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OPTIMAL DIMENSIONAL SYNTHESIS OF CENTRAL-LEVER STEERING LINKAGE USING NATURAL COORDINATES AS DESIGN VARIABLES

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Abstract. *Traditionally, in the dimensional synthesis of mechanisms, angles, lengths, and directed segments are mainly used as design variables and have seldom been questioned in the literature. This article uses natural coordinates as design variables in the optimal synthesis of a central-lever steering linkage, showing a particular advantage over traditional design variables. The design variables correspond to the Cartesian coordinates of the kinematic pairs to the left of the line of symmetry in the neutral position of the mechanism. In this way, the mode of assembly of the mechanism is defined, and the Ackermann condition in the neutral position is guaranteed. During the optimization process, the assembly of the mechanism may be physically impossible. To overcome this drawback, we add the residual of the constraints to the objective function multiplied by a penalty constant, ensuring that when minimizing the objective function, the constraints in the range of motion are satisfied. In addition, using the above penalty solves the problem of singular positions. The optimization problem is formulated as a multiobjective optimization problem. The primary objective function is the sum of the squared differences between the steering angle of the outer wheel and the ideal angle given by the Ackermann condition. The secondary objective function is the sum of the squared differences in the transmission angles and the right angle, equivalent to the sum of the squared pressure angles. The optimization problem was implemented in Matlab, considering three different weights of the objective functions. The steering angles are close to the Ackermann condition in the optimized mechanisms. In the cases where transmission angles were included, these angles were guaranteed to be within the established limits. Moreover, the procedure proposed in this work can be extended to more complicated mechanisms, such as the double butterfly mechanism, and the procedure could even be implemented in spatial mechanisms. The ideas presented here are not limited to steering mechanisms and can be extended to other applications.*

Keywords: *Optimal dimensional synthesis, steering linkage, natural coordinates, design variables.*

1. INTRODUCTION

The central-lever steering linkage, also known as bell-crank steering linkage, is common to tractors, ATVs, golf carts and the like, and to some trailers. It is also used in various types of passenger cars and vans Simionescu and Smith (2002). It seems that La Mancelle is believed to be the initial car that features a central-lever steering linkage. Around 50 units of this car have been produced, marking it as the first automobile to be manufactured in a series Simionescu *et al.* (2022). Duditza and Alexandru (1975) were the first to publish a report on the development of this steering mechanism. Their method is similar to Rao and Eng (1968), since they solve the synthesis equation for two specific points of precision. One point corresponds to the neutral position, while the second point represents the wheels turned approximately $2/3$ of their maximum range. The synthesis of the central-lever steering linkage using optimization is presented in Sancibrian *et al.* (2007); De-Juan *et al.* (2012). In these works the kinematic modeling is performed using natural coordinates which allows the analytical computation of the gradient of the objective function. Romero (2014) presents a procedure for the optimal synthesis of steering mechanisms by using natural coordinates and genetic algorithms. The proposed method is applied to traditional steering mechanisms, including the central lever steering linkage. Núñez and Villalba (2018) propose an optimal synthesis method using genetic algorithms. In this paper, the kinematics of the central-lever steering mechanism is solved using the bilateration method proposed by Rojas (2012).

In mechanism optimization, the design variables are usually the link lengths. However, an alternative approach is to consider the initial values of the coordinates as design variables, which can offer numerous advantages in certain applications. The main advantage is that the initial configuration of the mechanism is known. Therefore, the same configuration is maintained throughout the range of motion Lio *et al.* (2000). When the design variables are based on the lengths of the links, the initial configuration of the mechanism is not known beforehand. This lack of prior knowledge can lead to difficulties during assembly of the mechanism, as multiple assembly modes may exist in many cases. On the other hand, using natural coordinates as design variables provides an additional advantage. Natural coordinates are less susceptible to small changes during optimization. In contrast, even slight alterations in link lengths can result in substantial changes in the mechanism configuration, potentially leading to singularity problems during the optimization process Lio *et al.* (2000).

This paper presents a method for the optimal dimensional synthesis of a central-lever steering mechanism using the initial value of the natural coordinates as design variables. The rest of the paper is structured as follows. The design variables used in optimization are presented in Section 2. The kinematic analysis is carried out in Section 3. The formulation of the optimization problem is presented in Section 4. The numerical results of optimization are presented in Section 5. Finally, the conclusions are presented in Section 6.

