COBEM 2023 27TH INTERNATIONAL CONGRESS OF MECHANICAL ENGENEERING

ABCM Brazilian Society of Mechanical Sciences and Engineering
27th ABCM International Congress of Mechanical Engineering
December 04-08, 2023, Florianópolis-SC, Brazil

COB-2023-1504
DESIGN OF SHAPE FUNCTIONS FOR ELIMINATING THE GIBBS PHENOMENON

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Abstract. *One of the major challenges in approximate methods is the mitigation of spurious oscillations from the obtained solution and its derivatives, near the boundary of the problem domain, known as the Gibbs phenomenon. Eventual errors caused by such oscillations may be reduced but not eliminated as the number of degrees of freedom is increased. Because of the Gibbs phenomenon, the Fourier series may face difficulties in representing even sufficiently smooth functions defined on a compact interval, when their periodic extension have discontinuities at the interval boundaries. Such difficulties can be properly surmounted expanding the modified function $f-\varphi$, instead of merely expanding the original function f. The suitably chosen function φ helps improve the convergence of the approximation by enforcing not only the continuity of the periodic extension of $f-\varphi$ but also of its derivatives. Once the auxiliary function φ is stated, one says that the series of f has been enriched because of its modification towards a faster convergence. Aiming to design complete sets of shape functions that provide fast convergence according to the theory of Fourier series and eliminate the Gibbs phenomenon up to derivatives of order p, this work proposes a rational approach to construct hierarchic sets of shape functions of any C^q continuity ($q = 0, 1, 2, \ldots$) derived from proper C^p periodic extension of $f-\varphi$.*

Keywords: *Gibbs phenomenon, improved convergence, hierarchic set*

1. INTRODUCTION

The theory of Fourier series grounds much of modern applied mathematics in a wide range of fields, from signal analysis to differential equations, serving as fundamentals for solution procedures. Because of the Gibbs phenomenon, the Fourier series may face difficulties in representing even sufficiently smooth functions defined on a compact interval, when their periodic extension have discontinuities at the interval boundaries. The higher the degree of discontinuity, the worse the convergence. Such difficulties can be properly surmounted, fortunately, by the artifice of expanding the modified function $f-\varphi$, instead of merely expanding the original function f. The suitably chosen function φ helps improve the convergence of the approximation by enforcing not only the continuity of the periodic extension of $f-\varphi$ but also of its derivatives (Lanczos, 1966). The function $f-\varphi$ has, on the compact interval where f is defined, the same smoothness of f. If the periodic extension of $f-\varphi$ is made C^p continuous, the Gibbs phenomenon is absent from the Fourier expansion of that extension itself and from their derivatives up to order p (Jones and Hardy, 1970; Baszenski et al., 1995). Once the auxiliary function φ is stated, one says that the series of f has been enriched because of its modification towards a faster convergence. All supplementary terms provided by φ can be used not only to improve the solution but also to satisfy the boundary conditions. It must be emphasized, however, that the role of the enrichment has not been undertaken consciously in some works, as commented in Li (2002). In structural analysis, the use of the above idea was apparently first sketched out in the works of Iguchi (1936, 1937, 1938) and Leggett (1941) on stability and vibration of thin plates, where φ was stated in a suitable polynomial form based on term-by-term differentiation aspects of $f-\varphi$.

In the search for approximations, a very attractive procedure to achieve computational efficiency is offered by a complete hierarchic set of shape functions that can accommodate the geometric boundary conditions by simply removing some functions from the set (Yshii et al., 2018). In this type of procedure, the set is arranged into a useful form so that its first components (nodal functions) are designed to mainly enforce boundary values, and the remaining ones (hierarchic or non-nodal functions) are conceived to refine the approximation. Aiming to conceive complete sets that provide fast convergence according to the theory of Fourier series, this work proposes a rational approach to construct hierarchic sets of shape functions of any C^q continuity ($q = 0, 1, 2, \ldots$) derived from proper C^p periodic extension of $f-\varphi$. In particular, a set of enriched sines, which have its performance illustrated by means of Ritz solutions of a beam bending problem, must be made to obey $q = (p-1)/2$.

