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COB-2023-0722 (XXXX is the identification number of the final paper) The Use of Cells for Accurate Body Forces Modelling in the Isogeometric Boundary Element Method (IGABEM): Comparison with the Galerkin Vector Approach and Insights for Nonlinear Problems.

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Abstract. *The Boundary Element Method (BEM) is a commonly used numerical technique for solving elastostatic problems. In most physical and engineering problems the inclusion of body forces in the analysis is of vital importance. In the present work, two different approaches for modelling body forces are presented: the Galerkin Vector Approach and the domain discretization method via cells. The first approach involves transforming the integral of the body force over the domain into integrals over the boundary. The second method is based on the concept of subdividing the domain into a set of cells. Presumably, the cell-based method is considered time-consuming, however, it is considered one of the most reliable methods for analyzing highly nonlinear problems. Furthermore, in problems where the domain discretization cannot be approximated by a boundary integral, such as in problems where the internal variables assume a significant role (plasticity or damage, for example), the use of cells becomes indispensable. Thus, a numerical example via the Isogeometric Boundary Element Method (IGABEM) of both methodologies is presented. Linear quadrilateral cells are proposed. Practical guidance on the implementation of domain discretization is provided. In addition, a brief idea of how to apply the cells method to future nonlinear problems is given.*

Keywords: *BEM, body forces, galerkin vector, cells.*

1. INTRODUCTION

Elastostatic problems are of concern in many physical and engineering applications. The description of such problems might be performed through differential or integral equations. For numerically solving the integral equations, the Boundary Element Method (BEM) is widely employed. In the BEM, the numerical discretization is conducted at reduced dimensionality of the problem, i.e., for problems in two spatial dimensions, for example, the discretization is on the boundary only. This reduced dimension enables smaller linear systems, less computer memory requirements, and consequently, more efficient computation (Cheng and Cheng, 2005).

Therefore, the resolution of boundary value problems for domains with complex geometry is simpler in terms of mesh generation (Sauter and Schwab, 2010). Besides, in order to directly link geometry generator software and the numerical analysis, the isogeometric analysis (IGA) uses the same basis functions that are used in CAD software (Simpson *et al.*, 2012).

Concerning specifically the modelling of volume forces acting on a body, in the BEM framework there are distinct approaches for taking this effect into consideration, such as the Galerkin Vector Approach and the domain discretization method via cells. The Galerkin vector strategy involves transforming the integral of the body force over the domain into integrals over the boundary. On the other hand, discretization via cells is based on the concept of subdividing the domain into a set of cells. Although this approach is presumed to be time-consuming and not considered very attractive because of the requirement of generating a mesh, it is considered the most reliable method for analyzing nonlinear problems (Aliabadi, 2002; Botta *et al.*, 2005).

By modeling the influence of the body forces on the mechanical response of a structure, the present study gives practical guidance to assist in the implementation of the domain discretization technique in BEM. Linear quadrilateral cells formulation is herein proposed. The outcomes are compared with analytical solutions and with the results given by the Galerkin vector strategy, which is used by the BEM community to consider the volume effects. A glimpse into the future application of this method to address nonlinear problems is outlined.

2. THE BOUNDARY ELEMENT METHOD (BEM)

Physical laws representing real problems can be described in terms of integral equations. As analytical solutions are available for only a few range of problems, numerical approximations for solving the boundary value problems are required. The most general and most commonly used method for solving integral equations is the Boundary Element Method (BEM). The boundary is discretized into a finite number of elements and the governing equations are formulated in terms of the boundary variables. The displacements and stress fields are obtained through the solution of the integral equation that describes the problem. In the following, the numerical approximation for the boundary integral equations of linear elasticity is presented.

2.1 Boundary integral equations of linear elasticity

One approach to derive the formulation of the elasticity problem is by applying Betti's reciprocity theorem for two self-equilibrated states (u_i, t_i, b_i) and (u_i^*, t_i^*, b_i^*), as follows:

$$\int_\Gamma t_i u_i^* d\Gamma + \int_\Omega b_i u_i^* d\Omega = \int_\Gamma t_i^* u_i d\Gamma + \int_\Omega b_i^* u_i d\Omega \tag{1}$$

where u_i and u_i^* are the displacements; t_i and t_i^* are tractions; and b_i and b_i^* are the body forces. The superscript $*$ stands for the fundamental solution of each field. The fundamental solution consider an infinite domain under the action of a singular load. Hence, the body force b_i^* corresponds to a point force in an infinite sheet represented by the Dirac delta function $\Delta(\mathbf{x}', \mathbf{x})$, as shown below:

$$b_i^* = \Delta(\mathbf{x}', \mathbf{x})\, e_i \tag{2}$$

in which $\mathbf{x}'$ is the point where the force is applied (the source point); x is the field point and e_i is a unit positive force vector in x_i direction. For the two-dimensional problems herein studied, e_i is a force per unit thickness. By applying the Delta Dirac sifting function's property in Eq. (2), the last integral shown in Eq. (1) becomes:

$$\int_\Omega b_i^* u_i d\Omega = \int_\Omega \Delta(\mathbf{x} - \mathbf{x}')\, e_i u_i d\Omega = u_i(\mathbf{x}')\, e_i \tag{3}$$

