27th ABCM International Congress of Mechanical Engineering
December 04-08, 2023, Florianópolis-SC, Brazil

NUMERICAL SOLUTION FOR ONE-DIMENSIONAL MULTIPHASE FLOWS IN PIPELINE-RISER SYSTEMS

Leonardo Alegretti Belaroza
Karl Peter Burr
Universidade Federal do ABC
leonardobelaroza@gmail.com, karl.burr@ufabc.edu.br

Abstract. *A numerical solver is designed and implemented to solve a mixture flow model for multiphase flow in a pipeline-riser system with inertia terms included. The mixture model consists of the gas and liquid mass conservation equations, a momentum equation for the liquid-gas mixture and an adequate closure law, which is a function of the multiphase flow pattern. Basically, two flow patterns are considered, stratified and non-stratified flow. For the flow in the riser, a drift-flux model, evaluated for the local conditions in the riser, is used as a closure law. The flow in the pipeline is assumed as always stratified, and a local balance among gas and liquid wall friction forces, gas-liquid interfacial friction force and gravity force furnishes a closure law. It is well known that 1D multiphase flow models may not have a clear hyperbolic nature and the difficulty to obtain a neat analytical expression for Jacobian matrix limits the formulation a Roe type Riemann solver (Quasi-Linear). To circumvent this issue, a slightly different approach is used. The system of equations is solved numerically using the Lax Wendroff method. This approach to solve the Riemann problem gives a numerichal squeme with second order accuracy. The code is validated and simulation results for different intermittent flow patterns in pipeline - rise systems are presented.*

Keywords: *pipeline-riser system, multiphase flow, intermittent flow patterns, numerical simulation, hyperbolic system of equations.*

1. INTRODUCTION

Pipeline-riser systems in offshore oil production units connecting the oil field to the offshore production structure may present parts with downward inclination followed by parts with upward inclination, like the riser, for example. For low gas and liquid flow rates from the oil well, this type of pipeline-riser configuration may present intermittent flow regimes instead of a steady state regime. These intermittent flow patterns are periodic or almost periodic. An example of such intermittent flow regimes is severe slugging, which has usually long periods, large pressure variation at the riser base and very large flow velocities at the riser top, what may cause the system operation shutdown. Details of the different intermittent flow patterns that may appear in multiphase flow in pipeline-riser are given in Malekzadeh *et al.* (2012). These intermittent flow pattern may also occur in land based pipelines with downward inclined sections followed by upward inclined sections, and they are called terrain induced slugging in the literature.

Tools to predict the dynamic of the multiphase flow in pipeline-riser systems as a function of the system parameters and boundary conditions is necessary in the design of offshore oil production systems. Linear stability analysis predicts the regions in the system parameter and boundary condition space where the single stationary state looses stability, no steady state regime is possible and an intermittent flow regime sets in.

Azevedo *et al.* (2015) and Azevedo *et al.* (2017) used linear stability analysis to construct stability maps for the single stationary state in the system parameter and boundary condition space. Azevedo *et al.* (2015) studied the effect of different severe slugging mitigation mechanisms on the stationary state stability and Azevedo *et al.* (2017) studied the effect of the pipeline modeling in the stationary state stability. The flow model for the riser used in Azevedo *et al.* (2015) and Azevedo *et al.* (2017) is a isothermal mixture model with the drift-flux model as the closure law. Bendiksen *et al.* (1987) correlation was used for the distribution parameter and slip velocity of the drift-flux model. In Azevedo *et al.* (2015), the pipeline was modeled as a control volume and the flow assumed as always stratified with constant void fraction in time and space. In Azevedo *et al.* (2017), the pipeline was modeled in three different ways. The first two approaches modeled the pipeline as a control volume, one with constant void fraction in time and space and the other with void fraction constant in space but varying with time. The third approach models the pipeline as a distributed system and uses a mixture model for the multiphase flow, but with a local equilibrium relation for stratified flow as the closure law for the mixture model.

Linear stability analysis is able to predict the regions in the system parameters and boundary conditions space where intermittent flow patterns may appear, but does not allow any additional information. Evolution equations obtained through asymptotic theory to capture the flow system periodic behaviour, as described in Burr and Baliño (2015), is valid only in a small region of the system parameter and boundary condition space close to the boundary between regions with and without steady state flow regime and furnishes estimates for the intermittency period and for the flow variables variation range.

Since linear stability analysis and instability evolution equations offer a limited knowledge about the multi-phase flow dynamics in pipeline-riser systems, to better understand, predict and control the dynamics of such systems, tools of system dynamics theory could be helpful, but then the ability to perform correct numerical simulation of an adequate model of such systems is necessary.