2. IMPROVING THE CONVERGENCE OF FOURIER SERIES

Let f be a smooth function of x defined on the interval $0 \leq x \leq L$. The Fourier sine series expansion of f is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x \tag{1}$$

with $\lambda_n = n\pi/L$ and coefficients

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \lambda_n x \, dx. \tag{2}$$

In order to establish the sine series (1) the function f is first extended from the interval $0 \leq x \leq L$ to the interval $-L \leq x \leq 0$ as an odd function, and then it is extended to the whole real x as a periodic function of period $2L$. The series will converge to f on the open interval $0 < x < L$ and to the average values $[f(0+) + f(0-)]/2 = 0$ and $[f(L+) + f(L-)]/2 = 0$ at the jump discontinuities of the extension. The series oscillates strongly next to such jumps with an error in the form of under- and overshoots known as Gibbs phenomenon. In general, the smoother the periodic extension is, the faster its Fourier coefficients b_n tend to zero as $n \to \infty$. Thus, it would be of great interest if the smoothness of the periodic extension could be improved by, say, controlling the continuity of the extension itself and of its derivatives. If the periodic extension of f is C^p continuous ($p = 1, 3, 5, \ldots$), the coefficients b_n will be of order $n^{-(p+2)}$. This key assumption about the continuity seems to be unreal for practical purpose.

Fortunately, one can employ the artifice of expanding the modified function $f - \varphi$, instead of merely expanding the function f, to satisfy the assumption just made (Lanczos, 1966). The auxiliary function φ, which must be continuous and differentiable up to the desired order $p + 2$, must be chosen to ensure

$$f^{(k)}(0) = \varphi^{(k)}(0) = \alpha_k \qquad f^{(k)}(L) = \varphi^{(k)}(L) = \beta_k \qquad k = 0, 2, \ldots, p-1 \tag{3}$$

where α_k and β_k are parameters to be identified, $f^{(0)}(x) = f(x)$ and $f^{(k)}(x) = d^k f(x)/dx^k$. Since the periodic extension of $f - \varphi$ is C^p continuous, it can be much smoother than the periodic extension of f. That is, the above procedure improves the convergence of the Fourier series of $f - \varphi$ because the Gibbs phenomenon is not present in the series itself or in its derivatives up to the order p. Thus, the series (1) is said to have been enriched by the auxiliary function φ in

$$f(x) = \varphi(x) + \sum_{n=1}^{\infty} b_n \sin \lambda_n x \tag{4}$$

because it is modified for a faster convergence. The auxiliary function φ can be stated in many ways. For illustration, suppose $k = 0, 2$ ($p = 3$) so that

$$\begin{aligned} &f^{(0)}(0) = \varphi^{(0)}(0) = \alpha_0 \qquad f^{(2)}(0) = \varphi^{(2)}(0) = \alpha_2 \\ &f^{(0)}(L) = \varphi^{(0)}(L) = \beta_0 \qquad f^{(2)}(L) = \varphi^{(2)}(L) = \beta_2. \end{aligned} \tag{5}$$

The coefficients b_n will be of order n^{-5} for the above number of enrichment constraints (Lanczos, 1966). Assuming the simplest choice

$$\varphi(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3, \tag{6}$$

the imposition of (5) gives

$$\varphi(x) = \alpha_0 \varphi_{l0}(x) + \alpha_2 \varphi_{l2}(x) + \beta_0 \varphi_{r0}(x) + \beta_2 \varphi_{r2}(x) \tag{7}$$

for which

$$\varphi_{l0}(x) = 1 - \frac{x}{L} \qquad \varphi_{l2}(x) = \frac{L^2}{6}\left[-2\frac{x}{L} + 3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right]$$

$$\varphi_{r0}(x) = \frac{x}{L} \qquad \varphi_{r2}(x) = \frac{L^2}{6}\left[-\frac{x}{L} + \left(\frac{x}{L}\right)^3\right]. \tag{8}$$

Note that φ_{l1}, φ_{r1}, φ_{l3}, φ_{r3} have the properties

$$\varphi_{lk}^{(j)}(0) = \varphi_{rk}^{(j)}(L) = \delta_{jk} \qquad \varphi_{lk}^{(j)}(L) = \varphi_{rk}^{(j)}(0) = 0 \qquad j, k = 0, 2 \tag{9}$$

where δ_{jk} is the Kronecker delta. The generalization of (7) for any enrichment level reads

$$\varphi(x) = \sum_{k=0,2,\ldots}^{p-1} [\alpha_k \varphi_{lk}(x) + \beta_k \varphi_{rk}(x)], \tag{10}$$

with

$$\varphi_{rk}(x) = \sum_{j=0,2,\ldots}^{k} \left(\frac{x}{L}\right)^{j+1} \frac{L^{j+1} R_{j+1}}{(j+1)!} \qquad \varphi_{lk}(x) = \varphi_{rk}(L - x). \tag{11}$$