Moreover, the displacement and the traction fields of where the point load is applied are given by:

$$u_i^* = U_{ij}^*(\mathbf{x}', \mathbf{x})\, e_j \tag{4}$$

$$t_i^* = T_{ij}^*(\mathbf{x}', \mathbf{x})\, e_j \tag{5}$$

where U_{ij}^* and T_{ij}^* are, respectively, the displacement and the traction fundamental solutions.

By replacing Eqs. (3-5) into Eq. (1), one obtains:

$$u_i(\mathbf{x}') + \int_\Gamma T_{ij}^*(\mathbf{x}', \mathbf{x})\, u_j(\mathbf{x}) d\Gamma = \int_\Gamma U_{ij}^*(\mathbf{x}', \mathbf{x})\, t_j(\mathbf{x}) d\Gamma + \int_\Omega U_{ij}^*(\mathbf{x}', \mathbf{x})\, b_j(\mathbf{x}) d\Omega \tag{6}$$

The above equation is the well-known Somigliana identity and it is valid for any source point located inside the domain Ω. In order to resolve the boundary value problem, a limit analysis is carried out to bring the source point onto the boundary, providing the following:

$$C_{ij} u_j(\mathbf{x}') + \int_\Gamma T_{ij}^*(\mathbf{x}', \mathbf{x})\, u_j(\mathbf{x}) d\Gamma = \int_\Gamma U_{ij}^*(\mathbf{x}', \mathbf{x})\, t_j(\mathbf{x}) d\Gamma + \int_\Omega U_{ij}^*(\mathbf{x}', \mathbf{x})\, b_j(\mathbf{x}) d\Omega \tag{7}$$

where C_{ij} corresponds to the free term, assuming the value of 0 for points out of the domain; identity I for points inside the domain and $1/2I$ for points located in a smooth boundary.

For obtaining the stresses field, the Somigliana identity is derived with respect to the source point $\mathbf{x}'$. By taking into account the strain-displacement relation and Hooke's law in the derived expression, the subsequent integral equation is established:

$$\sigma_{ij}(\mathbf{x}') + \int_\Gamma S_{kij}^*(\mathbf{x}', \mathbf{x})\, u_k(\mathbf{x}) d\Gamma = \int_\Gamma D_{kij}^*(\mathbf{x}', \mathbf{x})\, t_k(\mathbf{x}) d\Gamma + \int_\Omega D_{kij}^*(\mathbf{x}', \mathbf{x})\, b_k(\mathbf{x}) d\Omega \tag{8}$$

For the sake of simplicity, all the fundamental kernels for both boundary and internal values can be found in basic BEM references, such as in Brebbia and Dominguez (1994) , Aliabadi (2002) , Telles (2012) and Brebbia and Walker (2016).

2.2 Numerical discretization

A boundary discretization for a finite number of elements is carried out, where for each element e, the geometry, the displacements, and the tractions fields are described as a function of the nodal values, as shown:

$$\mathbf{x}^e = \sum_{\alpha=1}^{m} N_\alpha(\xi)\mathbf{x}_\alpha \tag{9}$$

$$\mathbf{u}^e = \sum_{\alpha=1}^{m} N_\alpha(\xi)\mathbf{u}_\alpha \tag{10}$$

$$\mathbf{t}^e = \sum_{\alpha=1}^{m} N_\alpha(\xi)\mathbf{t}_\alpha \tag{11}$$

In the above equations, the shape functions N_α are defined in terms of non-dimensional coordinates ξ $(-1 \leq \xi \leq 1)$ in the parametric space. The terms $\mathbf{x}_\alpha$, $\mathbf{u}_\alpha$ and $\mathbf{t}_\alpha$ are the function values at node α. In the present study, the Isogeometric Analysis (IGA) framework is employed, where the shape functions are defined using Non-Uniform Rational B-splines (NURBS). In a general form, these functions are derived by projecting the B-splines from the R^{d+1} space to the R^d space. Hence, the B-spline functions $N_{a,p}$ of degree p at the parametric coordinate ξ are defined recursively, as follows for $p = 0$ and with $1 \leq a \leq n$:

$$N_{a,0} = \begin{cases} 1 & \text{if } \xi_a \xi < \xi_{a+1} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