2. DESIGN VARIABLES

We choose the design variables as the values of the natural coordinates in the neutral configuration of the steering mechanism. Since the mechanism is symmetric, the design variables correspond to points P_1^0, P_2^0, and the y-coordinate of the fixed point A as shown in Fig. 1.

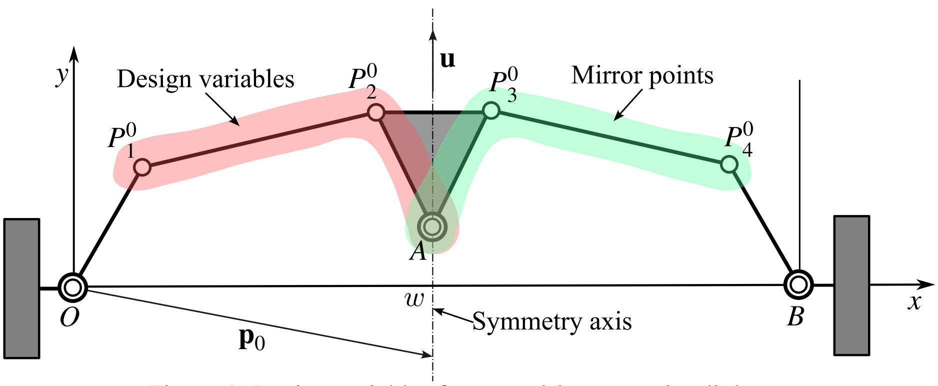

Figure 1: Design variables for central-lever steering linkage.

The x-coordinate of the point A is $x_A = w/2$, where w is the track. Then the vector of design variables can be written as

$$\mathbf{z} = \left[\begin{array}{ccccc} x_1^0 & y_1^0 & x_2^0 & y_2^0 & y_A \end{array}\right]^T. \tag{1}$$

In the kinematic analysis, it is necessary to know the mirror points at the neutral or initial position since the initial configuration is used as an initial guess in the solution of the constraint equations that define the kinematics of the mechanism. Mirror points can be computed as follows

$$\mathbf{p}_3^0 = \mathbf{p}_0 + \mathbf{M}(\mathbf{p}_0 - \mathbf{p}_2^0), \tag{2}$$

$$\mathbf{p}_4^0 = \mathbf{p}_0 + \mathbf{M}(\mathbf{p}_0 - \mathbf{p}_1^0), \tag{3}$$

where $\mathbf{M}$ is the mirror matrix, and $\mathbf{p}_0$ is a point belonging to the axis of symmetry. The mirror matrix is defined as

$$\mathbf{M} = \left[\begin{array}{cc} 1 - 2u_x^2 & -2u_x u_y \\ -2u_x u_y & 1 - 2u_y^2 \end{array}\right], \tag{4}$$

where u_x, u_y are the component of the unit vector $\mathbf{u}$ that defines the symmetry axis.

The use of the initial value of natural coordinates as design variables in the steering mechanism has the following advantages:

1. The initial configuration of the mechanism is known; therefore, the assembly of the mechanism in the neutral position is guaranteed, which allows for maintaining the same assembly mode throughout the entire range of movement of the steering mechanism.

2. In the neutral position of the steering mechanism, Ackermann's condition is precisely fulfilled due to the symmetrical initial configuration. However, when employing length-based design variables, it becomes necessary to penalize the objective function in order to approximate Ackermann's condition in the neutral position.

3. Maintaining the mechanism within the available space is significantly simpler when constraints can be directly defined at the lower and upper bounds of the design variables. Most optimization techniques make it simple to guarantee this. On the other hand, utilizing length-based design variables (constrained optimization) to constrain the space occupied by the mechanism can make the algorithm considerably more complex García-Marina *et al.* (2018).

3. KINEMATIC ANALYSIS

Figure 2 illustrates the central-lever steering linkage positioned arbitrarily, with the model represented using natural coordinates where the position of the point P_1 is determined based on the steering angle of the inner wheel δ_i.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\delta_i & -\sin\delta_i \\ \sin\delta_i & \cos\delta_i \end{bmatrix} \begin{bmatrix} x_1^0 \\ y_1^0 \end{bmatrix}. \tag{5}$$