The aim of this work is to develop a Riemann solver for the transient isothermal one-dimensional mixture flow model with a generic closure law. The form of the closure law changes along the pipeline and riser due to changes in the flow pattern and according to suitable transitions criteria. For the present work, only stratified and non-stratified flow patterns are considered. For the non-stratified flow pattern in the riser, a drift-flux model is used as the closure law and a local balance among gas and liquid wall friction forces, gas-liquid interfacial friction force and gravity force is used as a closure law for the assumed as always stratified flow in the pipeline. More general forms of the closure law maybe necessary. For example, simulation of severe slugging intermittency requires a closure law able to deal with counter-current flow at the riser during the end of the gas blowdown phase of this intermittency (see Malekzadeh *et al.* (2012) for a description of the severe slugging inttermitency phases). The drift-flux model with the Chexal-Lellouche correlation (see Chexal *et al.* (1977) and Hibiki (2019)) for the distribution parameter and slip velocity with an appropriate flooding line correlation can handle counter-current flows. Since the flow in the riser is mostly counter-current annular at this intermittency phase, a balance among the forces involved in this annular flow can also be used as closure law for the riser flow. Therefore, the closure law changes in space and time along the pipeline-riser system. As a consequence, a Roe matrix to be used in a Riemann solver has to be constructed numerically at each time step of the numerical simulation due to the complexity of the governing equations Jacobian matrix. The contribution of the present work in progress is the development and implementation of Roe matrix based Riemann solver for the one-dimensional isothermal mixture flow model with a generic space and time variable closure law.

However, for the numerical method to work, the system must be hyperbolic, meaning it must have real and distinct eigenvalues. Loss of hyperbolicity indicates inappropriate closure equations. The transient equations representing the mixture model form a nonlinear hyperbolic system of conservation laws in one space dimension, mathematically representing the expected wave behavior of the gas-liquid mixture. Hyperbolicity is a key feature of this system, governing the nature of the numerical methods used to solve it.

The Riemann solver is used to simulate numerically the air–water multiphase flow in pipeline-riser system with a catenary riser for the experimental conditions reported in Wordsworth *et al.* (1998).

In the next section the isothermal one-dimensional mixture model for air-water multiphase flow in a pipeline-riser system with a catenary riser is described and the appropriate closure law is discussed. The governing equations are written in a conservative form. The boundary conditions are also written in terms of appropriate conservative variables. The numerical approach using a Riemann solver is described in the third section. Results are presented in the fourth section, and consist of hyperbolicity maps for the system of governing equations in conservative form, and numerical evidence of the success or lack of success of the chosen numerical approach to capture the different types of intermittent patterns. Discussion and conclusions are given in the fifth section.

2. MATHEMATICAL FORMULATION

2.1 One-Dimensional Modeling of the Oil Pipeline-Riser System

The flow in the pipeline-riser system is assumed isothermal and one-dimensional. Therefore, all dependent variables are functions of time and of the pipeline-riser system arc length s. The liquid and gas are assumed compressible fluids. The multiphase flow in a pipeline-riser system with the catenary-shaped riser considered in Balino and Burr (2007) is modeled by a mixture model with inertia terms included and without mass exchange between phases. The model consists of the liquid and gas phases mass conservation equation, the linear momentum mixture conservation equation, similar to that used by Masella (1998) and a general closure law. The pipeline-riser system geometry and boundary conditions are illustrated in Fig. 1.

The liquid phase mass conservation equation is

$$-\frac{\partial}{\partial t}\left[(1-\alpha)\rho_l\right]+\frac{\partial}{\partial s}\left[(1-\alpha)\rho_l u_l\right]=0 \tag{1}$$

where α is the void fraction, ρ_l is the liquid phase density and u_l is the liquid phase velocity. The gas phase mass conservation equation is

$$\frac{\partial}{\partial t}\left(\rho_g\alpha\right)+\frac{\partial}{\partial s}\left(\alpha\rho_g u_g\right)=0 \tag{2}$$

where ρ_g is the gas phase density, and u_g is the gas phase velocity. The linear momentum equation for the liquid-gas mixture is

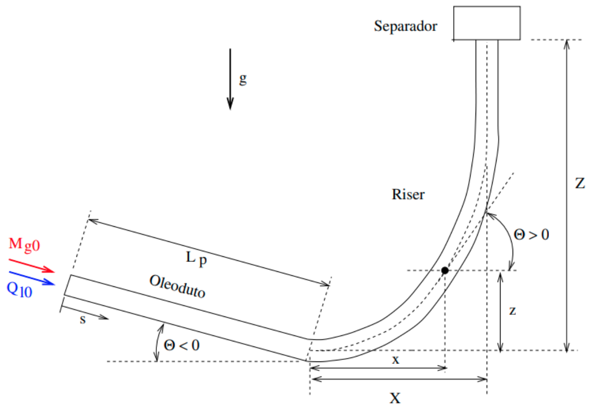


Figure 1. Oil Pipeline-Riser System

$$\frac{\partial}{\partial t}\left[(1-\alpha)\rho_l u_l + \alpha\rho_g u_g\right] + \frac{\partial}{\partial s}\left[(1-\alpha)\rho_l {u_l}^2 + \alpha\rho_g {u_g}^2\right] = -\rho_m\left[g\sin\theta(s) + \frac{2}{D} f_m\left(R_e, m, \epsilon/D\right) j|j|\right] \quad (3)$$

