in the state-space formulation. In state-space formulation, the system is described as:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{z}} \\ \dot{\mathbf{z}} \end{Bmatrix} + \begin{bmatrix} -\Omega\mathbf{G} & \mathbf{K} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{z}} \\ \mathbf{z} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \;\Rightarrow\; \mathbf{A}\dot{\mathbf{u}} + \mathbf{B}\mathbf{u} = \mathbf{0} \tag{3}$$

whose eigenvalue problem is stated as:

$$(\mathrm{i}\lambda_j \mathbf{A} + \mathbf{B})\,\phi_j = \mathbf{0} \tag{4}$$

where λ_j is the j-th eigenvalue of the system, ϕ_j is the j-th eigenvector of the system, and $\mathbf{A}$ and $\mathbf{B}$ are state-space matrices.

Furthermore, the bearings allow rotational motion only (lateral displacements are constrained), thus representing the configuration of simply-supported rotor. As a rule, we adopt 10 finite elements between two disks or between a disk and a bearing (a number of elements large enough to assure natural frequency convergence).

2.2 Uncertainty Analysis

As already mentioned, the positioning of the disks along the rotor is considered the input random variables of analysis (5 disks = 5 input random variables). The process of choosing the combination of values for the positions is based on

Table 1: Properties of the rotor in study.

Property	Value	Unit
shaft length (L)	600	mm
shaft diameter (D)	5	mm
disk diameter (D_D)	60	mm
disk thickness (E_D)	5	mm
Young modulus (E)	2.1×10^{11}	N.m^{-2}
material density (ρ)	7850	kg.m^{-3}

an assumed probability density function (PDF) (Kroese *et al.*, 2013). In this work, we adopted a uniform distribution of values for the position of the disks. The values vary in a range between a minimum and a maximum chosen values ($=$ *mean* ± 5 mm), where this mean value is the nominal periodic position for each disk. Therefore, any combination of position values has equal probability to occur in a given data sample (a conservative hypothesis). Such sample of values are created for each position of the disks, thus resulting in a set of five random variables $\mathbf{N_d} = [n_d^1, n_d^2, n_d^3, n_d^4, n_d^5]_{N_s \times 5}$, where N_s is the number of samples and n_d^i is the position of i-th disk. Given a combination of disk positions, four output random variables are obtained: the start and stop natural frequencies of the band gap and, consequently, its bandwidth and central frequency. This step is repeated for all N_s combinations. To assure convergence of Monte Carlo method, the second statistical moment of the output random variables are verified. To evaluate the relation between position uncertainty and the resultant gyroscopic effect, a stochastic Campbell diagram is built. Finally, a global sensitivity analysis (Saltelli *et al.*, 2000), using the Hoeffding–Sobol decomposition, is employed. This allows to partition the system response variance and to determine the Hoeffding–Sobol decomposition in terms of variance (Sobol, 1993), that means, to investigate the influence of each input random variable on the total output variance. In terms of equations:

$$S_i = \frac{Var[Y_i]}{Var[\mathbf{Y}]} \tag{5}$$

where, S_i is the first Sobol index and it quantifies the additive effect of i-th input separately concerning the total variance, $Var[\,]$ is the variance operator, Y_i is output random variable due to the i-th input and $\mathbf{Y}$ is the total output random variable due to all inputs together.

3. RESULTS

The properties of the rotor in study are listed in Table 1, a scheme is illustrated in Fig. 1. Figure 2 shows a Campbell diagram of the periodic rotor. In this case, all the position of the disks are the periodic nominal positions (positions without uncertainty). As one can see, a band gap zone, the highlighted blue zone, is formed in the system's frequency spectrum. As the rotating speed increases, the band gap tends to narrow due to the splitting of modes into backward and forward whirl modes, but this does not impair the use of the phenomenon in the rotating speed range analyzed, since the bandwidth of the band gap is still large enough to be used.