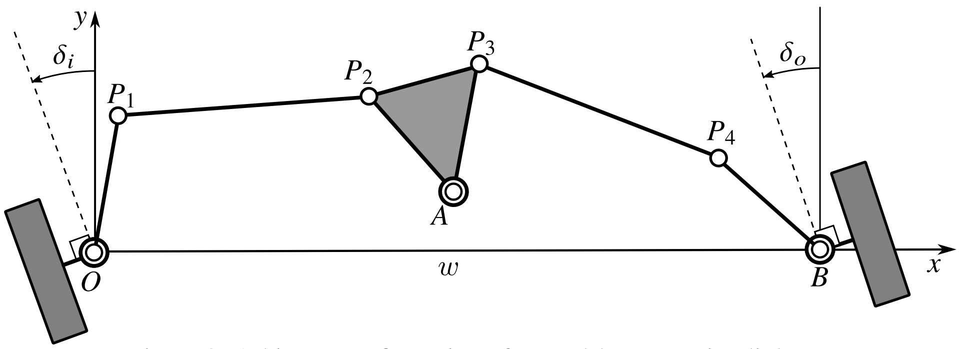

Figure 2: Arbitrary configuration of central-lever steering linkage.

The points P_2, P_3, and P_4 define the natural coordinates. Then the vector of natural coordinates can be written as

$$\mathbf{q} = \begin{bmatrix} x_2 & y_2 & x_3 & y_3 & x_4 & y_4 \end{bmatrix}^T. \tag{6}$$

The vector of kinematic constraints corresponds to three distance constraints for triangle $\triangle P_A P_2 P_3$, three distance constraints for links $P1P_2$, $P3P_4$, and $P4B$, respectively. Constant distances are formulated as functions of the initial natural coordinates. Then the vector of kinematic constraints can be written as:

$$\mathbf{\Phi}(\mathbf{q},\mathbf{z}) = \begin{bmatrix} (x_2-x_1)^2+(y_2-y_1)^2-[(x_2^0-x_1^0)^2+(y_2^0-y_1^0)^2] \\ (x_3-x_2)^2+(y_3-y_2)^2-[(x_3^0-x_2^0)^2+(y_3^0-y_2^0)^2] \\ (x_2-x_A)^2+(y_2-y_A)^2-[(x_2^0-x_A)^2+(y_2^0-y_A)^2] \\ (x_3-x_A)^2+(y_3-y_A)^2-[(x_2^0-x_A)^2+(y_2^0-y_A)^2] \\ (x_4-x_3)^2+(y_4-y_3)^2-[(x_2^0-x_1^0)^2+(y_2^0-y_1^0)^2] \\ (x_4-x_B)^2+(y_4-y_B)^2-[(x_1^0)^2+(y_1^0)^2] \end{bmatrix} = \mathbf{0}. \tag{7}$$

3.1 Kinematic position analysis

The most direct way to solve the position problem of Eq. (7) is to use the Newton-Raphson method. But in the case that in some positions the assembly of the mechanism is not physically possible, the Newton-Raphson method does not provide a consistent error measure to penalize the objective function in the optimal synthesis. The above drawbacks can be resolved if the position problem is solved by some suitable optimization method as shown in Jean-François Collard (2007). If an optimization method based on Newton's method is used, then the computational cost is not a problem, since, in general, whether the starting point is sufficiently close to the minimum, the method will converge quadratically towards the solution Frandsen *et al.* (2004).

We can formulate the kinematics of position as a least-square problem

$$\underset{\mathbf{q}}{\text{minimize}} \quad \frac{1}{2}\mathbf{\Phi}(\mathbf{q},\mathbf{z})^T\mathbf{\Phi}(\mathbf{q},\mathbf{z}). \tag{8}$$

The Newton iteration step $\Delta\mathbf{q}$ is computed by solving a linear equation $\mathbf{H}(\mathbf{q},\mathbf{z})\Delta\mathbf{q} = -\mathbf{\Phi}_{\mathbf{q}}^T\mathbf{\Phi}$; where $\mathbf{H}$ is the Hessian matrix and $\mathbf{\Phi}_{\mathbf{q}}$ is the Jacobian of the constraint vector $\mathbf{\Phi}$ respect to the vector of natural coordinates $\mathbf{q}$. The Jacobian and Hessian are easily calculated since the kinematic constraints are all quadratic.

The initial position of the mechanism is given by the initial value of the natural coordinates

$$\mathbf{q}_0 = \left[\begin{array}{cccccc} x_2^0 & y_2^0 & x_3^0 & y_3^0 & x_4^0 & y_4^0 \end{array}\right]^T, \tag{9}$$

and to determine the subsequent position, a small increment of the input direction angle $\delta_i$is made, and the new position is determined by solving Eq. (8) by the Newton method using $\mathbf{q}_0$ as the initial guess. In this way, we continue using the value of the immediately previous coordinates as initial guess to determine the new position until we reach the full range of motion of the mechanism.