For $p = 1, 2, 3...$, the recursive Cox-de Boor formula (Cox, 1972; De Boor, 1972) is used:

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \tag{13}$$

Finally, the projection of the B-splines is performed by an additional coordinate denoted as weight w_a, as given below:

$$\mathbf{c}(\xi) = \sum_{a=1}^{n} R_{a,p}(\xi)\mathbf{p}_a \tag{14}$$

where

$$R_{a,p} = \frac{N_{a,p}(\xi) w_a}{\sum_{\hat{a}=1}^{m} N_{\hat{a},p}(\xi) w_{\hat{a}}} \tag{15}$$

Therefore, for each integration point, the kernels $\mathbf{T}^*(\mathbf{x'},\mathbf{x})$ and $\mathbf{U}^*(\mathbf{x'},\mathbf{x})$ are evaluated in a matrix form in the following manner:

$$\boldsymbol{T}^* = \begin{bmatrix} t_{11}^* & t_{12}^* \\ t_{21}^* & t_{22}^* \end{bmatrix} \tag{16}$$

$$\boldsymbol{U}^* = \begin{bmatrix} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{bmatrix} \tag{17}$$

where the subindices i and j of t_{ij} and u_{ij} are the resulting tractions and displacements, respectively, in j direction due to the application of a unitary force in i direction.

A set of matrices for solving Eq. (7) is assembled relating all displacement and all the tractions components, by the following:

$$\mathbf{Hu} = \mathbf{Gt} + \mathbf{Db} \tag{18}$$

in which, **H** is a square matrix that includes all the integrals of the kernel $\mathbf{T}^*(\mathbf{x'},\mathbf{x})$, while **G** represents the integrals of the kernel $\mathbf{U}^*(\mathbf{x'},\mathbf{x})$ and **D** from the body force integral. The vectors **u**, **t**, and **b** contains the nodal displacements, tractions, and volume effects, respectively.

There are several methods developed for the evaluation of the last term of Eq. (18). In this study, two specific methods for evaluating such integral are considered: the Galerkin vector approach and domain discretization using cells. A comprehensive explanation of both methods is provided in the subsequent sections.

Thus, after prescribing the boundary conditions, Eq. (18) is rearranged according to:

$$\mathbf{Ax} = \mathbf{f} \tag{19}$$

where **x** corresponds to the vector of unknown degrees of freedom; **A** is a full and non-symmetric matrix; and **f** is an independent vector.

3. EVALUATION OF THE DOMAIN INTEGRAL

The domain forces, also known as body forces, act directly on the surface of an object under analysis and are distributed throughout the entire volume of the object. Examples of such forces include gravitational and centrifugal ones. Obtaining precise solutions that take into account the effect of such forces requires the precise modeling of these entities. According to Aliabadi (2002), in a BEM analysis, the evaluation of the body force integral given in Eq. (7) can be performed through several distinct methods. These methods include:

- Dual reciprocity;
- Multiple reciprocity;
- Domain discretization;
- Galerkin vector.

As already mentioned in this text, the last two methods are herein considered and their approach is detailed in the following sections.

3.1 Volume effects via the Galerkin vector strategy

The presence of forces acting on the body domain can be represented by definite integrals over the problem domain. For the particular case of gravitational forces, b is constant, so $b = \rho(\mathbf{x})g(\mathbf{x})$, where $\rho(\mathbf{x})$ and $g(\mathbf{x})$ are the density of the material and the gravity's acceleration, respectively. The Galerkin vector can be used to transform the domain body force into boundary ones. The displacement fundamental solution $\mathbf{U}^*(\mathbf{x'},\mathbf{x})$ is related to the Galerkin vector according to:

$$U^*_{ij}(\mathbf{x}',\mathbf{x}) = \phi_{ij,kk}(\mathbf{x}',\mathbf{x}) - \frac{1}{2(1-v)}\phi_{ik,jk}(\mathbf{x}',\mathbf{x}) \tag{20}$$

Thus, the domain integral can be written as:

$$B_i = \int_\Omega U^*_{ij}(\mathbf{x}',\mathbf{x})\, b_j(x) d\Omega = \int_\Omega \left[\phi_{ij,kk}(\mathbf{x}',\mathbf{x}) - \frac{1}{2(1-v)}\phi_{ik,jk}(\mathbf{x}',\mathbf{x})\right] b_j d\Omega \tag{21}$$