The parameters R_1, R_3, ... can be computed recursively from

$$\begin{aligned}
R_{k+1} &= \frac{1}{L} \\
R_{k-1} &= -\frac{L^2}{3!} R_{k+1} = -\frac{1}{6} L \\
R_{k-3} &= -\frac{L^2}{3!} R_{k-1} - \frac{L^4}{5!} R_{k+1} = \frac{7}{360} L^3 \\
R_{k-5} &= -\frac{L^2}{3!} R_{k-3} - \frac{L^4}{5!} R_{k-1} - \frac{L^6}{7!} R_{k+1} = -\frac{31}{15120} L^5 \\
&\vdots
\end{aligned} \tag{12}$$

3. HIERARCHIC SETS OF SHAPE FUNCTIONS

The concept of enriching Fourier series, as stated by (4), will be extended in this section to the construction of hierarchic sets of shape functions of C^q continuity. To provide a guideline for the set generation, one writes

$$f^{(j)}(0) = u_{lj} \qquad f^{(j)}(L) = u_{rj} \qquad j = 0, 1, \ldots, q \tag{13}$$

and expresses the function

$$f(x) = \psi(x) + \sum_{n=1}^{N} h_n H_n(x), \tag{14}$$

in a component form, with a finite number N of terms under the summation sign. Here

$$\psi(x) = \sum_{j=0}^{q} \left[u_{lj} \psi_{lj}(x) + u_{rj} \psi_{rj}(x)\right] \tag{15}$$

where

$$\psi_{lk}^{(j)}(0) = \psi_{rk}^{(j)}(L) = \delta_{jk} \qquad \psi_{lk}^{(j)}(L) = \psi_{rk}^{(j)}(0) = 0 \qquad j, k = 0, 1, \ldots, q \tag{16}$$

so that ψ takes the same values as f at the boundaries, and

$$H_n^{(j)}(0) = H_n^{(j)}(L) = 0 \qquad j = 0, 1, \dots, q. \tag{17}$$

To recast the enriched sine series (4) in the form of (14), the series is first rewritten as

$$f(x) = \sum_{k=0,2,\dots}^{p-1} [\alpha_k \varphi_{lk}(x) + \beta_k \varphi_{rk}(x)] + \sum_{n=1}^{N} b_n \sin \lambda_n x \tag{18}$$

and then made it equal to (14) with coefficients b_n, for convenience and without loss of generality, replaced by h_n:

$$\sum_{k=0,2,\dots}^{p-1} [\alpha_k \varphi_{lk}(x) + \beta_k \varphi_{rk}(x)] + \sum_{n=1}^{N} h_n \sin \lambda_n x = \sum_{j=0}^{q} \left[u_{lj} \psi_{lj}(x) + u_{rj} \psi_{rj}(x)\right] + \sum_{n=1}^{N} h_n H_n(x). \tag{19}$$

One must now take $p - 1 = 2q$ [i.e., $q = (p-1)/2$] to make the number of enrichment parameters α_k and β_k equal to the number of boundary parameters u_{lj} and u_{rj}. In matrix form, expression (19) reads

$$\boldsymbol{\varphi}^T \boldsymbol{\gamma} + \boldsymbol{\phi}^T \mathbf{h} = \boldsymbol{\psi}^T \mathbf{u} + \mathbf{H}^T \mathbf{h} \tag{20}$$

where

$$\begin{aligned}
\boldsymbol{\phi} &= \lfloor \sin \lambda_1 x \quad \sin \lambda_2 x \quad \cdots \quad \sin \lambda_N x \rfloor^T \\
\mathbf{h} &= \lfloor h_1 \quad h_2 \quad \cdots \quad h_N \rfloor^T \\
\mathbf{H} &= \lfloor H_1(x) \quad H_2(x) \quad \cdots \quad H_N(x) \rfloor^T
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\boldsymbol{\varphi} &= \lfloor \varphi_{l0}(x) \quad \varphi_{l2}(x) \quad \cdots \quad \varphi_{l(2q)}(x) \quad \varphi_{r0}(x) \quad \varphi_{r2}(x) \quad \cdots \quad \varphi_{r(2q)}(x) \rfloor^T \\
\boldsymbol{\gamma} &= \lfloor \alpha_0 \quad \alpha_2 \quad \cdots \quad \alpha_{2q} \quad \beta_0 \quad \beta_2 \quad \cdots \quad \beta_{2q} \rfloor^T \\
\boldsymbol{\psi} &= \lfloor \psi_{l0}(x) \quad \psi_{l1}(x) \quad \cdots \quad \psi_{lq}(x) \quad \psi_{r0}(x) \quad \psi_{r1}(x) \quad \cdots \quad \psi_{rq}(x) \rfloor^T \\
\mathbf{u} &= \lfloor u_{l0} \quad u_{l1} \quad \cdots \quad u_{lq} \quad u_{r0} \quad u_{r1} \quad \cdots \quad u_{rq} \rfloor^T.
\end{aligned} \tag{22}$$

