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COMBINED ANALYTICAL AND NUMERICAL STUDY OF LAMINAR MAGNETOHYDRODYNAMICS IN A BACKWARD-FACING STEP CHANNEL.

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Abstract. *The aim of this study is to investigate the complex reciprocal action between a flow of electrically conductive fluid and an externally applied magnetic field in a backward-facing step channel. This problem is of great interest in a variety of engineering applications, such as in the design of magnetohydrodynamic (MHD) power generators and flow control devices. In this geometry, the flow encounters a sudden expansion, which leads to the formation of a recirculation zone behind the step. The presence of an external magnetic field can significantly affect the flow and alter this region. We present one formulation for the backward-facing step problem, governed by the equations of Navier-Stokes and the transport of magnetic field, originating in the classical electromagnetism equations. To solve this formulation with General Integral Transform Technic (GITT) we use two second order eigenvalue problems to the components of the magnetic field, horizontal (B_1) and vertical (B_2), and one forth order eigenvalue problem to solve the formulation associated to the velocity. With that we find hybrid solutions with better computational cost and recover the boundary conditions associated with the main problem. Results are compared with the extensive literature for benchmarking purposes.*

Keywords*: Magnetohydrodynamics (MHD), Navier-Stokes Equations, Integral Transforms (GITT), backward-facing step.*

1. INTRODUCTION

Study of magnetohydrodynamic (MHD) flow in ducts is of great interest because of the its growing applications in several engineering areas. MHD pumping, power generation and metallurgy are some examples where its potential has been explored. In actual applications, for example, MHD control is being used to optimize drug absorption through arterial blood flow (Alghamdi et al., 2021). Streamfunction formulation and scalar magnetic function were previously adopted to analyse, through the finite difference method, the MHD entry region of a parallel-plate channel, as we see in (Brandt and Gillis, 1966).

In parallel to the application of purely numerical methods, the so-called GITT (Generalized Integral Transform Technique) is a mathematical approach that has gained attention in keeping all the characteristics of an analytical solution, together with the strength of numerical methods for solution of ordinary differential equations (Cotta, 1993) and (Santos, Quaresma e Lima, 2001). In (Lima and Rêgo, 2013), (Pontes et al., 2016) and (Assad et al., 2016) one and two-dimensional MHD flows in parallel-plate channels were analyzed via the integral transform approach [4-6] for low magnetic Reynolds numbers (Rem << 1), i.e., it was admitted that the magnetic field inside the flow field was not altered by the induced one.

It was observed in (Assad, 2016) that the eigenvalue problem used in the methodology is a 2nd differential equation order and does not recover the 4 necessary boundary conditions (two for the horizontal component and two for the vertical component of the magnetic field) to solve the transport equations of the magnetic field that will be shown below. The main objective of the present work is to show a solution associated with a 4th order eigenvalue problem for the velocity field and a 2nd order eigenvalue problem for each component of the magnetic field, thus recovering, after application of the inversion formulas, the boundary conditions of the original problem.

2. MATHEMATICAL FORMULATION

We consider the flow along the symmetry plane of a rectangular step channel of width w and height h. The step height is a, and the two lateral/vertical walls are electrically conductive while the two horizontal ones are isolated. The incompressible, steady-state MHD laminar flow inside the 2-D backward-facing step channel is represented in (Figure 1). The physical properties are constant and the flow is submitted to an external transverse magnetic field, B_0^*, that can be altered inside the channel by induced currents as the flow develops. Under these assumptions, the main potentials can be described as $\vec{V}^* \equiv (u^*, v^*, 0)$, $\vec{B}^* \equiv (B_1^*, B_2^*, 0)$, $\vec{E}^* \equiv (0, 0, E_z^*)$, $\vec{J}^* \equiv (0, 0, J_z^*)$. u^* and v^* are the velocity components, $B_1^* = B_x^*$ and $B_2^* = B_y^* + B_0^*$ are the induced/applied magnetic fields components, E_z^* is the applied/measured electrical field component, and J_z^* is the induced current density.
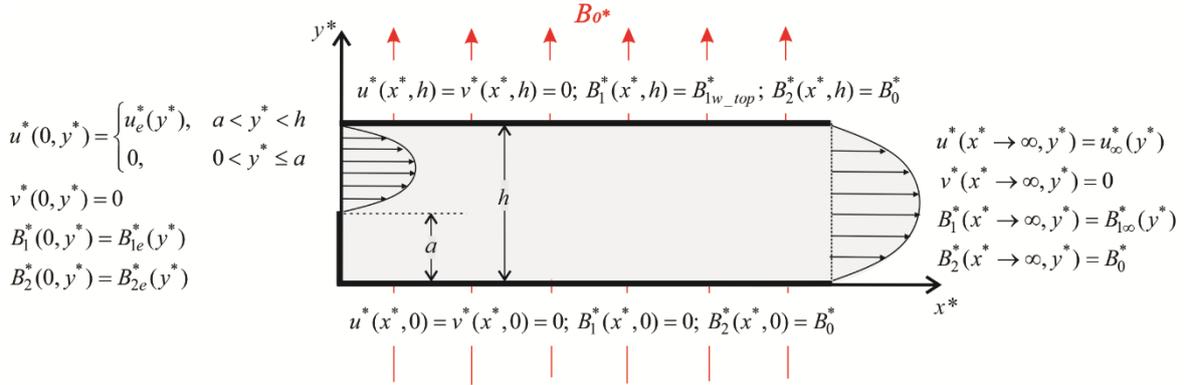

