COBEM 2023 27TH INTERNATIONAL CONGRESS OF MECHANICAL ENGENEERING

ABCM Brazilian Society of Mechanical Sciences and Engineering
27th ABCM International Congress of Mechanical Engineering
December 04-08, 2023, Florianópolis-SC, Brazil

COB-2023-1136
STOCHASTIC OPTIMIZATION OF A ROTATING MACHINE THROUGH RELIABILITY-BASED DESIGN OPTIMIZATION

Eduardo Henrique de Paula
Helio Fiori de Castro
Laboratory of Rotating Machinery, School of Mechanical Engineering, University of Campinas, Mendeleyev Street, 200, Cidade Universitária, Campinas, SP, 13083-860, Brazil
e166756@dac.unicamp.br, heliofc@fem.unicamp.br

Abstract. *Rotating machines are components of significant importance for engineering, being fundamental for the operation of the most different mechanisms present in society. Given the importance of rotors for the industry, their operating parameters must be studied and optimized, reducing their manufacturing and operational cost. At the same time, their reliability must be ensured, preventing unexpected failures from interrupting their operation or causing accidents with their operators. Most studies involving the optimization of these components consider their parameters as deterministic. However, just like in any real system, their parameters are stochastic, presenting variability in the material properties, loads, and geometry. To consider this variability in the optimization project, it is necessary to employ stochastic or robust optimization. These methods seek an optimal point for the rotor operating parameters in which, even if their properties vary, the parameter to be optimized does not vary considerably. Therefore, the reliability of the rotor's operation and the robustness of the project are guaranteed. In this paper, a detailed analysis of the optimization of a rotating machine will be conducted through the Reliability-Based Design Optimization (RBDO) method. RBDO is an optimization method that focuses on achieving the robustness of the design by seeking an optimal condition for its operating parameters while ensuring that its failure criteria are not exceeded, guaranteeing a predefined level of reliability. To validate this method, the Jeffcott rotor with a non-central disc and flexible bearings will be studied. A detailed application of RBDO methods in formulating and solving the optimization problem will be presented. The convergence of the optimization results and the computational cost of different approaches will be discussed, aiming to identify the most recommended ones. Once this methodology is validated for a simpler model, further research can be developed to expand its application to more complex rotor models.*

Keywords: *rotating machines, uncertainty quantification, reliability-based design optimization, Jeffcott rotor*

1. INTRODUCTION

Rotating machines are essential for operating various mechanisms in industry, including equipment like pumps, turbines, motors, and compressors. Depending on their application, these machines consist of a rotating shaft coupled to at least one element, like gears, pulleys, discs, or propellers. The shaft is supported by at least two bearings that enable its rotational movement. Given the importance of rotating machines, optimizing their operating parameters to reduce costs, improve performance, and ensure their reliability is imperative.

Like in any other real-world system, the parameters of rotating machines are stochastic and may vary due to factors such as rotor material, loads, manufacturing and assembly processes, and wear and tear resulting from usage. According to Cafeo and Thomas (1997), this variability can make optimization challenging and lead to conditions that significantly differ from projections, making it crucial to use statistical tools to deal with this problem. Among the tools available for considering variability in the optimization project, are stochastic and robust optimization, which seek an optimal point for the system operating parameters in which, even if the system's properties vary, the optimized parameter does not vary significantly, ensuring design reliability and robustness.

Since reliability is as critical as parameter variability in rotating machines, this study has been developed on the optimization of rotating machine parameters through the Reliability-Based Design Optimization (RBDO), which is a method that ensures the robustness of the project by seeking an optimal operating condition for its parameters while ensuring that the failure criteria are not exceeded, thereby ensuring a predefined level of reliability. This condition is achieved by jointly performing a reliability analysis and solving an optimization problem. Natarajan and Santhakumar (1995) claim that this method is relatively recent and mainly applied to structures, so one of the objectives of this paper is to validate its applicability to rotating systems.

Several approaches can be utilized to solve the optimization problem linked to the RBDO. Their development, which will be presented in this paper, is based on a manual (UQLab, 2022). The authors of this manual collaborated with other colleagues from ETH Zurich to develop UQLab™ (Marelli and Sudret, 2014), a framework for uncertainty quantification based on MATLAB™ that includes RBDO as one of its main modules. This framework will be used in this study.