where ρ_m is the liquid-gas mixture density, given by

$$\rho_m = (1-\alpha)\rho_l + \alpha\rho_g. \quad (4)$$

$\theta(s)$ is the local inclination along the pipeline-riser system. For the pipeline section (descending section), $\theta(s) < 0$ and constant, while for the riser section $\theta(s) \geq 0$ and variable since the riser has a catenary shape. The term g represents the gravitational acceleration, D is the diameter of the pipeline and riser, ϵ is the pipe wall roughness, j is the total superficial velocity, f_m is the friction factor of the liquid-gas mixture, which depends on the mixture Reynolds number and the relative roughness (ϵ/D). The mixture Reynolds number is given by

$$R_{e,m} = \frac{\rho_m D|j|}{\mu_m}, \quad (5)$$

with $\mu_m = \mu_g\alpha + (1-\alpha)\mu_l$. We also have that

$$\begin{aligned} j &= j_l + j_g, && (6)\\ j_l &= (1+\alpha)u_l, && (7)\\ j_g &= \alpha u_g, && (8)\end{aligned}$$

where j_l is the liquid phase superficial velocity and j_g is the gas phase superficial velocity.

For the assumed as always stratified flow pattern in the pipeline a local balance among gas and liquid wall friction forces, gas-liquid interfacial friction force and gravity force evaluated at local conditions, given in reference Balino and Burr (2010), is the adopted closure law. For the riser, the flow pattern is always non-stratified (usually a slug pattern) and a drift-flux model with distribution coefficient C_d and the slip velocity U_d is used as the closure law. The correlation for the distribution coefficient C_d and for the slip velocity U_d is given in the reference Bhagwat (2014). Therefore, the closure law in both cases are algebraic relations among the variables $\alpha, \rho_l, \rho_g, u_l$ and u_g and is represented as a functional relation among these variables in the form

$$\Phi(\alpha, \rho_l, \rho_g, u_l, u_g) = 0. \quad (9)$$

2.1.1 Thermodynamic Relations

In the model, both phases (liquid and gas) are considered as compressible fluids. The liquid phase density is given by:

$$\rho_l = \rho_{l,0} + \frac{p - p_{l,0}}{C_l^2}, \quad (10)$$

where $\rho_{l,0}$ and $p_{l,0}$ are given constants, and C_l is the liquid sound speed. The gas phase density is given by:

$$\rho_g = \frac{p}{C_g^2}, \tag{11}$$

where C_g is the gas sound speed. These equations describe the density variations of the two phases with respect to the pressure, taking into account compressibility effects characterized by the sound speeds of each phase.

2.1.2 Boundary Conditions

The boundary conditions are specified at the pipeline entrance and at the riser top. The gas and liquid mass flow rates at the beginning of the pipeline are assumed known and the separator pressure is also assumed known. These boundary conditions are given by the equations:

$$\rho_l j_l = \frac{\dot{M}_{l0}}{A} \text{ at } s = 0 \tag{12}$$

$$\rho_g j_g = \frac{\dot{M}_{g0}}{A} \text{ at } s = 0 \tag{13}$$

$$P = P_S \text{ at } s = L_r + L_p \tag{14}$$

where $\dot{M}_{l0}$ is the mass flow rate of liquid at the pipeline inlet, $\dot{M}_{g0}$ is the mass flow rate of gas at the pipeline inlet, P_S is the pressure in the separator (top of the riser), L_p is the pipeline length and L_r is the riser length.

2.2 Governing Equations in Conservative Form

Equations (1), (2) and (3) can be written in conservative form as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial s} = \mathbf{S} \tag{15}$$

where

$$\mathbf{U} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} (1-\alpha)\rho_l \\ \alpha\rho_g \\ (1-\alpha)\rho_l u_l + \alpha\rho_g u_g \end{Bmatrix} \tag{16}$$

$$\mathbf{F} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{Bmatrix} (1-\alpha)\rho_l u_l \\ \alpha\rho_g u_g \\ (1-\alpha)\rho_l u_l^2 + \alpha\rho_g u_g^2 + P \end{Bmatrix} \tag{17}$$

$$\mathbf{S} = \begin{Bmatrix} s_1 \\ s_2 \\ s_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\rho_m \sin\theta(s) - \frac{1}{2}\rho_m f_m(R_{e,m}, \epsilon/D) j|j| \end{Bmatrix} \tag{18}$$

where $\mathbf{U}$ is the vector of conserved variables, $\mathbf{F}$ is the vector of convective fluxes, and $\mathbf{S}$ is the vector of sources.