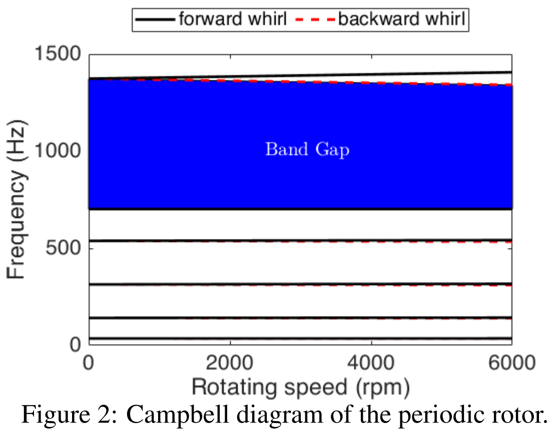

Figure 2: Campbell diagram of the periodic rotor.

Figure 3 shows the histograms of input random variables, for all cases an uniform distribution (with a number of samples of $N_s = 2000$) was assumed and the range of this distribution is equal the periodic position ± 5 mm (10 mm of total range). Furthermore, in Fig. 4 is shown the convergence of Monte Carlo method through the estimation of the second statistical moment, in which the convergence can be noted for $\approx N_s > 1000$. This assures the convergence of the method and allows us to continue to the stochastic analysis of the output variables.

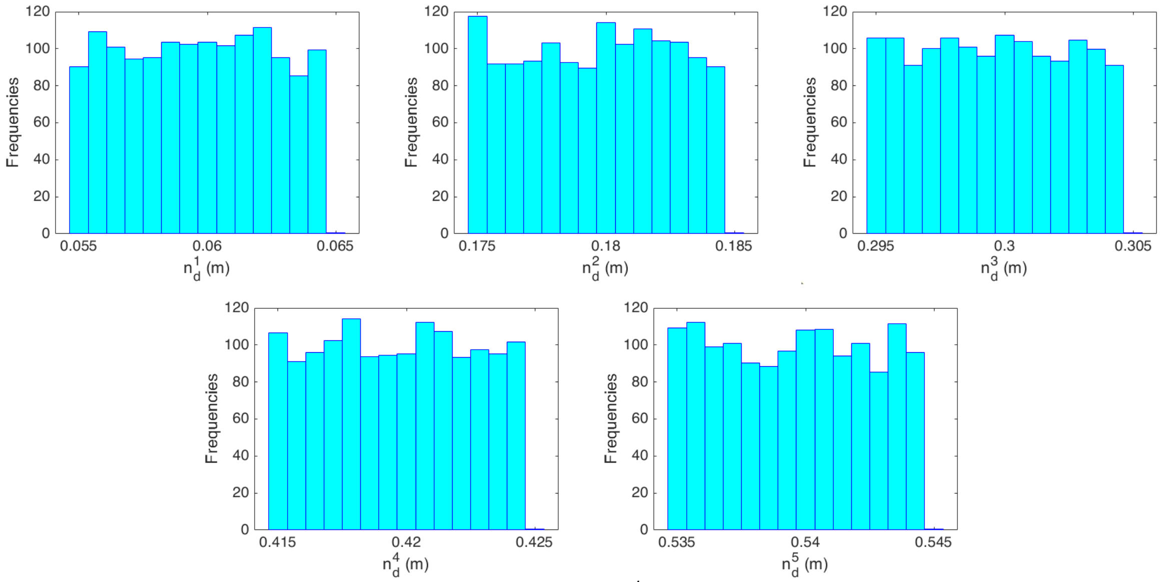

Figure 3: Histograms of generated input random variables: n_d^i represents the position along the rotor of th i-th disk.