Figure 1. Schematic representation of the backward-facing step channel and the boundary conditions

Writing the components of velocity in terms of an streamfunction $(u = \partial\psi/\partial y, v = -\partial\psi/\partial x)$, that eliminates terms associated with pressure, facilitates the visualization of current flow lines, reduces the number of equations to be solved and increases rates of numerical convergence in relation to the problems solved in terms of primitive variables, the governing (Navier-Stokes and eletromagnetic) equations are writen in dimensionless form as follows:

$$\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial x\partial y^2}\right) - \frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y^3} + \frac{\partial^3\psi}{\partial y\partial x^2}\right) = \frac{1}{Re}\left(\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4}\right) -$$
$$\frac{Ha^2}{Re\,Re_m}\left[B_2\left(\frac{\partial^2 B_2}{\partial x\partial y} - \frac{\partial^2 B_1}{\partial y^2}\right) - B_1\left(\frac{\partial^2 B_1}{\partial x\partial y} - \frac{\partial^2 B_2}{\partial x^2}\right) + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\left(\frac{\partial B_2}{\partial y} + \frac{\partial B_1}{\partial x}\right)\right] \tag{1}$$

$$0 = \frac{\partial^2\psi}{\partial y^2}B_2 + \frac{\partial\psi}{\partial y}\frac{\partial B_2}{\partial y} + \frac{\partial^2\psi}{\partial x\partial y}B_1 + \frac{\partial\psi}{\partial x}\frac{\partial B_1}{\partial y} + \frac{1}{Re_m}\left(\frac{\partial^2 B_1}{\partial x^2} + \frac{\partial^2 B_1}{\partial y^2}\right) \tag{2}$$

$$0 = \frac{\partial^2 \psi}{\partial x^2} B_1 + \frac{\partial \psi}{\partial x} \frac{\partial B_1}{\partial x} + \frac{\partial^2 \psi}{\partial x \partial y} B_2 + \frac{\partial \psi}{\partial y} \frac{\partial B_2}{\partial x} - \frac{1}{Re_m} \left(\frac{\partial^2 B_2}{\partial x^2} + \frac{\partial^2 B_2}{\partial y^2} \right) \tag{3}$$

The corresponding boundary conditions is:

$$x = 0 \quad \begin{cases} \psi(0, y) = \begin{cases} \psi_e(y), & q < y < 1 \\ 0, & 0 < y \le q \end{cases} \\ \dfrac{\partial \psi}{\partial x}(0, y) = 0 \\ B_1(0, y) = B_{1e}(y) \\ B_2(0, y) = B_{2e}(y) \end{cases} \qquad x \to \infty \quad \begin{cases} \psi(x \to \infty, y) = \psi_\infty(y) \\ \dfrac{\partial \psi}{\partial x}(x \to \infty, y) = 0 \\ B_1(x \to \infty, y) = B_{1\infty}(y) \\ B_2(x \to \infty, y) = B_{2\infty}(y) \end{cases} \tag{4 a-h}$$

$$y = 0 \quad \begin{cases} \psi(x,0) = 0 \\ \dfrac{\partial \psi}{\partial y}(x,0) = 0 \\ B_1(x,0) = 0 \\ B_2(x,0) = 1 \end{cases} \qquad y = 1 \quad \begin{cases} \psi(x,1) = 1 - q \\ \dfrac{\partial \psi}{\partial y}(x,1) = 0 \\ B_1(x,1) = B_{1w\_top} \\ B_2(x,1) = 1 \end{cases} \tag{5 a-h}$$

We will not present a similar scalar function for the components of magnetic field because we want to use a singular eigenvalue problem for each one.