Evaluation of (20) and its derivatives up to order q at the boundaries gives

$$\bar{\boldsymbol{\varphi}} \boldsymbol{\gamma} + \bar{\boldsymbol{\phi}} \mathbf{h} = \bar{\boldsymbol{\psi}} \mathbf{u} \tag{23}$$

with

$$\begin{aligned}
\bar{\boldsymbol{\varphi}} &= [\boldsymbol{\varphi}(0) \quad \boldsymbol{\varphi}^{(1)}(0) \quad \cdots \quad \boldsymbol{\varphi}^{(q)}(0) \quad \boldsymbol{\varphi}(L) \quad \boldsymbol{\varphi}^{(1)}(L) \quad \cdots \quad \boldsymbol{\varphi}^{(q)}(L)]^T \\
\bar{\boldsymbol{\phi}} &= [\boldsymbol{\phi}(0) \quad \boldsymbol{\phi}^{(1)}(0) \quad \cdots \quad \boldsymbol{\phi}^{(q)}(0) \quad \boldsymbol{\phi}(L) \quad \boldsymbol{\phi}^{(1)}(L) \quad \cdots \quad \boldsymbol{\phi}^{(q)}(L)]^T \\
\bar{\boldsymbol{\psi}} &= [\boldsymbol{\psi}(0) \quad \boldsymbol{\psi}^{(1)}(0) \quad \cdots \quad \boldsymbol{\psi}^{(q)}(0) \quad \boldsymbol{\psi}(L) \quad \boldsymbol{\psi}^{(1)}(L) \quad \cdots \quad \boldsymbol{\psi}^{(q)}(L)]^T.
\end{aligned} \tag{24}$$

From properties (16), $\bar{\boldsymbol{\psi}}$ is an identity matrix. Assuming that $\bar{\boldsymbol{\varphi}}$ is nonsingular, the enrichment parameters can be identified by means of

$$\boldsymbol{\gamma} = \bar{\boldsymbol{\varphi}}^{-1} (\mathbf{u} - \bar{\boldsymbol{\phi}} \mathbf{h}) \tag{25}$$

whose substitution into (20) yields

$$(\boldsymbol{\varphi}^T \bar{\boldsymbol{\varphi}}^{-1}) \mathbf{u} + (\boldsymbol{\phi}^T - \boldsymbol{\varphi}^T \bar{\boldsymbol{\varphi}}^{-1} \bar{\boldsymbol{\phi}}) \mathbf{h} = \boldsymbol{\psi}^T \mathbf{u} + \mathbf{H}^T \mathbf{h}. \tag{26}$$

Both sides of (26) are exactly the same if

$$\boldsymbol{\psi} = \bar{\boldsymbol{\varphi}}^{-T} \boldsymbol{\varphi} \qquad \mathbf{H} = \boldsymbol{\phi} - \bar{\boldsymbol{\phi}}^T \boldsymbol{\psi}. \tag{27}$$

Once an enrichment level is stated and $\boldsymbol{\varphi}$ identified, one is able to construct the set of shape functions

$$\mathbf{F}_s = \begin{Bmatrix} \boldsymbol{\psi} \\ \mathbf{H} \end{Bmatrix} \tag{28}$$

of C^q continuity related to hierarchic coefficients h_n. The arrays $\boldsymbol{\psi}$ and $\mathbf{H}$ are then said to contain nodal and hierarchic (non-nodal) functions, respectively. The difference between them is only of computational interest, because the presence of nodal functions is advantageous when performing boundary conditions. Relations (27) show: (a) how the enrichment feature, originally contained in $\boldsymbol{\varphi}$ according to Fourier series, spreads into $\boldsymbol{\psi}$ and $\mathbf{H}$; (b) that the components of $\boldsymbol{\psi}$ are functions of the same class of that used in $\boldsymbol{\varphi}$. Because $\mathbf{H}$ and all its derivatives up to order q vanish at the boundaries, it is said that the hierarchic set $\mathbf{F}_s$ is C^q continuous. Since the coefficients b_n of the enriched sine series are of order $n^{-(p+2)}$, the hierarchic coefficients h_n related to $\mathbf{F}_s$ will be of order $n^{-(2q+3)}$.