3.2 Synthesis error

The primary synthesis error is defined as the difference between the outer wheel steering angle and the ideal angle given by the Ackerman condition.

$$\varepsilon = (\delta_o - \delta_{oA}). \tag{10}$$

Where δ_o is the generated angle turned by the output wheel and δ_{oA} is the angle according to the Ackermann condition

$$\delta_{oA} = \cot^{-1}\left(\frac{w}{l} + \cot\delta_i\right), \tag{11}$$

where l is the wheelbase of the vehicle.

The secondary synthesis error is defined as the differences between the transmission angles and the right angle, which is equivalent to the pressure angles. The secondary synthesis error can be written as follows

$$\boldsymbol{\alpha} = \left[\begin{array}{c} (\mu_1 - \frac{\pi}{2}) \\ (\mu_4 - \frac{\pi}{2}) \end{array}\right], \tag{12}$$

where μ_1 and μ_4 are the transmission angles, see Fig. 3

$$\mu_1 = \arccos\left(\frac{x_1(x_1 - x_2) + y_1(y_1 - y_2)}{\sqrt{[(x_1^0)^2 + (y_1^0)^2][(x_2^0 - x_1^0)^2 + (y_2^0 - y_1^0)^2]}}\right), \tag{13}$$

$$\mu_4 = \arccos\left(\frac{(w - x_4)(x_3 - x_4) + y_4(y_4 - y_3)}{\sqrt{[(x_1^0)^2 + (y_1^0)^2][(x_2^0 - x_1^0)^2 + (y_2^0 - y_1^0)^2]}}\right). \tag{14}$$

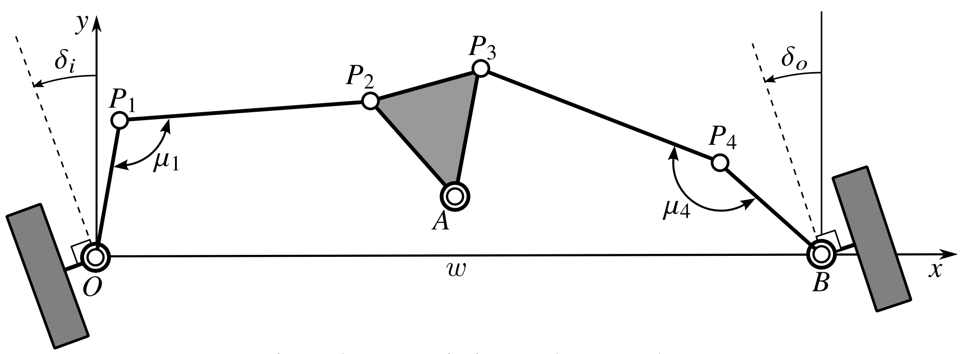

Figure 3: Transmission angles μ_1 and μ_4.

The optimum value of the transmission angle is 90°. In fact, the value of the angle itself is not as significant as the deviation from its optimum value, since, for transmission to be constant, the junction transmission must remain constant, i.e., the connection between the elements would have to be rigid, a characteristic that is physically difficult to achieve due to the variation of angles in the kinematic pairs. However, there is also no unanimity when it comes to establishing acceptable limits. Some studies, compiled by Balli and Chand (2002), suggest different upper and lower limits, such as upper and lower limits, such as 45°-135°.

4. OPTIMIZATION

In principle, the objective function can be defined as the squared sums of the synthesis errors

$$\nu \sum_{i=1}^{n} \varepsilon_i^2 + (1-\nu) \sum_{i=1}^{n} \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_i \ , \tag{15}$$

where ν is a weight variable that can take values from 0 to 1. A value of $\nu = 1$ means that only the Ackermann condition is taken into account in the optimization, and a value of $\nu = 0$ means that only transmission angles are taken into account. However, during the optimization process, there may be values of the design variables where it is not physically possible to assemble the mechanism. Although the value of the synthesis error is calculated, it has no physical meaning. To overcome this drawback, we can penalize the objective function with the sum of the minimum values of the objective function in the position problem Jean-François Collard (2007). Then the optimization problem can be defined as follows