In the above formulation, the following dimensionless groups were employed:

$$(x, y, q) = \frac{(x^*, y^*, a)}{h}, \ (u, v) = \frac{(u^*, v^*)}{\bar{U}_e}, \ P = \frac{P^*}{\rho \bar{U}_e^2}, \ Re = \frac{\bar{U}_e h}{\nu}, \ (B_x, B_y) = \frac{(B_x^*, B_y^*)}{B_0^*} Ha, \ B_1 = \frac{B_1^*}{B_0^*} = B_x$$

$$B_2 = \frac{B_2^*}{B_0^*} = (B_y + 1), \ E_z = \frac{E_z^*}{\bar{U}_e B_0^*}, \ Ha = B_0^* h \left(\frac{\sigma}{\mu} \right)^{1/2}, \ J_z = \frac{J_z^*}{\sigma \bar{U}_e B_0^*}, \ Re_m = \mu_m \sigma \bar{U}_e h \equiv \frac{\bar{U}_e h}{\nu_m}$$

In this groups, $\bar{U}_e$ is the average flow velocity at the channel entrance, Ha is the Hartmann number, Re is the Reynolds number (characteristic length is the height of channel (h)), Re_m is the magnetic Reynolds number, ρ is the fluid density, μ and ν are the dynamic and kinematic fluid viscosity, respectively, σ is the fluid electrical conductivity, μ_m is the fluid magnetic permeability, and ν_m is the fluid magnetic diffusivity.

2.1 Solution Methodology

To eliminate any non-homogeneous boundary conditions in the direction to be integral transformed, undesired in light of the integral transform technique (GITT), each original field is split-up in two parts: a filtered field (subscript *H*), with homogeneous boundary conditions, and a filter field (subscript *F*) that carries the original non-homogeneity:

$$\psi(x, y) = \psi_H(x, y) + \psi_F(y) \tag{6}$$

$$B_1(x, y) = B_{1H}(x, y) + B_{1F}(y) \tag{7}$$

$$B_2(x, y) = B_{2H}(x, y) + B_{2F}(y) \tag{8}$$

For the streamfunction and magnetic field, the filter employed is the solution of the fully developed flow field, *i.e.* $\psi_F(y) = \psi_\infty(y)$, $B_{1F}(y) = B_{1\infty}(y)$ and $B_{2F}(y) = B_{2\infty}(y)$. Previous works of great support attest to the effectiveness of filtering fields: (Machado and Cotta, 1995), (Santos et al., 2001) and (Assad, 2021).

2.2 Integral transformation

The main idea behind the integral transformation method is to write all fields as eigenfunctions expansions (inverse/integral transform pairs):

$$\psi_H(x,y) = \sum_{i=1}^{\infty} \tilde{Y}_i(y)\,\bar{\psi}_{Hi}(x)\,, \qquad \bar{\psi}_{Hi}(x) = \int_0^1 \tilde{Y}_i(y)\,\psi_H(x,y)\,dy \tag{9, 10}$$

$$B_{1H}(x,y) = \sum_{i=1}^{\infty} \tilde{C}_i(y)\,\bar{B}_{1H}(x)\,, \qquad \bar{B}_{1H}(x) = \int_0^1 \tilde{C}_i(y)\,B_{1H}(x,y)\,dy \tag{11, 12}$$

$$B_{2H}(x,y) = \sum_{i=1}^{\infty} \tilde{H}_i(y)\,\bar{B}_{2H}(x)\,, \qquad \bar{B}_{2H}(x) = \int_0^1 \tilde{H}_i(y)\,B_{2H}(x,y)\,dy \tag{13, 14}$$

$\tilde{Y}_i(y)$, $\tilde{C}_i(y)$ and $\tilde{H}_i(y)$ are the streamfunction and components of magnetic field eigenfunctions, respectively, obtained from auxiliary eigenvalue problems related to homogeneous versions of the original problems. $\bar{\psi}_{Hi}(x)$, $\bar{B}_{1H}(x)$ and $\bar{B}_{2H}(x)$ are the respective transformed potentials.

Now, application of the operators $\int_0^1 \tilde{Y}_i(y)\cdots dy$, $\int_0^1 \tilde{C}_i(y)\cdots dy$ and $\int_0^1 \tilde{H}_i(y)\cdots dy$ on the governing equations, and use of the orthogonality properties of the eigenfunctions and inversa formulae, leads to the following coupled system of ODE equations in the x-direction:

$$
\begin{aligned}
\frac{d^4\bar{\psi}_{Hi}}{dx^4}(x) = &\ \mathrm{Re}\left[\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\begin{array}{l} A_{ijk}^{\psi}\left(\dfrac{d^3\bar{\psi}_{Hj}}{dx^3}(x)\bar{\psi}_{Hk}(x) - \dfrac{d\bar{\psi}_{Hj}}{dx}(x)\dfrac{d^2\bar{\psi}_{Hk}}{dx^2}(x) \right) \\ + \left(B_{ijk}^{\psi} - E_{ijk}^{\psi} \right)\left(\dfrac{d\bar{\psi}_{Hj}}{dx}(x)\bar{\psi}_{Hk}(x) \right) \end{array} \right) + \sum_{j=1}^{\infty}\left(\begin{array}{l} C_{ijF}^{\psi}\dfrac{d^3\bar{\psi}_{Hj}}{dx^3}(x) + \left(D_{ijF}^{\psi} - F_{ijF}^{\psi} \right)\dfrac{d\bar{\psi}_{Hj}}{dx}(x) \\ - \dfrac{2}{\mathrm{Re}}G_{ij}^{\psi}\dfrac{d^2\bar{\psi}_{Hj}}{dx^2}(x) \end{array} \right) \right. \\[1em]
&+ \frac{Ha^2}{\mathrm{Re}_m}\left[\begin{array}{l} -\displaystyle\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(K_{ijk}^{\psi} + O_{ijk}^{\psi} \right)\bar{B}_{1Hj}(x)\dfrac{d\bar{B}_{1Hk}}{dx}(x) \\ +\displaystyle\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} I_{ijk}^{\psi}\left(\begin{array}{l}\dfrac{d\bar{B}_{1Hj}}{dx}(x)\dfrac{d\bar{B}_{2Hk}}{dx}(x) + \bar{B}_{1Hj}(x)\dfrac{d^2\bar{B}_{2Hk}}{dx^2}(x) \\ +\left(\eta_{1i}^2 I_{ijk}^{\psi} - L_{ijk}^{\psi} \right)\bar{B}_{1Hj}(x)\bar{B}_{2Hk}(x) \end{array}\right) \\ +\displaystyle\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} J_{ijk}^{\psi}\left(\dfrac{d\bar{B}_{2Hj}}{dx}(x)\bar{B}_{2Hk}(x) + \bar{B}_{2Hj}(x)\dfrac{d\bar{B}_{2Hk}}{dx}(x) \right) \end{array} \right] \\
&\left. - \left(\begin{array}{l} \displaystyle\sum_{j=1}^{\infty}\left(\left(M_{ijF}^{\psi} + Q_{ijF}^{\psi} \right)\dfrac{d\bar{B}_{1Hj}}{dx}(x) - \alpha_i^2 T_{ijF}^{\psi}\bar{B}_{1Hj}(x) \right) + \\ \displaystyle\sum_{j=1}^{\infty}\left(\begin{array}{l} S_{ijF}^{\psi}\dfrac{d\bar{B}_{2Hj}}{dx}(x) - \left(N_{ijF}^{\psi} + R_{ijF}^{\psi} \right)\bar{B}_{2Hj}(x) \\ + P_{ijF}^{\psi}\dfrac{d^2\bar{B}_{2Hj}}{dx^2}(x) \end{array} \right) \end{array} \right) - U_{iFF} \right] \\
&- H_{iF} - \mu_i^4 \bar{\psi}_{Hi}(x)
\end{aligned}
\tag{15}
$$

$$
\frac{d^2\bar{B}_{1Hi}}{dx^2}(x) = -\mathrm{Re}_m\left\{ \begin{array}{l} \displaystyle\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(G_{ijk}^{B1} + I_{ijk}^{B1} \right)\left(\dfrac{d\bar{\psi}_{Hj}}{dx}(x)\bar{B}_{1Hk}(x) \right) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(A_{ijk}^{B1} + E_{ijk}^{B1} \right)\left(\bar{\psi}_{Hj}(x)\bar{B}_{2Hk}(x) \right) \\ +\displaystyle\sum_{j=1}^{\infty}\left[B_{ijF}^{B1}\bar{\psi}_{Hj}(x) + \left(H_{ijF}^{B1} + J_{ijF}^{B1} \right)\dfrac{d\bar{\psi}_{Hj}}{dx}(x) \right] + \sum_{j=1}^{\infty}\left[\left(C_{ijF}^{B1} + F_{ijF}^{B1} \right)\bar{B}_{2Hj}(x) \right] + D_{iFF}^{B1} \end{array} \right\} + \alpha_i^2 \bar{B}_{1Hj}(x) - K_{iF}^{B1}
\tag{16}
$$