The completeness of the set $\{\sin \lambda_n x\}$ with respect to the odd periodic extension of $f - \varphi$ implies the completeness of $\mathbf{F}_s$ with respect to f (Tolstov, 1976). To construct a hierarchical set (28) of C^q continuity from enriched sine functions, one should follow the steps outlined below.

1. Identify the functions φ_{lk} and φ_{rk} $(k = 0, 2, \dots, 2q)$.
2. Obtain the array $\boldsymbol{\varphi}$ from φ_{lk}, φ_{rk} and the array $\boldsymbol{\phi}$ from the sine set of functions $\{\sin \lambda_n x\}$.
3. Evaluate the matrices $\bar{\boldsymbol{\varphi}}$ and $\bar{\boldsymbol{\phi}}$.
4. Compute the arrays $\boldsymbol{\psi}$ of nodal functions and $\mathbf{H}$ of hierarchic functions in accordance with (27).

4. ILLUSTRATIVE EXAMPLE

A clamped beam of length L, subjected to an exponentially distributed load $q_0 e^{2x}$ and referred to the coordinate x measured along its axis, is depicted in Figure 1. According to the linear Euler-Bernoulli beam theory (Reddy, 2002), the transverse displacement $v(x)$ of the beam corresponds to the stationary point $\delta\Pi = 0$ of the potential energy

$$\Pi(v) = \int_0^L \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 + q_0 e^{2x} v \right] dx. \tag{29}$$

This example requires $\delta\Pi = 0$ subjected to the (geometric) boundary conditions

$$v = \frac{dv}{dx} = 0 \quad \text{at } x = 0 \qquad\qquad v = \frac{dv}{dx} = 0 \quad \text{at } x = L. \tag{30}$$

For constant bending rigidity EI, the exact solution reads

$$v_e(x) = -\frac{q_0}{16EI} \left[e^{2x} - 2\frac{1 + L - (1 - L)e^{2L}}{L^3} x^3 + \frac{3 + 4L - (3 - 2L)e^{2L}}{L^2} x^2 - 2x - 1 \right]. \tag{31}$$

In the Ritz method (Reddy, 2002), the solution is approximately sought in the component form (14), which can be rearranged as

$$v(x) \approx \begin{Bmatrix} \boldsymbol{\psi} \\ \mathbf{H} \end{Bmatrix}^T \begin{Bmatrix} \mathbf{u} \\ \mathbf{h} \end{Bmatrix} \tag{32}$$

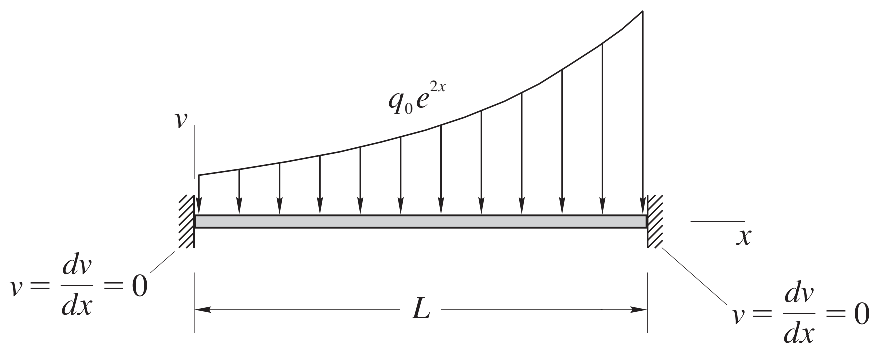

Figure 1. Clamped beam subjected to an exponentially distributed load.