By applying the Divergence theorem and taking the constant term b_j out of the integral, one obtains:

$$B_i = b_j \int_\Gamma \left\{\phi_{ij,k} - \frac{1}{2(1-\nu)}\phi_{ik,j}\right\} n_k d\Gamma = \int_\Gamma P_i d\Gamma \tag{22}$$

where the solution of the Galerkin vector is given by

$$\phi_{ki} = \frac{-1}{8\pi G} r^2 \ln(r) \delta_{ki} \tag{23}$$

In the above relation, the δ_{ki} and r are the Kronecker Delta and the distance from the source point **x'** to field one **x**, respectively. Thus, after substituting (20) into (22) and after several mathematical manipulations, one obtains:

$$P_i = \frac{r}{8\pi\mu}\left\{\left[2\ln\frac{1}{r} - 1\right]\left(b_i n_k r_{,k} - \frac{1}{2(1-v)} b_k r_{,k} n_i\right)\right\} \tag{24}$$

Notwithstanding, the domain term must be included in the internal stress values, that is:

$$\int_\Omega D_{kij} b_k d\Omega = \int_\Gamma D_{ij} d\Gamma \tag{25}$$

where

$$\begin{aligned} D_{ij} =& \frac{1}{8\pi}\left\{2n_m r_{,m}\left(b_i r_{,j} + b_j r_{,i}\right)\right. \\ &+ \frac{1}{1-v}\left[v\delta_{ij}\left(2n_m r_m b_k r_{,k} + \left[1 - 2\ln\frac{1}{r}\right] b_m n_m\right) - b_m r_{,m}\left(n_i r_j + n_j r_i\right)\right. \\ &\left.\left.+ \frac{1-2v}{2}\left(1 - 2\ln\frac{1}{r}\right)\left(b_i n_j + b_j n_i\right)\right]\right\} \end{aligned} \tag{26}$$

Therefore, the domain integral that contains the body force effect is replaced by a boundary one in this strategy.

3.2 Domain effects via cells

In this method, part of the domain where the body forces are applied is divided into a number of cells. The volume integral is estimated by summing the contributions of M cells of domain Ω_m throughout the domain Ω. In other words, the body force integral becomes:

$$\int_\Omega U_{ij}^*\left(\mathbf{x}', \mathbf{x}\right) b_j(\mathbf{x}) d\Omega = \sum_{m=1}^{M} \int_{\Omega_m} U_{ij}^*\left(\mathbf{x}', \mathbf{x}\right) b_j(\mathbf{X}) d\Omega_m \tag{27}$$

The above integral is generally well behaved and the numerical Gauss quadrature is used to carry out the integration.

For generating the cell elements, the basic idea coming from the Finite Element Method (FEM) community is addressed. Thus, linear quadrilateral cells are herein proposed, as shown in Fig. (1). The bivariate shape functions for the linear elements are defined through the Lagrangian polynomials, according to the following:

$$N_1(\xi,\eta) = \frac{1-\xi}{2}\frac{1-\eta}{2} \tag{28}$$

$$N_2(\xi,\eta) = \frac{1+\xi}{2}\frac{1-\eta}{2} \tag{29}$$

$$N_3(\xi,\eta) = \frac{1+\xi}{2}\frac{1+\eta}{2} \tag{30}$$

$$N_4(\xi,\eta) = \frac{1-\xi}{2}\frac{1+\eta}{2} \tag{31}$$

By the first-order derivatives of the relations shown (Eqs. 28-31), the Jacobian of transformation for the bivariate space can be obtained through the following:

$$J = J_\xi J_\eta \tag{32}$$

where $J_\xi = \partial\mathbf{x}/\xi$ and $J_\eta = \partial\mathbf{x}/\eta$.

Once the shape functions are known, the geometry of each cell can be approximated by the following interpolation:

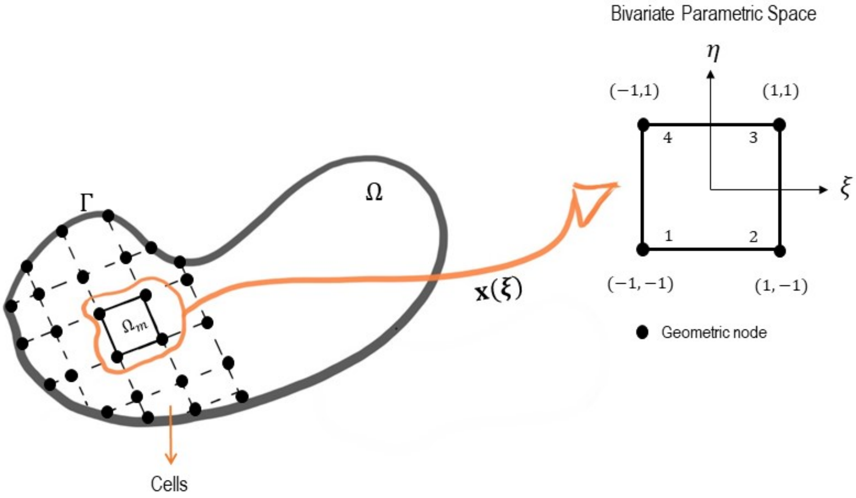


Figure 1. Domain discretization into cells and the corresponding coordinates mapping to the bivariate parametric space.

$$\mathbf{x} = \sum_{i=1}^{4} N_i(\xi, \eta)\mathbf{x}_i \tag{33}$$

where $\mathbf{x}_i$ refers to the x_i and y_i coordinates for each local node of the quadrilateral element.

Similarly, the gravitational force field is approximated by:

$$\mathbf{b} = \sum_{i=1}^{4} N_i(\xi, \eta)\mathbf{b}_i \tag{34}$$

in which $\mathbf{b}_i$ refers to the nodal values.

4. NUMERICAL EXAMPLE

This section provides an illustrative numerical example for which the analytical solution is known. The methodologies discussed in this study have been implemented in a FORTRAN 90 code. The purpose is to validate the implemented cells and to contrast the outcomes with the Galerkin vector approach. This second method is usually preferred to treat domain terms once no domain discretization is required. However, having in hands a form of validation for the formulation of the cell is crucial for its utilization in non-linear problems. Therefore, in section 5, a BEM formulation where the use of cells is indispensable is presented.

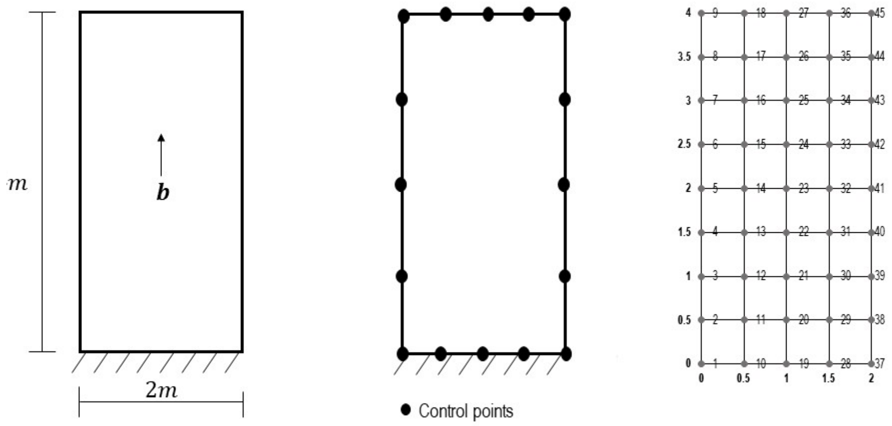
4.1 Rectangular strip subjected to self-weight

The example given in Pérez-Gavilán and Aliabadi (2001) is herein discussed, as presented in Fig. (2a). Firstly, the boundary displacements and the stress field for points inside the domain are obtained via cells. The results are compared with the analytical solution of the problem. The particular solution when $\nu = 0.0$ leads to:

$$\begin{aligned} &u_1 = 0 && u_2 = \frac{\rho g}{2E} x_2^2 \\ &\sigma_{11} = \sigma_{12} = 0 && \sigma_{22} = \rho g x_2 \end{aligned} \tag{35}$$

where **x** refers to the coordinates, ρ is the weight density, and g is the gravitational acceleration.

The adopted IGABEM discretization is composed of 8 NURBS of order $p = 2$, totalizing 16 equally spaced control points (Fig. (2b)), and consequently 16 collocation points. The adopted material properties are: Young modulus = 1000000.0 kg/m² and weight density = 2,400 kg/m³. As shown in Fig. (2c), the cells mesh comprises 32 quadrilateral cells.



(a) (b) (c)

Figure 2. (a) Rectangular strip under self-weight (b) control points position and (c) cells mesh.