$$\begin{aligned} &\underset{\mathbf{z}}{\text{minimize}} && g(\mathbf{z}) = \nu \sum_{i=1}^{n} \varepsilon_i^2 + (1-\nu) \sum_{i=1}^{n} \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_i + \rho \sum_{i=1}^{n} f_i(\mathbf{q}, \mathbf{z}) \\ &\text{subject to} && \underline{\mathbf{z}} \leq \mathbf{z} \leq \bar{\mathbf{z}} \\ &\text{while solving} && f_i(\mathbf{q}, \mathbf{z}) = \min_{\mathbf{q}} \frac{1}{2} \boldsymbol{\Phi}(\mathbf{q}, \mathbf{z})^T \boldsymbol{\Phi}(\mathbf{q}, \mathbf{z}) \end{aligned} \tag{16}$$

where ρ is a weighted factor, $\underline{\mathbf{z}}$ and $\bar{\mathbf{z}}$ are the lower and upper bounds of design variables respectively.

The main advantage of this formulation is that the objective function is always continuous and always differentiable in the design space. This property allows us to use gradient-based algorithms, which are local methods but often more efficient than direct searches or stochastic algorithms proposed by Jean-François Collard (2007).

5. NUMERICAL RESULTS

Let δ_i vary from -27° to 40°, δ_o varying correspondingly from -40^o to 27°, therefore, the presicion points n are 68, taking into account that the variation between angles was 1°, and all 68 points uniformly spaced along the δ_i axis. Furthermore, we set $w = 1.5$ m, and set $l = 2$ m as in De-Juan *et al.* (2012). The interior-point Algorithm of the Matlab toolbox was used to solve the optimization problem. The optimization parameters used in optimization are shown in Tab. 1, and the optimal design variables for $\nu = 1$, $\nu = 0.99$, and $\nu = 0.98$ are shown in Tab. 2.

Table 1: Optimization parameters.

Design variables	**Lower bounds**	**Upper bounds**	**Initial design**	**Units**
x_1^0	-0.1	0.2	-0.05	[m]
y_1^0	-0.21	0.21	0.2	[m]
x_2^0	0.45	1.05	0.65	[m]
y_2^0	-0.21	0.21	-0.2	[m]
y_A	-0.21	0.21	0.1	[m]

Table 2: Optimal central-lever steering linkage for $\nu = 1$, $\nu = 0.99$, and $\nu = 0.98$.

Variables	**Optimal** $\nu = 1$	**Optimal** $\nu = 0.99$	**Optimal** $\nu = 0.98$	**Units**
x_1^0	0.199919404796439	0.120114138346766	0.104053369666822	[m]
y_1^0	0.209921867719301	0.205386225093828	0.205804414020592	[m]
x_2^0	0.735751052876773	0.749820138454908	0.750662734866057	[m]
y_2^0	-0.209417857418622	-0.190898778315802	-0.178687610853892	[m]
y_A	0.0063752534567032	0.0344023418580578	0.044140972595826	[m]

Figure 4(a) illustrates the initial steering mechanism used in the optimization. While Figs. 4(b) to 4(d) illustrate the optimal steering mechanism for $\nu = 1$, $\nu = 0.99$, and $\nu = 0.98$, respectively. Figure 5 shows the convergence of the objective function in which a value of 1.393614×10^{-3} was reached in 107 iterations for $\nu = 1$, the objective function value of 7.542790×10^{-1} in 42 iterations for $\nu = 0.99$, and the objective function value of 1.15055 in 38 iterations for $\nu = 0.98$.

Figure 6 shows the structural error for the initial and optimal mechanisms for $\nu = 1$, $\nu = 0.99$, and $\nu = 0.98$, respectively. The maximum absolute steering error is 18.36° for the initial mechanism, 0.016° for the optimal solution