$$
\frac{d^2\bar{B}_{2Hi}}{dx^2}(x) = \mathrm{Re}_m\left\{ \begin{array}{l} \displaystyle\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} A_{ijk}^{B2}\left(\dfrac{d\bar{\psi}_{Hj}}{dx}(x)\bar{B}_{2Hk}(x) + \bar{\psi}_{Hj}(x)\dfrac{d\bar{B}_{2Hj}}{dx}(x) \right) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} D_{ijk}^{B2}\left(\dfrac{d^2\bar{\psi}_{Hj}}{dx^2}(x)\bar{B}_{1Hk}(x) + \dfrac{d\bar{\psi}_{Hj}}{dx}(x)\dfrac{d\bar{B}_{1Hk}}{dx}(x) \right) \\ +\displaystyle\sum_{j=1}^{\infty}\left[B_{ijF}^{B2}\dfrac{d\bar{\psi}_{Hj}}{dx}(x) + E_{ijF}^{B2}\dfrac{d^2\bar{\psi}_{Hj}}{dx^2}(x) \right] + \sum_{j=1}^{\infty}\left[C_{ijF}^{B2}\dfrac{d\bar{B}_{2Hj}}{dx}(x) \right] + \dfrac{1}{\mathrm{Re}_m}\beta_i^2 \bar{B}_{2Hj}(x) \end{array} \right\}
\tag{17}
$$

The inlet and outlet boundary conditions are equally transformed to yield:

$$\begin{cases} \overline{\psi}_{Hi}(0) = \overline{g}_i = \int_0^1 \tilde{Y}_i(y)\psi_H(0,y)dy = \int_0^1 \tilde{Y}_i(y)[g_e(y) - g_\infty(y)]dy \\ \left.\dfrac{d\overline{\psi}_{Hi}}{dx}\right|_{x=0} = 0 \\ \overline{B}_{H1}(0,y) = -\overline{h}_{1i} = -\int_0^1 \tilde{B}_{1i}(y)\,B_{1F}(0,y)dy \\ \overline{B}_{H2}(0,y) = -\overline{h}_{2i} = -\int_0^1 \tilde{B}_{2i}(y)\,B_{2F}(0,y)dy \end{cases} ; \begin{cases} \overline{\psi}_{Hi}(x_\infty) = 0 \\ \left.\dfrac{d\overline{\psi}_{Hi}}{dx}\right|_{x=x_\infty} = 0 \\ \overline{B}_{H1}(x \to \infty, y) = 0 \\ \overline{B}_{H2}(x \to \infty, y) = 0 \end{cases} \qquad (18, 19)$$

To obtain the numerical results, a computer code was developed in FORTRAN 90 language. From (IMSL Fortran Numerical Library, 2010), subroutine BVPFD, which is especially suitable for solving systems of stiff ordinary differential equations, was employed to solve the coupled system (Eqs. 15-19). A relative error target of 10^{-5} was employed as criterion for convergence of the transformed potentials. The integral coefficients appearing on these equations, as shows (Assad et al., 2016), were evaluated through routine QDAGS, according to the library mentioned above. The difficulty of locating x_∞ was circumvented by a change of coordinates ($\eta = 1 - e^{-cx}$, $c > 0$) to transform the domain $x \in [0,\infty[$ to $\eta \in [0,1]$.

3. RESULTS AND DISCUSSION

For validation of the computational code used in the formulation of the present work, we show, in Figure 2, the vorticity graph (ω) in two positions along the channel, x=7 and x=15, showing that the results found are in total agreement with (Gartling, 1990). This figure shows that the results found validate the present formulation, showing the effectiveness of the G.I.T.T..
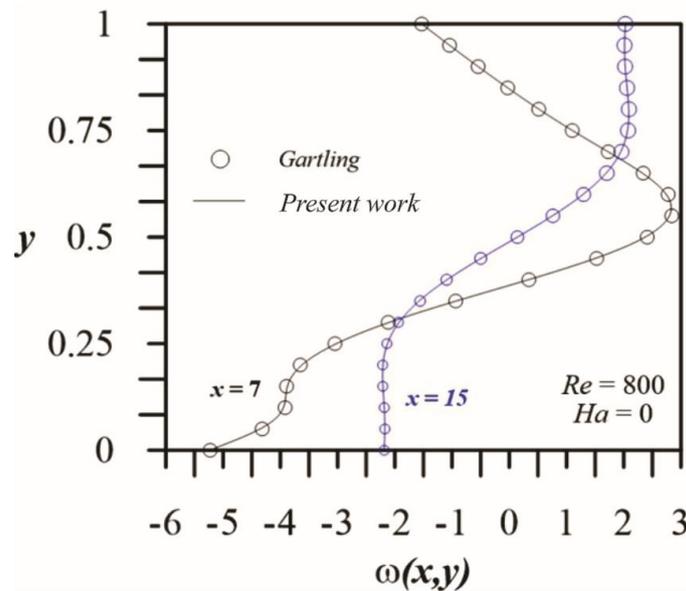

Figure 2. Vorticity for X=7 and X=15 comparing results from Gartling (1990) and the present work.