The primary component of a rotating machine is its rotor, which generates rotational motion by rotating around its axis. Accurately modeling the rotor and its interaction with associated bearings and structures is crucial to approximate the behavior of a rotating system. A more detailed modeling approach leads to a system that is closer to its real operation, but this also results in a more complex model that needs to be solved by sophisticated computational techniques with high costs. The main goal of this paper is to study the optimization of a rotating machine through RBDO, presenting the methodology for formulating and solving the problem, discussing the convergence of results, and comparing the computational costs of different approaches.

Since this study does not require a highly detailed rotor model, the Jeffcott rotor, also known as the Laval rotor, is studied in this paper. This simple model was proposed by Gustaf de Laval in 1895 and later improved by Jeffcott in 1919 (Visnadi, 2018). Due to its simplicity, which approximates it to an ideal rotor, it holds significant importance in the study of rotating systems. This model consists of a long, flexible shaft with negligible mass supported by a rigid structure with a rigid disc fixed at its longitudinal center. Jeffcott introduced the equivalent damping to the system, enabling the representation of the energy dissipated in the deformation of the shaft. In this paper, different RBDO approaches are applied to study this model, thoroughly investigating this method for future application in more complex models.

This paper is organized as follows. The methodology for formulating and solving a general RBDO problem is presented in Section 2. The dynamic modeling of the Jeffcott rotor with a non-central disc and flexible bearings is showed in Section 3. The formulation of the RBDO problem for the Jeffcott rotor is presented in Section 4. The results of this study are presented and discussed in Section 5. Finally, the conclusions of this paper and discussions about potential correlated future works are provided in Section 6.

2. FORMULATION AND SOLUTION OF AN RBDO PROBLEM

This section presents a discussion of the RBDO methodology for optimizing a stochastic system. As this method is based on system reliability, it is essential to understand the concepts of the limit state function $g(\mathbf{X})$ and the failure probability $P_f = \mathbb{P}(g(\mathbf{X}) \leq 0)$. The limit-state function determines the safe and failure regions of the state space by defining whether the system is failing ($g(\mathbf{X}) \leq 0$) or operating safely ($g(\mathbf{X}) > 0$). The failure probability represents, given the uncertainties of the state parameters, the likelihood that the system is in a failed state.

2.1 Formulation of an RBDO problem

An RBDO problem aims to minimize the cost $\mathbf{c}(\mathbf{d})$ of a system while ensuring it meets specified performance constraints that present uncertainties. A general formulation of this problem is shown in Eq. (1), where the cost function $\mathbf{c}$ is minimized concerning the design variables $\mathbf{d} \subset \mathbb{R}^{M_d}$, where $f_j, j = 1, \dots, s$ are the soft constraints (simple functions that define the design space) and $g_k, k = 1, \dots, n$ are the hard constraints (limit-state functions that describe the performance of the system). In the limit-state function concept, a system is in a failure state when $g_k(\mathbf{x}(\mathbf{d}), \mathbf{z}) \leq 0$ for any design $\mathbf{d}$, where $\mathbf{x}$ and $\mathbf{z}$ are realizations of the random variables $\mathbf{X} \in \mathbb{R}^{M_d} \sim f_{\mathbf{X}|\mathbf{d}}$ and $\mathbf{Z} \in \mathbb{R}^{M_z} \sim f_{\mathbf{Z}}$. These variables correspond, respectively, to a set of random variables related to the design variables, which may represent manufacturing tolerances, and to the environmental variables, which are parameters that may be random and cannot be controlled by the designer, such as system loads (UQLab, 2022).

$$\mathbf{d}^* = \arg\min_{\mathbf{d}\in\mathbb{D}} \mathbf{c}(\mathbf{d}) \; subject \; to \; \begin{cases} f_j(\mathbf{d}) \leq 0, & j = \{1, \dots, s\} \\ \mathbb{P}(g_k(\mathbf{X}(\mathbf{d}), \mathbf{Z}) \leq 0) \leq \bar{P}_{f_k}, & k = \{1, \dots, n\} \end{cases}, \tag{1}$$