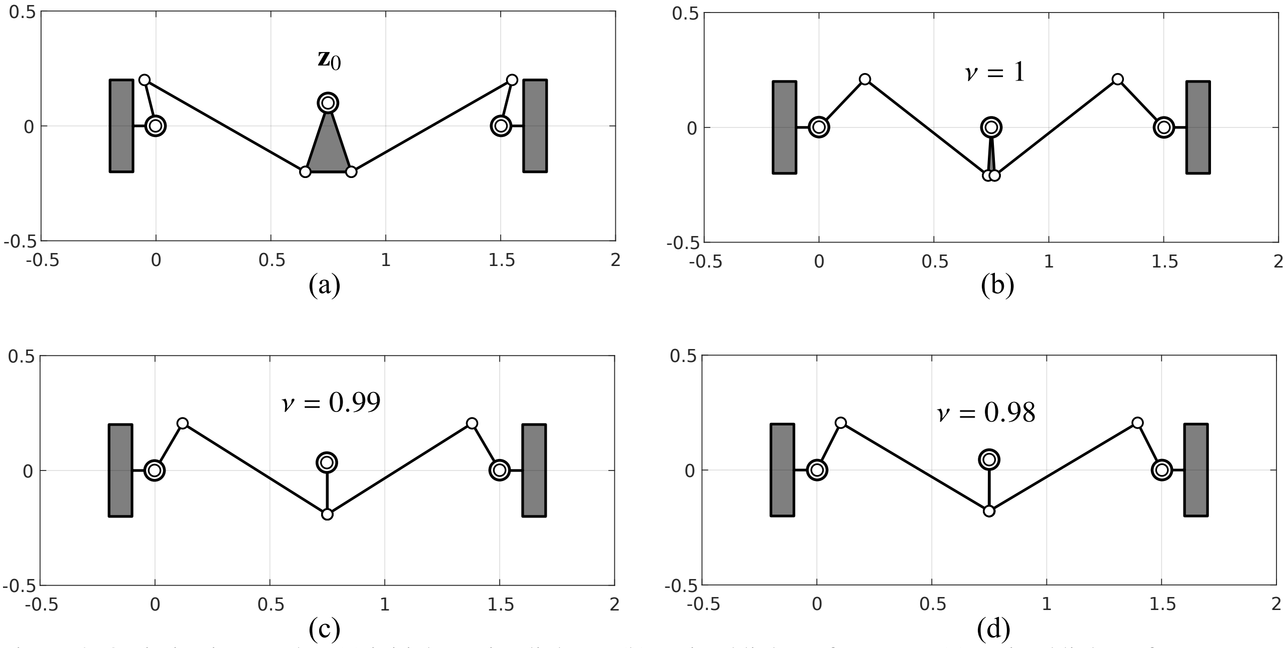

Figure 4: Optimization results: (a) initial steering linkage, (b) optimal linkage for $\nu = 1$, (c) optimal linkage for $\nu = 0.99$, and (d) optimal linkage for $\nu = 0.98$.

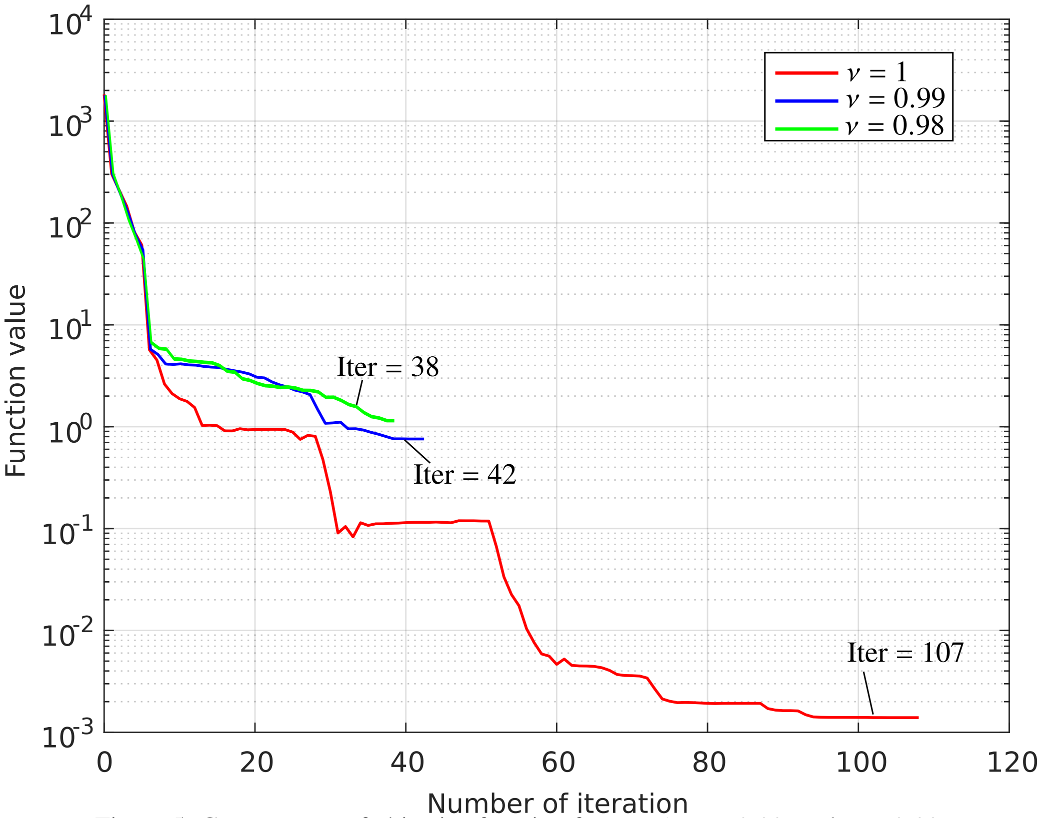

Figure 5: Convergence of objective function for $\nu = 1$, $\nu = 0.99$, and $\nu = 0.98$.