To show validation with Ha≠0, Figure 3-a is shown, which indicates the total agreement of the velocity profile, for $x=0.5$, $Ha=2$, $Re=20$ and $Re_m=25$, between the formulation described by (Assad, 2021), in formulation with the magnetic scalar function (β) and that of this work (with the two components B_1 and B_2). Figure 3-b also shows the behavior of the longitudinal velocity component in the centerline of the channel, for $Re=100$, $Ha=10$ and $Re_m=1$, showing a satisfactory agreement between the results of the two formulations.
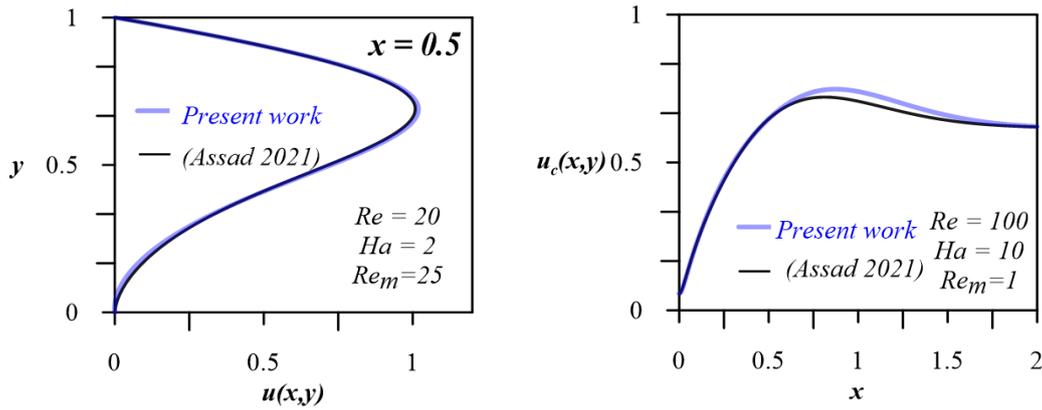
Figure 3. a) Velocity profile for $x=0.5$, $Re=20$, $Ha=2$ and $Re_m=25$, comparing (Assad, 2021) and the present work; b) Longitudinal component of velocity in the canal centerline for $Re=100$, $Ha=10$ and $Re_m=1$ (same literature).

For a comparison between the formulations of the present work and that of (Assad, 2021), Figure 4 shows the difference between the behaviors of the longitudinal component of the magnetic field (B_1), showing that the current formulation becomes more suitable for the solution of the problem when the flow is in development ($x=1$). The figure shows that the present formulation correctly recovers the value of B_1 in the contours ($B_1(x,0)=B1(x,1)=0$), which did not occur with the previous formulation, and indicates the correct sinusoidal variation in vertical domais ($y \in [0,1]$). It shows the behavior of $B_1(1,y)$, $Ha=2$, $Re=20$ and $Re_m=10$, in both formulations.
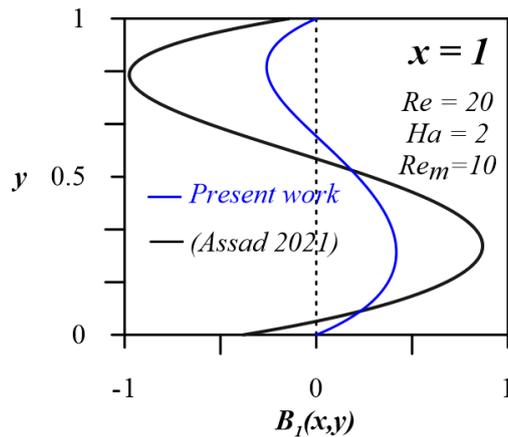

Figure 4. Behavior of B1(1,y), Ha=2, Re=20 and Rem=10, in both formulations.

To reiterate the validation of the computational code, the following case shows that the physics behind the non-MHD flow in a channel of parallel plates without step ($a=0$ in Figure 1) and with uniform profile at the entrance, is also evaluated correctly with the method from GITT, used here. Two references are used in the comparison: i) Wang and Longwell (1963) who use purely numerical methods and compare, as well as this work, with ii) Schlichting (1960) who describes through boundary layer the results for laminar flow in a channel of parallel plates. Tables 1 and 2 show, respectively, the comparison between values of the longitudinal velocity component (u), between the central line (y=0.5) and the vicinity of the lower wall (y=0.25), in two different positions along the channel (x=3.33314 and x=7.50000).