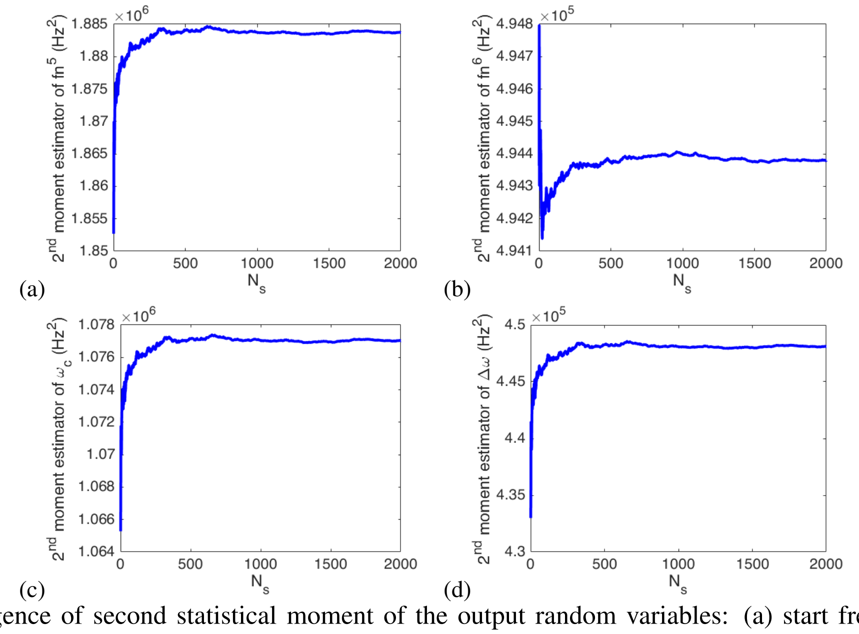

Figure 4: Convergence of second statistical moment of the output random variables: (a) start frequency f_n^5, (b) stop frequency f_n^6, (c) band gap central frequency ω_c and (d) band gap bandwidth $\Delta\omega$.

In Fig. 5, the histograms (light blue) with the resultant estimated PDFs (dark blue) of the output random variables are shown. As one can see, the start natural frequency of the band gap (f_n^5 in Fig. 5(a)) is not affected by the variation on the position of the disks, this can be noted by its very small standard deviation ($\hat{\sigma}_{f_n^5} = 0.45$ Hz). In Fig. 5(b), which represents the stop frequency of the band gap, the influence of a variation on the position of the disks is considerable, but still not too large ($\hat{\sigma}_{f_n^6} = 9.05$ Hz). Since the central frequency and the bandwidth of the band gap are dependent of these two natural frequencies (i.e., $\omega_c = (f_n^5 + f_n^5)/2$ and $\Delta\omega = f_n^6 - f_n^5$), the same pattern is followed in Fig. 5(c) and (d), in which we see a small standard deviation of both.

Moreover, in Fig. 6(a), the influence of the rotating speed is analyzed through a stochastic Campbell diagram. Note that in this figure the precession modes (FW and BW) are separeted to a better visualization, this was made by plotting the backward modes with a negative rotating speed, so, it is worth to mention that this only has a visualization effect and the negative sign of the rotating speed has no physical meaning. Furthermore, in this diagram we see that the stochastic mean values converged to the periodic values which were obtained with the deterministic model, that means, the mean

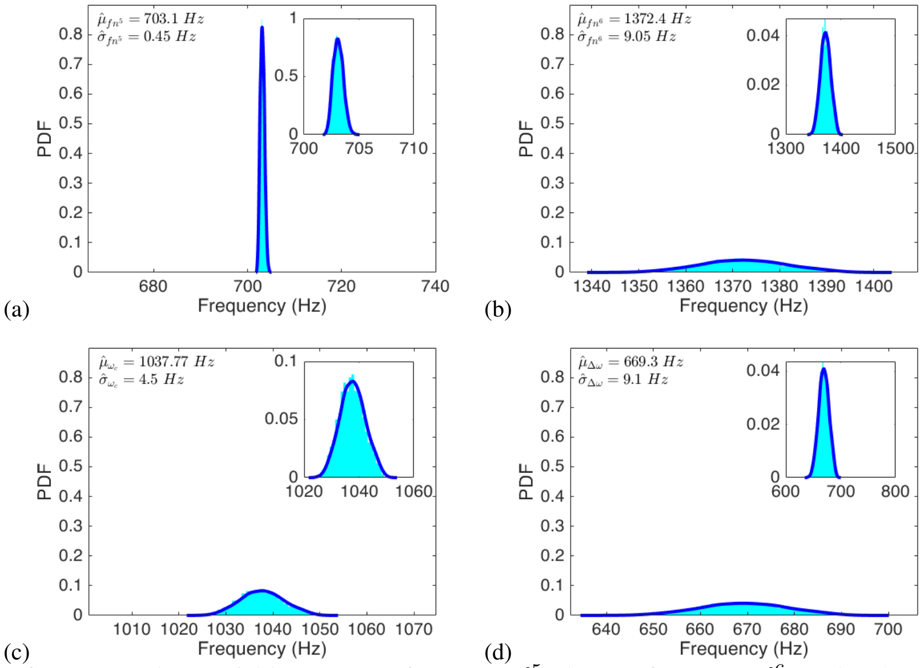

Figure 5: PDFs of output random variables: (a) start frequency f_n^5, (b) stop frequency f_n^6, (c) band gap central frequency ω_c and (d) band gap bandwidth $\Delta\omega$.