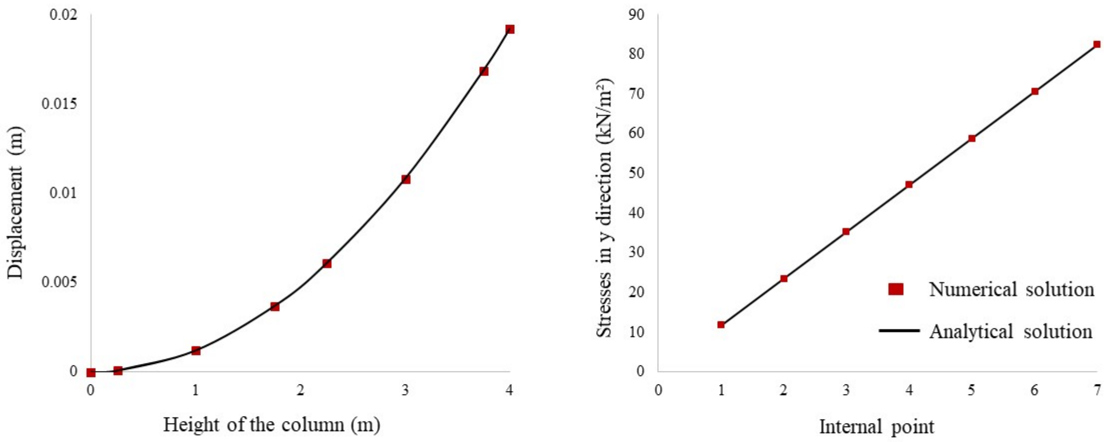


Figure 3. (a) Displacements evolution in the boundary (b) stress field in direction y for seven internal points

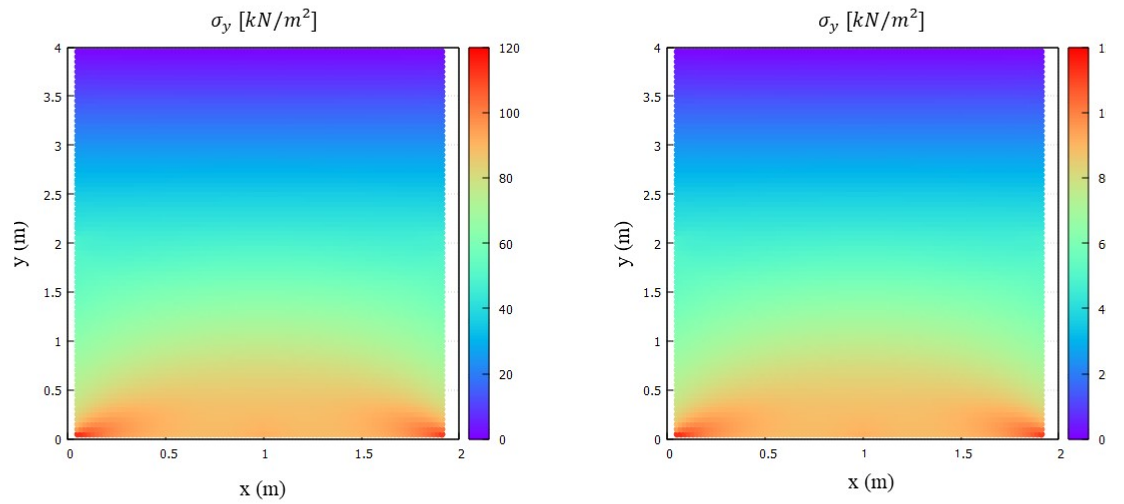
In Fig. (3a), the evolution of displacements is shown for each collocation point. Seven internal points are selected for analyzing the stress field in the y direction, and the result is presented in Fig. (3b). The points coordinates are: (1, 0.5), (1, 1), (1, 1.5), (2, 2), (2, 2.5), (2, 3) and (3, 3.5).

It is evident that the approximation using cells provides a highly accurate method for achieving the analytical solution. To compare the outcomes obtained through the Galerkin vector approach and cells, the same example is analyzed considering $\nu = 0.3$. The stress field for each method is shown in Fig. (4). Unlike when $\nu = 0.0$, the field is non-uniform. This is expected because of the Poisson effect, which results in stresses acting in all directions. It can be observed that both approximations yield identical solutions, thereby confirming the validity of the cell method.

5. DOMAIN DISCRETIZATION VIA CELLS: INSIGHTS FOR NON-LINEAR PROBLEMS

As presented in the numerical example, different methodologies yielded to the same solution. In general, the BEM community prefers to avoid any domain discretization: boundary integrals are preferred always as possible to address a certain problem. Nevertheless, in some cases, specially in highly non-linear problems, avoiding such type of domain discretization is not permissible. Examples are cases where a internal variable assumes an important role for the mechanical behaviour description, as in plasticity or damage evolution.