The convective flux vector, given by Eq. (17), is written in terms of the primitive variables $\alpha, \rho_l, \rho_g, u_l, u_g$ and P and since a numerical method is used to advance $\mathbf{U}$ in time, at each new time steps the discrete form of the convective fluxes need to be evaluated and, consequently, at each time step the value of the primitive variables need to be known. The methodology to obtain the primitive variables from $\mathbf{U}$ is described below. First, Eqs. (11) and (10) are substituted in the first two lines of Eq. (16). The resulting two equations are then solved for the void fraction α and for the pressure P in terms of u_1 end u_2. Since α is a solution of a quadratic equation, the branch which gives $0 \leq \alpha \leq 1$ is considered. Once, P is known, Eqs. (11) and (10) furnish, respectively, ρ_g and ρ_l. Now, with α, ρ_l and ρ_g known and u_3 given, the third line of Eq. (16) furnishes an algebraic relation between u_l and u_g. This algebraic relation is used to write, for example, u_l as a function of u_g (u_g as a function of u_l) and the result is substituted in Eq. (9), which ends up as a non-linear algebraic equation for u_g (for u_l). This non-linear algebraic equation is solved numerically using a root search method. Finally, once u_g (u_l) is found, the algebraic relation obtained from the third line of Eq. (16) gives u_l (u_g).

2.2.1 Boundary Conditions in Terms of Conservative Variables

The boundary conditions given by equations (12)-(14) in terms of the primitive variables can be written in terms of the conservative variables according to the equations:

$$\frac{u_1}{\rho_{l,s}} - \frac{u_2}{\rho_{g,s}} - 1 = 0 \text{ at } s = L_p + L_r \tag{19}$$

$$\frac{\dot{M}_{l0}}{A} + \frac{\dot{M}_{g0}}{A} - u_3 = 0 \text{ at } s = 0 \tag{20}$$

$$\Phi\left(\alpha(u_1,u_2), \rho_l(u_1,u_2), \rho_g(u_1,u_2), \frac{\dot{M}_{l0}}{Au_1}, \frac{\dot{M}_{g0}}{Au_2}\right) = 0 \text{ at } s = 0 \tag{21}$$

with $\rho_{l,s}$ and $\rho_{g,s}$ as, respectively, the liquid and gas density at the top of the riser. They are evaluated through the thermodynamic relations (Eqs. (10) and (11)) at the separator pressure, which is a boundary condition. With the densities ρ_l and ρ_g known at the top of the riser, the second line of Eq. (16) is solved for α and the result is substituted in the first line of Eq. (16), what results in Eq. (19). The two terms at the right side of the third line of Eq. (16) at the pipeline inlet are, respectively, equal to $\dot{M}_{l0}/A$ and $\dot{M}_{l0}/A$, what implies in Eq. (20). Since α, ρ_l, ρ_g can be easily obtained in terms of u_1 and u_2 by solving the first two lines of Eq. (16) and with Eqs. (10) and (11), $u_l = \dot{M}_{l0}/(Au_1)$ follows from Eq. (12) and the first line of Eq. (16), and $u_g = \dot{M}_{g0}/(Au_2)$ follows from Eq. (13) and the second line of Eq. (16), Eq. (21) results by imposing the closure law at $s = 0$ (pipeline inlet).

2.3 The Jacobian Matrix

Equation (15) can be rewritten in its non-conservative form as:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial s} = \mathbf{S}\,(\mathbf{U}), \tag{22}$$

where **A** represents the Jacobian matrix of the flux vector **F** and is defined as:

$$\mathbf{A} \equiv \left[\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right] = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3} \\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3} \end{bmatrix} \tag{23}$$

Constructing the Jacobian matrix **A** is a difficult process as Eq. (23) suggests. Since **F** is given in terms of primitive variables according to Eq. (17), to obtain $\partial f_i/\partial u_j, i,j = 1,2,3$ the components $f_i, i = 1,2,3$ have to be differentiated with respect to the primitive variables, and afterwards, the primitive variables have to be differentiated with respect to the conservative variables $u_j, j = 1,2,3$. The procedure to construct matrix **A** is analogous to the procedure described in Santim and Rosa (2015). The present work can be seen as a generalization of what was done in Santim and Rosa (2015). They used as closure law a drift-flux model with a very simple correlation for the distribution parameter and drift velocity, but the present work considers a general closure law represented by a functional relation among primitive variables according to Eq. (9), what implies that the elements of matrix **A** are given in terms of the partial derivatives of the functional relation represented by Eq. (9). Santim and Rosa (2015) obtained an analytical expression for matrix **A** in terms of the primitive variables, but for the present work this is not possible due to the nonlinear nature of the closure law.

3. Numerical model

3.1 The Roe linearization

The process of linearization becomes essential due to the non-constant nature of the coefficients in the matrix **A**, which represents the Jacobian of the flux function. These coefficients, dependent on α, ρ_g, u_l, and u_g, exhibit variations in both time and space. In order to accurately calculate the fluxes at each cell interface, it becomes necessary to linearize the matrix **A** within each cell. This linearized form, denoted as $\hat{\mathbf{A}}$, provides the average value of **A** as a function of the left and right states of the conservative variables, expressed as $\hat{\mathbf{A}}(u_L, u_R)$.By linearizing the problem, the original nonlinear equations given by Eq.22 are replaced with a linearized problem represented by Eq. 24.

$$\frac{\partial \mathbf{U}}{\partial t} + \hat{\mathbf{A}}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial s} = \mathbf{S}\,(\mathbf{U}) \tag{24}$$