Table 1. Comparative values of the longitudinal component of velocity for x=3.33314

y	Schlichting (Boundary Layer)	Wang (FDM)	Present work (GITT)
0.025	0.172	0.171	0.174
0.05	0.332	0.331	0.338
0.1	0.619	0.618	0.588
0.15	0.858	0.855	0.821
0.2	1.041	1.040	1.011
0.25	1.175	1.171	1.158

0.3	1.265	1.259	1.264
0.35	1.317	1.311	1.335
0.4	1.336	1.340	1.380
0.45	1.346	1.353	1.400
0.5	1.347	1.357	1.407

Table 2. Comparative values of the longitudinal component of velocity for x=7.50000

y	Schlichting (Boundary Layer)	Wang (FDM)	Present work (GITT)
0.025	0.156	0.154	0.148
0.05	0.302	0.298	0.288
0.1	0.568	0.564	0.546
0.15	0.797	0.793	0.772
0.2	0.969	0.965	0.966
0.25	1.143	1.140	1.129
0.3	1.261	1.259	1.260
0.35	1.347	1.347	1.361
0.4	1.404	1.406	1.433
0.45	1.436	1.440	1.475
0.5	1.446	1.451	1.488

To directly show the magnetohydrodynamic coupling we show: i) Figures 5a-b which indicate that the magnetic Reynolds number does not significantly affect the velocity profile ($u(x,y)$) but does affect the magnetic field ($B_1(x,y)$) and ii) Figures 6a-b which show that who affects the velocity profile, but not the magnetic field, is the Hartmann number (see Eqs 15 and 16). The parameter values are indicated in the figures cited above.
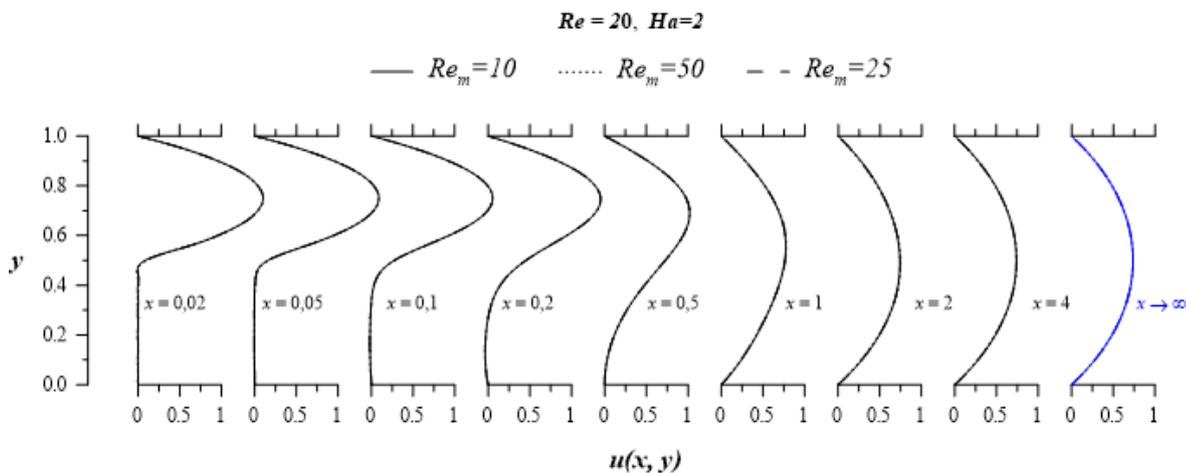

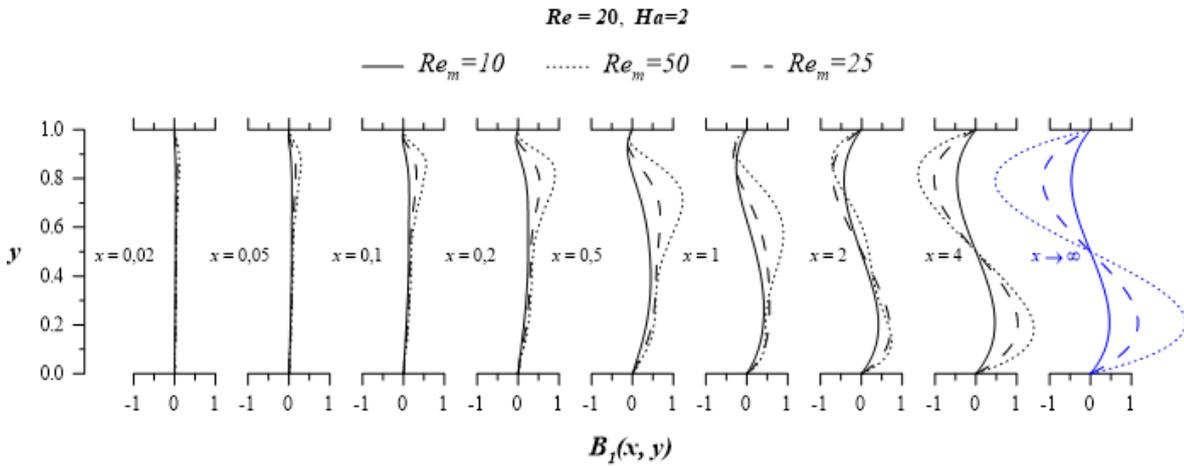Figure 5a. Development of the longitudinal velocity component, for *Re = 20, Ha = 2* and different *Re_m*.