with $\boldsymbol{\psi}$ and $\mathbf{H}$ defined in (27). All the coefficients collected in $\mathbf{u}$ are related to values of v and its derivatives at the boundaries, whereas those collected in $\mathbf{h}$ have no physical meaning. Here, the polynomials φ_{l0}, φ_{l2}, φ_{r0}, φ_{r2} given in (8) are used as the components of $\boldsymbol{\varphi}$ for $q = 1$ (i.e., $p = 3$). The boundary conditions (30) are satisfied by simply removing $\boldsymbol{\psi}$ from (32). The solution of the linear system of algebraic equations obtained from $\delta\Pi = 0$, after substitution of $v \approx \mathbf{H}^T\mathbf{h}$ into the potential energy (29), gives the coefficients h_n shown in Table 1 for $L = 1$ (in some consistent but unspecified units). Replacing f by the exact solution (31) and φ by the polynomial expression (7), the coefficients b_n of the sine series (4) are then evaluated and reported in the last line of Table 1. Clearly, the Ritz coefficients h_n converge to the exact coefficients b_n as the number of hierarchic functions goes from 1 to 40. This is a numerical evidence that (27) makes the enrichment embedded in the trigonometric series to be directly inherited by the hierarchic set.

Table 1. The Ritz coefficients associated with $\mathbf{F}_s$ of C^1 continuity for the clamped beam.

NDF	$\bar{h}_1$	$\bar{h}_2$	$\bar{h}_3$	$\bar{h}_4$	$\bar{h}_5$	$\bar{h}_6$	$\bar{h}_7$
1	−5.71250						
2	−5.71250	0.16348					
3	−6.07762	0.16348	−0.02844				
4	−6.07762	0.17918	−0.02844	0.00506			
5	−6.16507	0.17918	−0.03168	0.00506	−0.00214		
10	−6.21367	0.18731	−0.03348	0.00608	−0.00253	0.00077	−0.00043
20	−6.23509	0.18894	−0.03427	0.00628	−0.00270	0.00083	−0.00050
40	−6.24054	0.18941	−0.03448	0.00634	−0.00275	0.00084	−0.00051
Exact	−6.24237	0.18957	−0.03454	0.00636	−0.00276	0.00085	−0.00052

Note: NDF = number of degrees of freedom; $\bar{h}_n = 160EIh_n/q_0$

5. CONCLUSIONS

Elimination of the Gibbs phenomenon in trigonometric series by means of auxiliary polynomials is used in this work to construct hierarchic sets of shape functions with improved convergence properties. Hierarchic sets of enriched sines, of any desired C^q continuity, are then constructed. The knowledge of the whole family of hierarchic sets $\mathbf{F}_s$, here established for any degree of continuity, may prove to be useful for numerical methods.

6. REFERENCES

Baszenski, G., Delvos, F. J. and Tasche, M., 1995. "A united approach to accelerating trigonometric expansions". *Computers & Mathematics with Applications*, Vol. 30, pp. 33–49.
Iguchi ,S., 1936. "Allgemeine lösung der knickungsaufgabe für rechteckige platen". *Ingenieur-Archiv*, Vol. 7, pp. 207–215.
Iguchi ,S., 1937. "Die biegungsschwingungen der vierseitig eigenspannten rechteckigen platte". *Ingenieur-Archiv*, Vol. 8, pp. 11–25.
Iguchi ,S., 1938. "Die knickung der rechteckigen platte durch schubkräfte". *Ingenieur-Archiv*, Vol. 9, pp. 1–12.
Jones, W. B., Hardy, G., 1970. "Accelerating convergence of trigonometric approximations". *Mathematics of Computation*, Vol. 24, pp. 547–560.
Lanczos, C., 1966. *Discourse on Fourier Series*. Oliver and Boyd, Edinburgh.
Leggett, D. M. A., 1941. "The buckling of a square panel under shear when one pair of opposite edges is clamped, and the other pair is simply supported". *Reports and Memoranda* (No. 1991). Aeronautical Research Committee, London.
Li, W. L., 2002. "Reply to: Discussion on "Free vibrations of beams with general boundary conditions"". *Journal of Sound and Vibration*, Vol. 257, pp. 593–595.
Reddy, J. N., 2002. *Energy Principles and Variational Methods in Applied Mechanics* (2nd Edition). John Wiley, Hoboken.
Tolstov, G. P., 1976. *Fourier Series*. Dover, New York.

Yshii, L. N., Lucena Neto, E., Monteiro, F. A. C. and Santana, R. C., 2018. “Accuracy of the buckling predictions of anisotropic plates”. *Journal of Engineering Mechanics*, Vol. 144, p. 04018061.

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