Only a matrix $\hat{\mathbf{A}}(\mathbf{u}_L, \mathbf{u}_R)$ that fulfills the subsequent conditions is considered. The conditions as originally proposed by Roe (1981) are:
(i) It constitutes a linear mapping of the vector space $\mathbf{u}$ to the vector space $\mathbf{F}$;
(ii) As $\mathbf{u}_L \to \mathbf{u}_R \to \mathbf{u}, \hat{\mathbf{A}}(\mathbf{u}_L, \mathbf{u}_R) \to \mathbf{A}(\mathbf{U})$, where $\mathbf{A} = \partial\mathbf{A}/\partial\mathbf{U}$;
(iii) For any $\mathbf{U}_L$, $\mathbf{U}_R$, $\hat{\mathbf{A}}(\mathbf{U}_L, \mathbf{U}_R) \times (\mathbf{U}_L - \mathbf{U}_R) = \mathbf{F}_L - \mathbf{F}_R$;
(iv) The eigenvectors of $\hat{\mathbf{A}}$ are linearly independent.

3.2 Construction of matrix $\hat{\mathbf{A}}$

The difficulty in constructing the matrix $\hat{\mathbf{A}}$ lies entirely in condition (iii). The existence of a satisfactory $\hat{\mathbf{A}}$ which satisfies condition (iii) follows from the mean value theorem. Let ξ be a parameter that varies linearly between 0 and 1 along a straight path connecting $\mathbf{U}_L$ to $\mathbf{U}_R$, so that

$$\mathbf{U}(\xi) = \mathbf{U}_L + \xi\left(\mathbf{U}_R - \mathbf{U}_L\right); \tag{25}$$

$$d\mathbf{U} = \left(\mathbf{U}_R - \mathbf{U}_L\right) d\xi \tag{26}$$

then

$$\mathbf{F}\left(\mathbf{U}_R\right) - \mathbf{F}\left(\mathbf{U}_L\right) = \int_0^1 \frac{d\mathbf{F}}{d\mathbf{U}} d\xi = \int_0^1 \mathbf{A}(\mathbf{U}(\xi)) d\xi \left(\mathbf{U}_R - \mathbf{U}_L\right) = \hat{\mathbf{A}}\left(\mathbf{U}_R - \mathbf{U}_L\right) \tag{27}$$

whence

$$\hat{\mathbf{A}} = \int_0^1 \mathbf{A}(\mathbf{U}(\xi)) d\xi \tag{28}$$

This integral represents the matrix $\hat{\mathbf{A}}$, which satisfies condition (iii) and depends on the path parameter ξ. To numerically build the Roe matrix satisfying the conditions (i) to (iv), a three point Gauss-Legendre quadrature formula is used. The Legendre polynomials $P_n(\xi)$ are calculated from the Rodrigues formula (see Abramowitz *et al.* (1988)).

Here follows the procedure for constructing the Roe matrix numerically using Gauss Legendre quadrature:

1. Choose the number of quadrature points, which corresponds to the desired degree of precision for the numerical approximation. Here the three point Gauss Legendre quadrature is used.
2. Calculate the roots ξ_i and the weights w_i of the 3rd degree Legendre polynomial. These roots and weights are given in the table 1:
3. Evaluate the matrix $\mathbf{A}(\mathbf{U}(\xi))$ at each grid point ξ_i to obtain $\mathbf{A}_i = \mathbf{A}(\mathbf{U}(\xi_i))$.

ξ_i	$1/2 - \frac{\sqrt{15}}{10}$	$1/2$	$1/2 + \frac{\sqrt{15}}{10}$
n	$\frac{5}{18}$	$\frac{8}{18}$	$\frac{5}{18}$

Table 1. Table with root values and weights for the degree 3 Legendre polynomial given by Abbott (2005)

4. Approximate the integral in Equation (28) using Gauss-Legendre quadrature:

$$\hat{\mathbf{A}} = \sum_{i=1}^{4} w_i \mathbf{A}_i. \tag{29}$$

6. The resulting sum is the approximate Roe matrix $\hat{\mathbf{A}}$.

In summary, the three point Gauss-Legendre quadrature was used to construct a Roe matrix satisfying conditions (i) to (iv) outlined above.

3.3 Approximate Riemann solver of Roe applied to a drift-flux model

To apply the Roe-type Riemann resolution algorithm for the multiphase flow in the pipeline-riser system modelled according to the mixture model with generic closure law expressed by Eq. (9), a numerical approach which involves the