For such representation, initial applied strain or stress fields are notary essential. For materials with non-linear behavior, the incremental procedure for solving the problem makes use of such a technique. Works from Venturini and Brebbia (1984), Barbirato *et al.* (1999), Botta *et al.* (2003), Lima Junior (2006), Peixoto (2016) are examples where part of



(a) (b)

Figure 4. Stress field in y direction via (a) Galerkin vector approach (b) cells.

the domain is discretized into cells for approximating the domain term. In the following subsection, the altered integral equation for the elastoplastic problem, considering the initial fields, is presented.

5.1 Integral representations for problems with initial fields

The Somigliana's identity given in (7) is increased by an integral with initial stress terms, i.e.:

$$C_{ij}u_j\left(\mathbf{x}'\right)+\int_{\Gamma}T^*_{ij}\left(\mathbf{x}',\mathbf{x}\right)u_j(\mathbf{x})d\Gamma=\int_{\Gamma}U^*_{ij}\left(\mathbf{x}',\mathbf{x}\right)t_j(\mathbf{x})d\Gamma+\int_{\Omega}U^*_{ij}\left(\mathbf{x}',\mathbf{x}\right)b_j(\mathbf{x})d\Omega+\int_{\Omega}\varepsilon^*_{\mathrm{ijk}}\left(\mathbf{x}',\mathbf{x}\right)\cdot\sigma^0_{\mathrm{jk}}(\mathbf{x})\mathrm{d}\Omega \tag{36}$$

where $\varepsilon^*_{\mathrm{ijk}}(\mathbf{x}',\mathbf{x})$ is the fundamental kernel due to the initial stress and it is given by

$$\varepsilon^*_{\mathrm{ijk}}(\mathbf{x}',\mathbf{x})=\frac{-1}{8\pi(1-\mathrm{v})\mu r}\left[(1-2\nu)\left(\mathrm{r}_{,\mathrm{k}}\delta_{\mathrm{ij}}+\mathrm{r}_{,\mathrm{j}}\delta_{\mathrm{ik}}\right)-\mathrm{r}_{,\mathrm{i}}\delta_{\mathrm{jk}}+2\mathrm{r}_{,\mathrm{i}}\mathrm{r}_{,\mathrm{j}}\mathrm{r}_{,\mathrm{k}}\right] \tag{37}$$

The integral representation for the stresses is increased of a new domain integral as well, as shown:

$$\begin{aligned}\sigma_{ij}\left(\mathbf{x}'\right)+\int_{\Gamma}S^*_{kij}\left(\mathbf{x}',\mathbf{x}\right)u_k(\mathbf{x})d\Gamma&=\int_{\Gamma}D^*_{kij}\left(\mathbf{x}',\mathbf{x}\right)t_k(\mathbf{x})d\Gamma+\int_{\Omega}D^*_{kij}\left(\mathbf{x}',\mathbf{x}\right)b_k(\mathbf{x})d\Omega\\&+\int_{\Omega}E^*_{ijkl}(\mathbf{x}',\mathbf{x})\sigma^0_{kl}(\mathbf{x})d\Omega-\frac{1}{8(1-\nu)}[2\sigma^0_{ij}(\mathbf{x}')+(1-4\nu)\sigma^0_{mm}(\mathbf{x}')\delta_{ij}]\end{aligned} \tag{38}$$

where the kernel E^*_{ijkl} is expressed as

$$\begin{aligned}\mathrm{E}_{\mathrm{ijkl}}(\mathbf{x}',\mathbf{x})=&\frac{1}{4\pi(1-v)\mathrm{r}^2}\left\{(1-2\mathrm{v})\left[\delta_{\mathrm{il}}\delta_{\mathrm{jk}}+\delta_{\mathrm{jl}}\delta_{\mathrm{ik}}-\delta_{\mathrm{ij}}\delta_{\mathrm{kl}}+2\delta_{\mathrm{ij}}\mathrm{r}_{,\mathrm{k}}\mathrm{r}_{,1}\right]\right.\\&\left.+2v\left[\delta_{\mathrm{ik}}\mathrm{r}_{,\mathrm{j}}\mathrm{r}_{,1}+\delta_{\mathrm{jl}}\mathrm{r}_{,\mathrm{i}}\mathrm{r}_{,\mathrm{k}}+\delta_{\mathrm{il}}\mathrm{r}_{,\mathrm{j}}\mathrm{r}_{,\mathrm{k}}+\delta_{\mathrm{jk}}\mathrm{r}_{,\mathrm{i}},\mathrm{r}_{,l}\right]+2\delta_{\mathrm{kl}}\mathrm{r}_{,\mathrm{i},\mathrm{r}}\mathrm{r}_{,\mathrm{j}}-8\mathrm{r}_{,\mathrm{i},\mathrm{j},\mathrm{j},\mathrm{r}}\mathrm{r}_{l}\mathrm{r}_{,l}\right\}\end{aligned} \tag{39}$$

It is important to highlight that the consideration of initial strain instead of stress equivalent and consequently also possible. Further details can be found in Telles (2012) and Venturini (2012).