Based on Eq. (1), it is clear that the probabilistic constraints for the RBDO of a system require the failure probability $\mathbb{P}(g_k(\mathbf{X}(\mathbf{d}), \mathbf{Z})) \leq 0)$ to be below a specific limit $\bar{P}_{f_k}$, which is defined during the design phase. For each system constraint, the failure probability can be expressed by Eq. (2), where $\mathbf{W} = \{\mathbf{X}, \mathbf{Z}\}^T \in \mathbb{R}^M \sim f_{\mathbf{W}}$ is a vector that contains design and environmental variables.

$$P_{f_k}(\mathbf{d}) = \mathbb{P}(g_k(\mathbf{W}) \leq 0) = \int_{g_k(\mathbf{w})\leq 0} f_{\mathbf{W}}(\mathbf{w})\, d\mathbf{w}, \tag{2}$$

2.2 Solution of an RBDO problem

Several methods are available for solving RBDO problems, which can be classified into three groups: two-level approaches, mono-level approaches, and decoupled approaches (UQLab, 2022). This subsection explains these methods, including their definition and mathematical formulations.

2.2.1 Generalized two-level approach

The generalized two-level approach (TL) is one of the most common techniques for solving RBDO problems. It consists of nested loops, where the outer loop explores the design space, and the inner loop computes the corresponding failure probability $P_{f_k}(\mathbf{d}^{(i)})$. A general-purpose optimization algorithm is then applied to solve Eq. (1). At each iteration, the problem's probabilistic constraint is evaluated by estimating the failure probability, as shown in Eq. (2), using simulation methods, such as Monte Carlo simulation, importance sampling, and subset simulation, or approximation methods, such as the first-order reliability method (FORM) and the second-order reliability method (SORM) (UQLab, 2022).

2.2.2 Reliability index approach

The reliability index approach (RIA) is a particular case of the generalized two-level approach, where the inner loop employs a FORM analysis. Although the implementation of this approach is simple and straightforward, its numerical efficiency is low, which is its primary disadvantage (UQLab, 2022).

This approach considers the problem's constraints based on its reliability index instead of its failure probability, rewriting the problem in Eq. (1) as per Eq. (3), where $\bar{\beta}_k = \Phi^{-1}(1-\bar{P}_{f_k})$ and $\beta_k = \Phi^{-1}(1-P_{f_k})$ are the target and structural reliability indexes of the k-th limit state, respectively, and Φ^{-1} is the inverse cumulative distribution function (CDF) of the standard normal distribution.

$$\mathbf{d}^* = \arg\min_{\mathbf{d}\in\mathbb{D}} \mathbf{c}(\mathbf{d}) \; subject\ to \; \begin{cases} f_j(\mathbf{d}) \leq 0, & j=\{1,\ldots,s\} \\ \bar{\beta}_k - \beta_k(\mathbf{X}(\mathbf{d}),\mathbf{Z}) \leq 0, & k=\{1,\ldots,n\} \end{cases}, \tag{3}$$

The reliability index corresponds geometrically to the distance from the origin point in the standard space $\|\mathbf{u}_k^*\|$, so that, $\beta_k = \|\mathbf{u}_k^*\|$. Therefore, the inner loop searches for the design point that satisfies Eq. (4), where $G_k(\mathbf{u}) = g_k(\tau^{-1}(\mathbf{u}))$. According to Rackwitz and Flessler (1978), specialized optimization algorithms are typically used to solve this equation, such as the Hasofer-Lind-Rackwitz-Fiessler algorithm (HLRF) and its improved version (iHLRF).

$$\mathbf{u}_k^* = \arg\min_{\mathbf{u}\in\mathbb{R}^M} \{\|\mathbf{u}\|, G_k(\mathbf{u}) \leq 0\}, \tag{4}$$

2.2.3 Performance measure approach

The performance measure approach (PMA) is another special case of the generalized two-level approach. It also consists of two loops, but the inner loop involves an inverse FORM analysis, making it possible to set a target reliability index and search for the minimum performance target point (MPTP). The RBDO problem can be reformulated by Eq. (5), where $\mathbf{w}_{MPTP_k} = \{\mathbf{x}_{MPTP_k}, \mathbf{z}_{MPTP_k}\}$ is the k-th MPTP associated with the design $\mathbf{d}$. The search is carried out in standard Gaussian space using Eq. (6), and the corresponding MPTP in the physical space is obtained by applying the mapping $\mathbf{w}_{MPTP_k} = \tau^{-1}(\mathbf{u}_{MPTP_k})$.