Figure 5b. Development of the longitudinal magnetic field component, for *Re = 20, Ha = 2* and different Re_m.
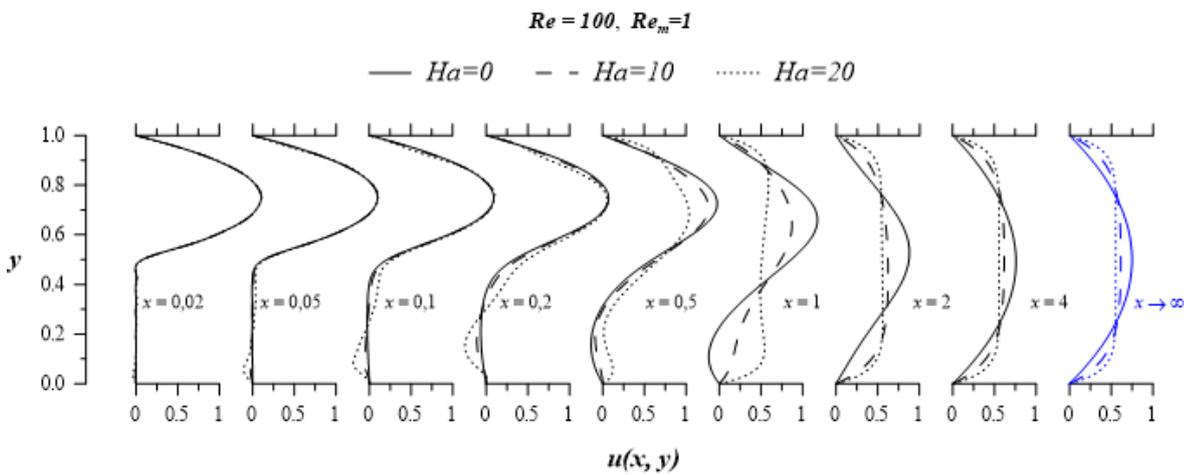

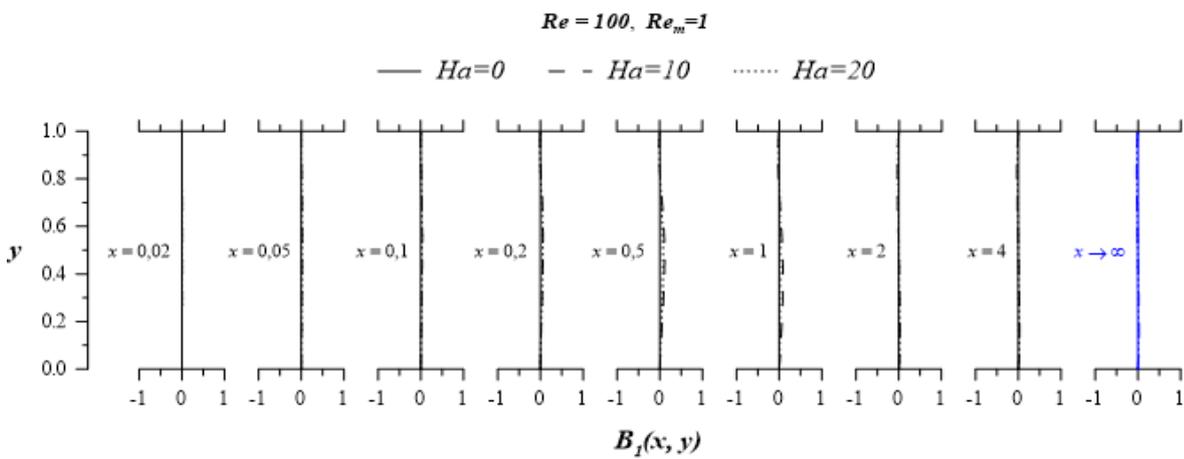Figure 6a. Development of the longitudinal velocity component, for *Re = 100, Rem = 1* and different *Ha*.

Figure 6b. Development of the longitudinal magnetic fiel component, for *Re = 100, Rem = 1* and different *Ha*.

Finally, we conclude that the formulation addressed in the present work, with two 2nd order eigenvalue problems, one for B_1 and another for B_2, in addition to the 4th problem. already known order for the velocity field, perfectly recovers the desired boundary conditions, solving the problem evidenced in (Assad, 2016) and (Assad, 2021). Validated with the literature, this formulation completes the suggested MHD analysis, constituting the epitome of this text and showing the robustness of the well-founded Generalized Integral Transform Technique (G.I.T.T.).

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