stochastic model converged to the deterministic model. Also, it is visible that only f_n^6 is affected by the rotating speed but the rotating speed does not affect the variance of this variable, since the statistical envelope only shifts and does not increases or decreases in size. Also, it is visible through the confidence interval of 95% that the variation on the position affects more f_n^6 than f_n^5. Lastly, in Fig. 6(b) the mode shapes of these natural frequencies are plotted and confirms this proposition, As one can see, the position of the disks in the mode shape associated with f_n^6 are really close to nodal points of the mode shape, so small perturbations on the position can lead to a big difference on the dynamics of disks and, consequently, on this natural frequency value, The same does not happen with the mode shape associated with f_n^5, where it is visible that small perturbations on the position, can lead to a very similar system.

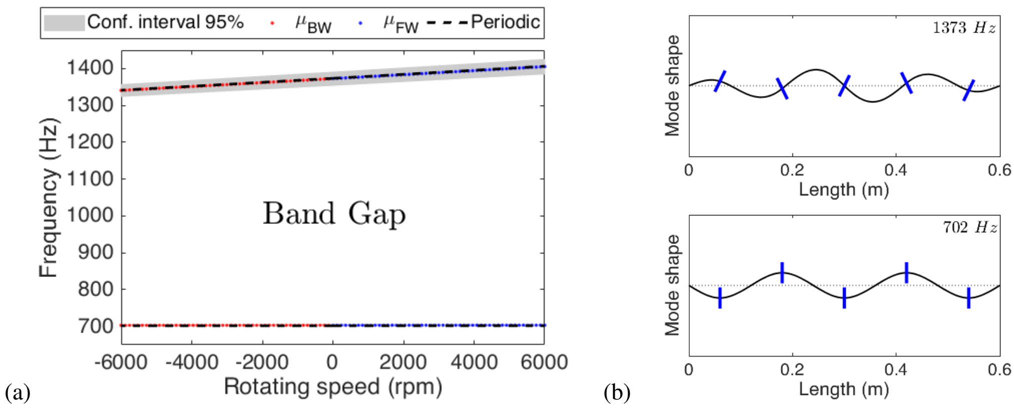

Figure 6: (a) Stochastic Campbell diagram (μ - mean value, BW - backward whril and FW - forward whirl) and (b) mode shapes of associated natural frequencies at null rotating speed.

Finally, a Sobol analysis of the system is shown in Fig. 7. In this analysis, the total variance of the output variables is partitioned and the influence of each input variable can be evaluated, and the reason why the the Sobol indices varies from 0 to 1 is because it is normalized by the total variance of the output random variables. Figure 7 shows the influence of each disk position on the total variance of the output variables. As one can see, the variance of f_n^5 is equally distributed through the disks, whereas for f_n^6 the influence of the central disk (n_d^3) is null and the disks near to the bearings have a small influence compared with their two neighbours (n_d^2 and n_d^4).

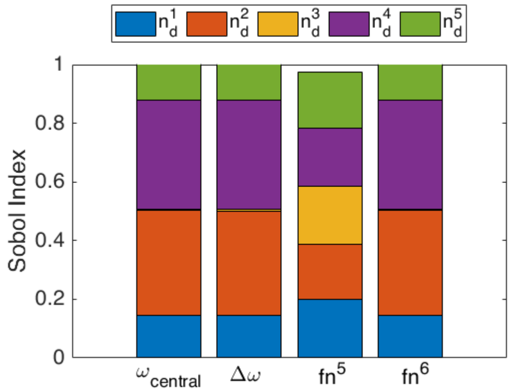

Figure 7: Sobol indices.

4. CONCLUSION

This work presents a methodology to quantify the uncertainty propagation in periodic rotating machines. The obtained results led to the following conclusions:

- in longitudinal periodic rotors it is possible to break the periodicity and still obtain band gaps;
- due to the mode shapes associated with the start and stop natural frequencies that delimits the band gap zone, only the stop natural frequency is affected by the variation on the position of disks;
- small variations on the disk position can be employed in order to accomplish design requirements. In this case, some disk's positions have more influence on the variability of the system than others;

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