with $\nu = 1$, the steering error of $0.284°$ for the optimal solution with $\nu = 0.99$, and $0.5°$ for the optimal solution with $\nu = 0.98$. Figure 7 shows the transmission angles for the initial steering mechanism and for the optimal mechanisms. Note that the transmission angles of the initial mechanism are quite deficient. The optimal mechanism with $\nu = 1$ although it approximates the Ackermann condition quite well, the transmission angles are a bit out of bounds for some positions. The optimal solutions with $\nu = 0.99$ and $\nu = 0.98$ are both reasonable solutions, and it is up to the designer which of the design requirements to sacrifice.

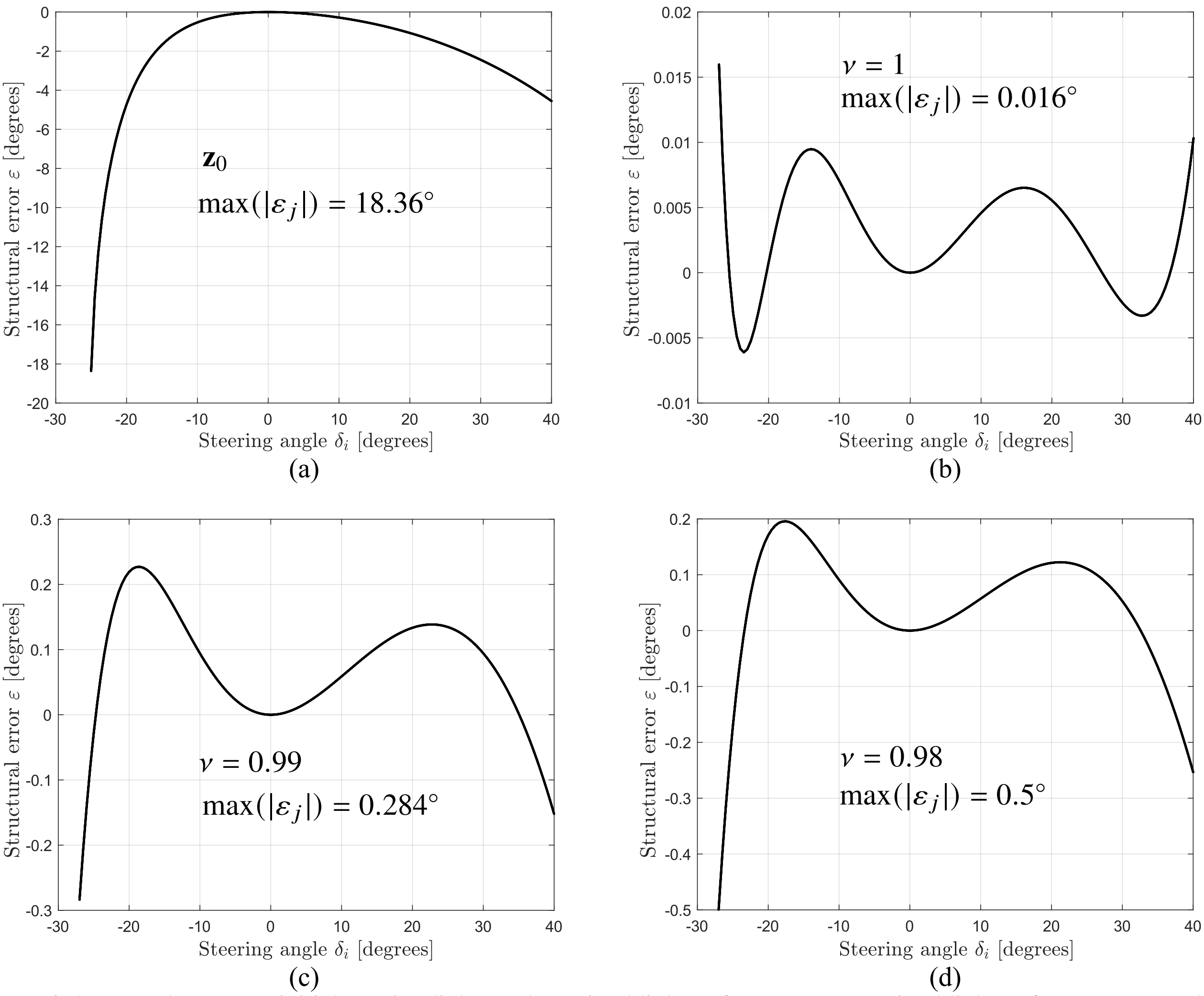

Figure 6: Structural error: (a) initial steering linkage, (b) optimal linkage for $\nu = 1$, (c) optimal linkage for $\nu = 0.99$ and (d) optimal linkage for $\nu = 0.98$.

6. CONCLUSION

This paper presents a procedure for the optimal dimensional synthesis of a central-lever steering linkage. In formulating the optimization problem, the values of the natural coordinates in the initial configuration are used as design variables. This approach offers several advantages over length-based design variables, including knowledge of the initial configuration of the mechanism, maintenance of the same configuration throughout the entire range of motion, exact compliance of the Ackermann condition in the neutral position, and the design variables allow direct restriction of the link size to the space available for the steering mechanism in the vehicle.