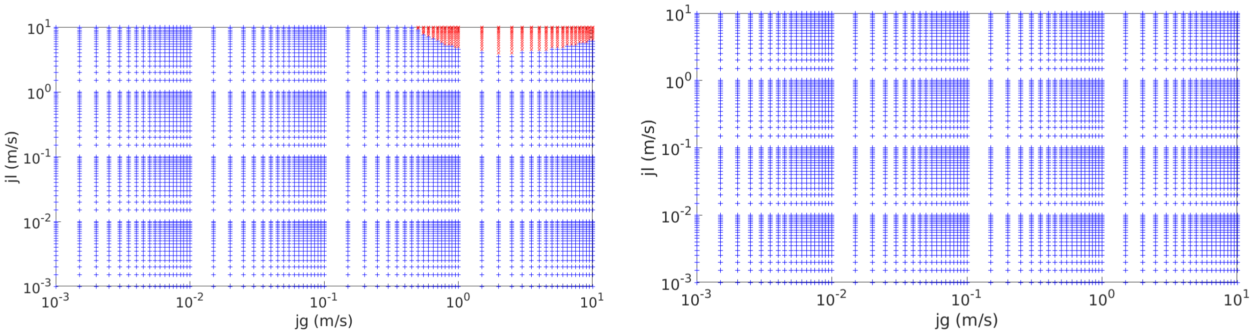
evaluation of the components of the $\mathbf{U}$ vector (conservative variables) is employed. These components, denoted u_i, are computed using a first-order upwind scheme and an explicit numerical method. The pipeline-riser geometry is discretized in $N-1$ intervals. Each grid point is the middle point of a cell. The subsequent procedures is adopted: 1. Given the initial conservative variables u_{1_i}, u_{2_i}, and u_{3_i} at each grid point, compute the corresponding primitive variables α_i, ρ_{l_i}, ρ_{g_i}, u_{l_i}, u_{g_i}, and P_i at each grid point according to the procedure described in the last paragraph of section 2.2
2. Calculate the average values of the primitive variables for the left side of the i-th cell:
- $\alpha_i^* = \frac{1}{2}(\alpha_i + \alpha_{i+1})$
- $\rho_{l,i}^* = \frac{1}{2}(\rho_{l_i} + \rho_{l_{i+1}})$
- $\rho_{g,i}^* = \frac{1}{2}(\rho_{g_i} + \rho_{g_{i+1}})$
- $u_{l,i}^* = \frac{1}{2}(u_{l_i} + u_{l_{i+1}})$
- $u_{g,i}^* = \frac{1}{2}(u_{g_i} + u_{g_{i+1}})$
- $P_i^* = \frac{1}{2}(P_i + P_{i+1})$
3. Calculate the parameters values in the mixture linear momentum equation at the i-th grid point:
- $\rho_{m,i} = (1-\alpha_i)\rho_{l,i} + \alpha_i\rho_{g,i}$
- $\sin\theta_i = \sin\theta(s_i)$
- $f_{m,i} = f_m(R_{e,m,i}; \epsilon/D)$
- $j_i = (1-\alpha_i)u_{l,i} + \alpha_i u_{g,i}$
- $R_{e,m,i} = \frac{\rho_{m,i} j_i D}{\mu_m}$
4. Compute the mixture model source term at the i-th grid point:
- $S_{1,i} = 0$
- $S_{2,i} = 0$
- $S_{3,i} = -\rho_{m,i}\sin\theta_i - 0.5\rho_{m,i} f_{m,i}|j_i|j^*$
5. Compute the Roe matrix $\hat{\mathbf{A}}$ using the Roe average values as defined in the Eq. 29
6. Compute the average conservative variables $\mathbf{U}_i^*$ at the left side of the i-th cell:
$\mathbf{U}_i^* = [(1-\alpha_i^*)\rho_{l,i}^*, \alpha_i^*\rho_{g,i}^*, (1-\alpha_i^*)\rho_{l,i}^* u_l^* + \alpha_i^*\rho_{g,i}^* u_{g,i}^*]$
7. Compute the average flux vector $\mathbf{F}_i^*$ at the left side of the i-th cell:
$\mathbf{F}_i^* = [(1-\alpha_i^*)\rho_{l,i}^* u_{l,i}^*, \alpha_i^*\rho_{g,i}^* u_{g,i}^*, (1-\alpha_i^*)\rho_{l,i}^*(u_{l,i}^*)^2 + \alpha_i^*\rho_{g,i}^*(u_{g,i}^*)^2 + P_i^*]$
8. Use the Roe Riemann solver to compute the numerical flux $\mathbf{F}_{i-1}^{Roe}$:
$\mathbf{F}_{i-1}^{Roe} = \frac{1}{2}(\mathbf{F}_{i-1}^* + \mathbf{F}_i^*) - \frac{1}{2}\hat{\mathbf{A}}(\mathbf{U}_{i-1}^*, \mathbf{U}_i^*)(\mathbf{U}_i^* - \mathbf{U}_{i-1}^*)$
9. Compute the source term vector $\mathbf{S}_i(\mathbf{U}_i)$ using the average variables $\mathbf{U}_i$.
10. Compute the time step using the Lax-Wendroff method:
$\Delta t = CFL \cdot \frac{\Delta x}{2\max(|\lambda|)}$
11. Update the solution at the interface:
$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x}(\mathbf{F}_{i-1}^{Roe} - \mathbf{S}_i(\mathbf{U}_i))$
15. Repeat steps 2-11 for all grid points in the pipeline-riser system geometry discretization.