$$\mathbf{d}^* = \arg\min_{\mathbf{d}\in\mathbb{D}} \mathbf{c}(\mathbf{d}) \; subject\ to \; \begin{cases} f_j(\mathbf{d}) \leq 0, & j=\{1,\ldots,s\} \\ g_k(\mathbf{x}_{MPTP_k}(\mathbf{d}), \mathbf{z}_{MPTP_k}) \geq 0, & k=\{1,\ldots,n\} \end{cases}, \tag{5}$$

$$\mathbf{u}_{MPTP_k} = \arg\min_{\mathbf{u}\in\mathbb{R}^M} \left\{G_k(\mathbf{u}), \|\mathbf{u}\| = \bar{\beta}_k\right\}, \tag{6}$$

According to Youn *et al.* (2005) and Cho and Lee (2011), three methods can efficiently solve this problem: the advanced mean value (AMV), conjugate mean value (CMV), and hybrid mean value (HMV).

2.2.4 Quantile estimate approach

This approach considers quantiles as a measure of conservatism of the solution instead of the failure probability. Recognizing that the failure probability can be expressed by Eqs. (7) and (8), the RBDO problem can be rewritten by Eq. (9), where $\alpha_k = \bar{P}_{f_k}$. These quantiles are estimated by Monte Carlo sampling.

$$\mathbb{P}(g_k(\mathbf{X}(\mathbf{d}),\mathbf{Z}) \leq 0) \leq \bar{P}_{f_k} \Leftrightarrow Q_{\alpha_k}(\mathbf{d}, g_k(\mathbf{X}(\mathbf{d}),\mathbf{Z})) \geq 0, \tag{7}$$

$$Q_{\alpha_k}(\mathbf{d}; g_k(\mathbf{X}(\mathbf{d}),\mathbf{Z})) = \inf\{q \in \mathbb{R} : \mathbb{P}(g_k(\mathbf{X}(\mathbf{d}),\mathbf{Z}) \leq q) \geq \alpha_k\}, \tag{8}$$

$$\mathbf{d}^* = \arg\min_{\mathbf{d}\in\mathbb{D}} \mathbf{c}(\mathbf{d}) \; subject\ to \; \begin{cases} f_j(\mathbf{d}) \leq 0, & j=\{1,\ldots,s\} \\ Q_{\alpha_k}(\mathbf{d}; g_k(\mathbf{X}(\mathbf{d}),\mathbf{Z})) \geq 0, & k=\{1,\ldots,n\} \end{cases}, \tag{9}$$

2.2.5 Mono-level approach

According to Liang *et al.* (2008), single loop approaches (SLA) avoid performing reliability analysis in each iteration of the optimization process by converting the problem into an equivalent single-loop deterministic process, which is achieved by adding the Karush-Kuhn-Tucker optimality conditions of the reliability analysis as additional constraints, resulting in Eqs. (10) and (11), where $\boldsymbol{\mu}_{\mathbf{X}}$ and $\boldsymbol{\mu}_{\mathbf{Z}}$ are the mean values of the design and environmental variables, respectively, and the vector $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_{\mathbf{X}}, \boldsymbol{\sigma}_{\mathbf{Z}}\}$ gathers the standard deviations of the design and environmental variables.

$$\mathbf{d}^* = \arg\min_{\mathbf{d}\in\mathbb{D}} \mathbf{c}(\mathbf{d}) \; subject\ to \begin{cases} f_j(\mathbf{d}) \leq 0, & j = \{1,\ldots,s\} \\ g_k(\mathbf{x}_{MPTP_k}(\mathbf{d}), \mathbf{z}_{MPTP_k}) \geq 0, & k = \{1,\ldots,n\} \end{cases}, \tag{10}$$

$$\mathbf{x}_{MPTP_k} = \boldsymbol{\mu}_{\mathbf{X}} - \boldsymbol{\sigma}_{\mathbf{X}} \cdot \bar{\beta}_k \cdot \boldsymbol{\alpha}_k \qquad \mathbf{z}_{MPTP_k} = \boldsymbol{\mu}_{\mathbf{Z}} - \boldsymbol{\sigma}_{\mathbf{Z}} \cdot \bar{\beta}_k \cdot \boldsymbol{\alpha}_k, \tag{11}$$