6. CONCLUSIONS AND OUTLOOKS

In this paper two different approaches for taking into account the volume effects are presented. As already mentioned, once the Galerkin vector strategy replaces a domain integral by a boundary one, it requires less computational effort and so it is preferable. However, for validation purposes, the same volume integral is approximated by a set of cells. Each cell approximates the body force field, and their contributions are accumulated at the end. Since there are no singularities in these integrals, numerical integration is performed using Gauss quadrature.

Similarly, the domain integrals considering initial fields can be approximated by this discretization. It is of paramount importance to remember that in non-linear problems complete domain discretization is not necessary. Instead, cells can be inserted only in the region where the internal variable evolves, i.e, where the material exhibits nonlinear behaviour. By focusing on these regions, computational resources are utilized efficiently, enhancing the accuracy of the results.

Therefore, through the utilization of cells solely in these specific regions, the computational cost is significantly reduced when compared to traditional domain discretization techniques. This allows to handle a larger range of more complex problems, ensuring an efficient analysis.

7. ACKNOWLEDGEMENTS

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8. REFERENCES

Aliabadi, M.H., 2002. *The boundary element method, volume 2: applications in solids and structures*, Vol. 2. John Wiley & Sons.

Barbirato, J.C.C. *et al.*, 1999. “Método dos elementos de contorno com a reciprocidade dual para a análise transiente tridimensional da mecânica do fraturamento”. *São Carlos. Tese (Doutorado)-Escola de Engenharia de São Carlos-Universidade de São Paulo*.

Botta, A.S., Venturini, W.S. and Benallal, A., 2005. “Bem applied to damage models emphasizing localization and associated regularization techniques”. *Engineering analysis with boundary elements*, Vol. 29, No. 8, pp. 814–827.

Botta, A.S. *et al.*, 2003. *Método dos elementos de contorno para análise de corpos danificados com ênfase no fenômeno da localização de deformações*. Ph.D. thesis, Universidade de São Paulo.

Brebbia, C.A. and Dominguez, J., 1994. *Boundary elements: an introductory course*. WIT press.

Brebbia, C.A. and Walker, S., 2016. *Boundary element techniques in engineering*. Elsevier.

Cheng, A.H.D. and Cheng, D.T., 2005. “Heritage and early history of the boundary element method”. *Engineering analysis with boundary elements*, Vol. 29, No. 3, pp. 268–302.

Cox, M.G., 1972. “The numerical evaluation of b-splines”. *IMA Journal of Applied mathematics*, Vol. 10, No. 2, pp. 134–149.

De Boor, C., 1972. “On calculating with b-splines”. *Journal of Approximation theory*, Vol. 6, No. 1, pp. 50–62.

Lima Junior, E.T.d., 2006. *Formulação do método dos elementos de contorno para análise de cascas abatidas*. Ph.D. thesis, Universidade de São Paulo.

Peixoto, R.G., 2016. “Análise de degradaçao material, bifurcaçao e transiçao entre descontinuidades fracas e fortes através do método dos elementos de contorno”.

Pérez-Gavilán, J. and Aliabadi, M., 2001. “A symmetric galerkin formulation and dual reciprocity for 2d elastostatics”. *Engineering analysis with boundary elements*, Vol. 25, No. 3, pp. 229–235.

Sauter, S.A. and Schwab, C., 2010. *Boundary element methods*. Springer.

Simpson, R.N., Bordas, S.P., Trevelyan, J. and Rabczuk, T., 2012. “A two-dimensional isogeometric boundary element method for elastostatic analysis”. *Computer Methods in Applied Mechanics and Engineering*, Vol. 209, pp. 87–100.

Telles, J.C.F., 2012. *The boundary element method applied to inelastic problems*, Vol. 1. Springer Science & Business Media.

Venturini, W.S., 2012. *Boundary element method in geomechanics*, Vol. 4. Springer Science & Business Media.

Venturini, W. and Brebbia, C., 1984. “Boundary element formulation for nonlinear applications in geomechanics”. *Applied Mathematical Modelling*, Vol. 8, No. 4, pp. 251–260.

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