CFL stands for the Courant-Friedrichs-Lewy condition. The time step is computed using the maximum absolute value of the larger eigenvalue in modulus among the eigenvalues of the Roe-matrices $\hat{\mathbf{A}}(\mathbf{U}_{i-1}^*, \mathbf{U}_i^*)$. This ensures stability and accuracy in the time integration. The solution at each grid is then updated according to the numerical fluxes and source terms, considering the Lax-Wendroff time discretization. The algorithm is repeated for all grid points to advance the solution in time.

4. RESULTS

A key issue for the numerical approach described above is the hyperbolicity of matrix $\mathbf{A}$. The elements of this matrix could be evaluated with the results given in the appendix A. To verify if matrix $\mathbf{A}$ has real eigenvalues a series of numerical experiments were performed. A grid over the plane $j_{l,0} \times j_{g,0}$ was considered for fixed pressure values. $j_{l,0}$ and $j_{g,0}$ are, respectively, the liquid and gas superficial velocities at the pipeline entrance and they were varied in the interval $[0.001, 10]$. The considered pressure range was from $1\ bar$ up to $4\ bar$ since the separator pressure at the experiments reported in Wordsworth *et al.* (1998) were $1\ bar$, $2\ bar$ and $3\ bar$. Once the pressure is specified, the liquid and gas densities are given by Eqs. (10) and (11) respectively and the void fraction α follows from the closure law. The adopted value for the gas and liquid sound speeds are, respectively, $300\ m/s$ and $1000\ m/s$. The reference liquid pressure $p_{l,0}$ and density $\rho_{l,0}$ are, respectively, $101.3\ kPa$ and $1000\ kg/m^3$. According to this numerical experiment, matrix $\mathbf{A}$ has always real eigenvalues in the pressure range specified above for $j_{l,0}$ and $j_{g,0}$ assuming values in the interval $[0.001, 10]$ and when the closure law is the drift-flux model. When the closure law is the local equilibrium relation used as the pipeline flow closure law, matrix $\mathbf{A}$ losses hyperbolicity for large values of the liquid and gas superficial velocities as illustrated in the hyperbolicity map in the $j_{g,0} \times j_{l,0}$ plane for pressure values of $1\ bar$ and $4\ bar$, given in Fig. 2. As pressure increases, the region in the $j_{g,0} \times j_{l,0}$ where matrix $\mathbf{A}$ losses stability decreases as can be seen in Fig. 2. To simulate the intermittent regimes SS1, SS2, SS3 and oscillatory, the region where matrix $\mathbf{A}$ losses hyperbolicity is not important, since the flow system stationary state is stable for these combination of liquid and gas superficial velocities values according to the stability map reported in Balino and Burr (2010).

Examples showing the ability of the numerical approach to capture the different intermittent regimes described in Malekzadeh *et al.* (2012) are given. Figure 25 of Balino and Burr (2010) reports stability and flow regime map for the pipeline-riser system reported in Wordsworth *et al.* (1998). The separator pressure is 2.013 bar and the values for the mass gas and liquid flow rates entering the pipeline are chosen based on Fig. 25 of Balino and Burr (2010). A Pair of the gas and liquid mass flow rate is chosen for each such flow regime reported in Fig. 25 of Balino and Burr (2010), namely, steady state, oscillatory regime, SS3, SS2 and SS1. The stationary state for each configuration mentioned above is evaluated numerically and used as initial condition for the numerical simulation. For the configuration in the stable region of Fig. 25 of Balino and Burr (2010) the simulation should not depart from the stationary state used as initial condition. For the configurations in the unstable region, the simulation results should depart from the initial condition and converge to the pattern related to the respective intermittency.



(a) 1 bar (b) 4 bar

Figure 2. Hyperbolicity maps. $(j_{l,0}, j_{g,0}) \in [0.001, 10] \times [0.001, 10]$.

5. DISCUSSION AND CONCLUSIONS

The main difficulty in the numerical approach outlined in this work is to obtain numerically the primitive variables from the conservative variables due to the different closure laws considered. For example, for large gas and liquid velocities the downward flow in the pipeline is not stratified and the drift-flux model with the distribution parameter and slip velocity given by correlation presented in Bhagwat (2014) could be used as a closure law. A transition criterion is used to decide if the downward flow in the pipe is stratified or not. If the flow is stratified, a local equilibrium relation is used, otherwise the drift-flux model with the distribution parameter and slip velocity given by the correlation presented in Bhagwat (2014) is used as closure law. Since the process to obtain the primitive variables from the conservative variables has a step which is the numerical solution of Eq. (9) for u_l or for u_g, difficulty arises when the sought value of u_l or u_g is close to the border between the stratified and non-stratified pattern for the pipeline downward flow. Numerical experiments showed that the root search algorithm jumped from values of u_l or u_g which correspond to stratified flow according to the transition criterion to values of u_l or u_g which correspond to non-stratified flow according to the transition criterion and no convergence was reached. To circumvent this situation additional information is needed, like the value of u_l or u_g from the previous time step or a neighbouring value. This approach was not pursued here, but to avoid this difficulty the pipeline downward flow was assumed as always stratified.