The normalized gradient $\boldsymbol{\alpha}_k$ of constraint k at the previous MPTP has its value updated only in iterations where the mean value of the design variables $\boldsymbol{\mu}_{\mathbf{X}}$ changes. If all random variables are independent and normally distributed, $\boldsymbol{\alpha}_k$ is given by Eq. (12). However, if these conditions are not met, the equivalent mean and standard deviations can be found using a non-linear transformation called the Rackwitz-Fiessler two-parameter equivalent normal method, which assumes that the cumulative distribution function Φ and the probability density function φ of the actual non-normal and the equivalent normal distributions are equal at the current MPTP, resulting in Eq. (13), where $\boldsymbol{\mu} = \{\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\mu}_{\mathbf{Z}}\}$, $\mathbf{w}_{MPTP_k} = \{\mathbf{x}_{MPTP_k}, \mathbf{z}_{MPTP_k}\}$ and the product and division shown are applied component-wise (UQLab, 2022).

$$\boldsymbol{\alpha}_k = \frac{\boldsymbol{\sigma} \cdot \nabla_{g_k}(\mathbf{x}_{MPTP_k}(\mathbf{d}), \mathbf{z}_{MPTP_k})}{\|\boldsymbol{\sigma} \cdot \nabla_{g_k}(\mathbf{x}_{MPTP_k}(\mathbf{d}), \mathbf{z}_{MPTP_k})\|}, \tag{12}$$

$$\boldsymbol{\sigma} = \frac{\varphi\left[\boldsymbol{\Phi}^{-1}(F_{\mathbf{W}}(\mathbf{w}_{MPTP_k}))\right]}{f_{\mathbf{W}}(\mathbf{w}_{MPTP_k})} \qquad \boldsymbol{\mu} = \mathbf{w}_{MPTP_k} - F_{\mathbf{W}}(\mathbf{w}_{MPTP_k}) \cdot \boldsymbol{\sigma}, \tag{13}$$

2.2.6 Decoupled approach

This approach solves a deterministic optimization problem followed by a reliability analysis. The sequential optimization and reliability assessment (SORA) method is implemented in UQLab^{TM}. According to Du and Chen (2004), this approach converts the probabilistic constraint into an equivalent deterministic constraint using the MPTP. At each cycle, the constraint is estimated by updating the design variables according to the MPTP obtained in the reliability analysis of the previous cycle. At cycle i, the problem in Eq. (1) is approximated by Eq. (14), where $\mathbf{s}_k^{(i)} = \mathbf{d}^{(i-1)} - \mathbf{x}_{MPTP_k}^{(i-1)}$ and $\mathbf{w}_{MPTP_k}^{(i-1)} = \left\{\mathbf{x}_{MPTP_k}^{(i-1)}, \mathbf{z}_{MPTP_k}^{(i-1)}\right\}$ represents the MPTP of the constraint k at cycle $i-1$, which is obtained through an inverse FORM analysis.

$$\mathbf{d}^* = \arg\min_{\mathbf{d}\in\mathbb{D}} \mathbf{c}(\mathbf{d}) \; subject\ to \begin{cases} f_j(\mathbf{d}) \leq 0, & j = \{1,\ldots,s\} \\ g_k(\mathbf{d} - \mathbf{s}_k^{(i)}, \mathbf{z}_k^{(i-1)}) \geq 0, & k = \{1,\ldots,n\} \end{cases}, \tag{14}$$