The optimization problem is formulated as a multi-objective optimization problem. The primary objective function corresponds to the squared differences between the steering angle of the outer wheel and the ideal angle prescribed by the Ackerman condition. The secondary objective function is the sum of the squared differences between the transmission angles and the right angles, equivalent to the sum squared of the pressure angles. The optimization problem is implemented in Matlab, with consideration of three different weights assigned to the objective functions. The optimized mechanisms exhibit steering angles that closely adhere to the Ackermann condition. When transmission angles are considered, they are guaranteed to remain within the specified limits. Furthermore, the proposed procedure in this study can be extended to more complex mechanisms, such as the double butterfly mechanism, and even applied to spatial mechanisms. The ideas presented in this work are not limited to steering mechanisms and can be extrapolated to other applications.

7. ACKNOWLEDGEMENTS

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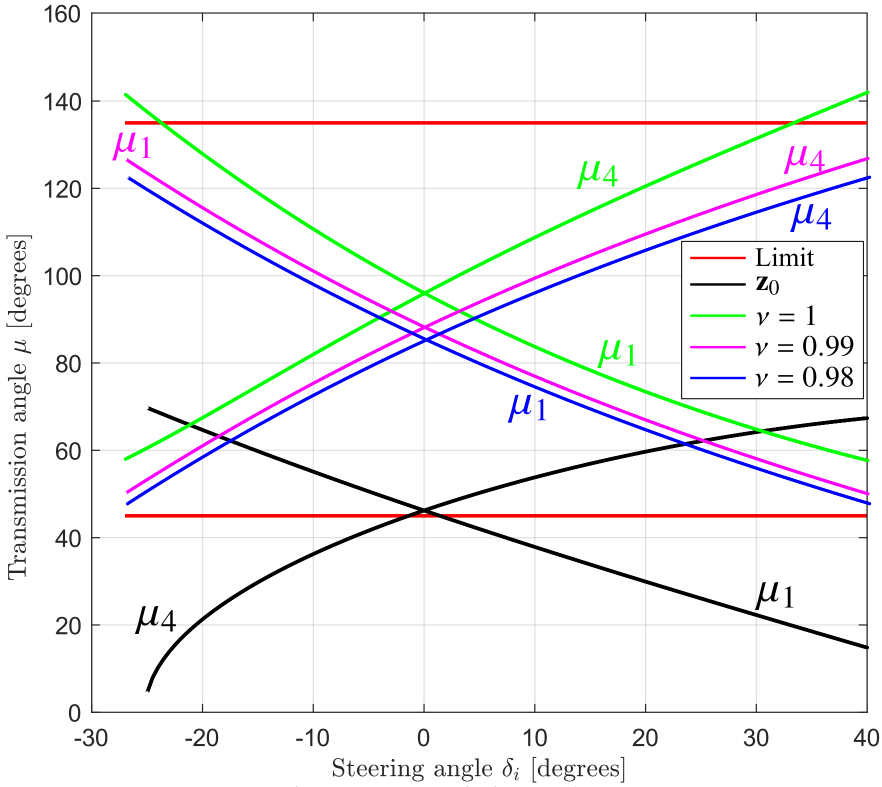

Figure 7: Transmission angles.

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