Another possible limitation could arise when the initial condition for the numerical simulation using the numerical approach outlined above is the stationary state for boundary conditions and system parameters located in the region of the system parameter and boundary condition space where intermittency type SS1 and SS2 exists (see Fig. 25 of Balino and Burr (2010), for example). According to Malekzadeh *et al.* (2012), at the end of the gas blowdown stage of SS1 or SS2 intermittency, the gas is not able to flush the liquid film up, and the liquid film at the riser wall falls back to the bottom of the riser. The gas still flows up, in the opposite direction of the liquid film, what corresponds to a counter-current flow, and the closure law adopted for the flow in the riser is not appropriate for such flow condition. The options are to use a more general correlation for the distribution parameter and slip velocity of the drift-flux model, like the Chexal–Lellouche correlation (see Chexal *et al.* (1977) and Hibiki (2019)) or a local equilibrium relation for annular flow.

Future work consists in implementing a more robust numerical approach to obtain the primitive variables values from the conservative variables values, what allows the use of more general closure laws with the possibility to change the type of closure law along the pipeline-riser system according to appropriate transitions criteria. This is a necessity to simulate, for example, severe slugging, since the flow in part of the riser turn out to be counter-current at the gas blowdown stage of the severe slugging intermittency where a different closure law is needed. Another line of research would be to work directly with the primitive variables instead of the conservative variables $u_j, j = 1, 2, 3$ and still use a Riemann solver approach to numerically simulate the isothermal transient multiphase flow in pipeline-riser systems.

6. REFERENCES

Abbott, P., 2005. “Tricks of the trade: Legendre-gauss quadrature”. *Mathematica*, Vol. 9, pp. 689–691.

Abramowitz, M., Stegun, I.A. and Romer, R.H., 1988. “Handbook of mathematical functions with formulas, graphs, and mathematical tables”.

Azevedo, G.R., Baliño, J. and Burr, K.P., 2015. “Linear stability analysis for severe slugging in air–water systems considering different mitigation mechanisms”. *International Journal of Multiphase Flow*, Vol. 73, pp. 238–250.

Azevedo, G.R., Baliño, J.L. and Burr, K.P., 2017. “Influence of pipeline modeling in stability analysis for severe slugging”. *Chemical Engineering Science*, Vol. 161, pp. 1–13.

Balino, J.L. and Burr, K.P., 2007. “Modeling and simulation of severe slugging in pipeline-riser systems”. *International Congress of Mechanical Engineering*, Vol. 19.

Balino, J.L. and Burr, K.P., 2010. “Modeling and simulation of severe slugging in air–water pipeline–riser systems”. *International Journal of Multiphase Flow*, Vol. 36, p. 643–660.

Bendiksen, K., Brandt, I. and Jacobsen, K., 1987. “Dynamic simulation of multiphase transportation systems”. *Multiphase Technology and Consequences for Field Development Forum*, Vol. 1.

Bhagwat, S.M., 2014. “A flow pattern independent drift flux model based void fraction correlation for a wide range of gas–liquid two phase flow”. *International Journal of Multiphase Flow*, Vol. 59, pp. 186–205.

Burr, K.P. and Baliño, J., 2015. “Evolution equation for weak hydrodynamic instabilities of two-phase flows in pipeline-riser systems”. In *Proceedings of the V Journeys in Multiphase Flows (JEM2015)*. Campinas, Brazil.

Chexal, B., Merilo, M., Maulbetsch, J., Horowitz, J., Harrison, J., Westacott, J., Peterson, C., Castner, W. and Schmidt, H., 1977. “Void fraction technology for design and analysis”. Technical report, Electric Power Research Institute.

Hibiki, T., 2019. “One-dimensional drift-flux correlations for two-phase flow in medium-size channels”. *Experimental and Computational Multiphase Flow*, Vol. 1, No. 2, pp. 85–100.

Malekzadeh, R., Henkes, R. and Mudde, R., 2012. “Severe slugging in a long pipeline–riser system: Experiments and predictions”. *International Journal of Multiphase Flow*, Vol. 46, pp. 9–12.

Masella, J.M., 1998. “Transient simulation of two-phase ¯ows in pipes”. *International Journal of Multiphase Flow*, Vol. 24, pp. 739–755.

Roe, P.L., 1981. “Approximate riemann solvers, parameter vectors, and difference schemes”. *Journal of Computational Physics*, Vol. 43, pp. 357–372.

Santim, C.G. and Rosa, E.S., 2015. “Roe-type riemann solver for gas–liquid flows using drift-flux model with an approximate form of the jacobian matrix”. *International Journal for Numerical Methods in Fluids*, Vol. 80, pp. 536–568.

Wordsworth, C., Das, I., Loh, W.L., McNulty, G., Lima, P.C. and Barbuto, F., 1998. “Multiphase flow behaviour in a catenary shaped riser, vol. i, ii & iii”. CALtec Report No.: CR 6820.

7. RESPONSIBILITY NOTICE

The following text, properly adapted to the number of authors, must be included in the last section of the paper:
The author(s) is (are) solely responsible for the printed material included in this paper.