3. JEFFCOTT ROTOR DYNAMICS

This section presents the dynamic modeling of the Jeffcott rotor with a non-central disc and flexible bearings, obtaining its dynamic response equations. According to Krämer (1993), this model has a non-central or outboard disc, which, together with other factors such as disc displacement, rotation around a radial axis, moment of inertia, and gyroscope effect, brings it closer to a real rotor. The rotor shaft has a circular cross-section, and its angular velocity remains constant and equal to Ω. The rotor experiences unbalanced vibration due to an unbalanced force resulting from the eccentricity of the center of mass and an unbalanced moment arising when the disc is not mounted perpendicular to the shaft's axis of rotation. This model is shown in Fig. 1 and has four degrees of freedom: the displacements x_1 and x_2 and the rotations ϕ_3 and ϕ_4. Table 1 shows the stochastic parameter values for this model and Tab. 2 displays the values of the parameters modeled as deterministic, as their variation was minor and had negligible impact on the model. Its equations of motion are shown in Eq. (15), where $m = \rho\pi(\frac{4R^2-D^2}{4})w$ is the rotor's mass, $I_d = \frac{m}{4}(\frac{4R^2-D^2}{4} + \frac{w^2}{3})$ is the rotor disc's diametral moment of inertia, $I_P = \frac{m(4R^2-D^2)}{8}$ is its polar moment of inertia, $F_U = me\Omega^2$ and $M_U = (I_d - I_P)\gamma\Omega^2$ are the unbalance force and moment, respectively, $g = 9.81$ m/s² is the gravity acceleration, and $k_{ij},\ i,j = \{1,\ldots,4\}$ are the model's force influence coefficients presented in Krämer (1993), where $a = \frac{2}{3}l$ and $b = l - a$ define the disc's position.

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & 0 & 0 & I_d \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\phi}_3 \\ \ddot{\phi}_4 \end{bmatrix} + \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & \hat{d} & I_p\Omega \\ 0 & 0 & -I_p\Omega & \hat{d} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \end{bmatrix} + \begin{bmatrix} k_{11} & 0 & 0 & k_{14} \\ 0 & k_{22} & k_{23} & 0 \\ 0 & k_{23} & k_{33} & 0 \\ k_{14} & 0 & 0 & k_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} F_U\cos(\Omega t) \\ F_U\sin(\Omega t) - mg \\ M_U\cos(\Omega t + \beta) \\ M_U\sin(\Omega t + \beta) \end{bmatrix}, \tag{15}$$

Considering the complex displacement $z = x_1 + jx_2$ and rotation $\psi = \phi_4 - j\phi_3$, Eq. (15) is rewritten as Eq. (16), where $k_{m_1} = \frac{k_{11}+k_{22}}{2}$, $k_{m_2} = \frac{k_{44}+k_{33}}{2}$, $k_{m_3} = k_{m_4} = \frac{k_{14}+k_{23}}{2}$, $k_{d_1} = \frac{k_{11}-k_{22}}{2}$, $k_{d_2} = \frac{k_{44}-k_{33}}{2}$, and $k_{d_3} = k_{d_4} = \frac{k_{14}-k_{23}}{2}$.

$$\begin{bmatrix} m & 0 \\ 0 & I_d \end{bmatrix} \begin{bmatrix} \ddot{z} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} d & 0 \\ 0 & (\hat{d} - jI_p\Omega) \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} k_{m_1} & k_{d_3} \\ k_{d_4} & k_{m_2} \end{bmatrix} \begin{bmatrix} z \\ \psi \end{bmatrix} + \begin{bmatrix} k_{d_1} & k_{m_3} \\ k_{m_4} & k_{d_2} \end{bmatrix} \begin{bmatrix} \bar{z} \\ \bar{\psi} \end{bmatrix} = \begin{bmatrix} F_U e^{j\Omega t} - jmg \\ -jM_U e^{j(\Omega t+\beta)} \end{bmatrix}, \tag{16}$$

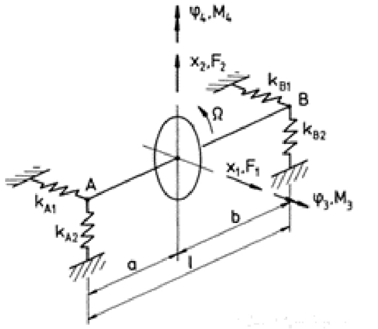

Figure 1. Jeffcott rotor with a non-central disc and flexible bearings (Krämer, 1993).

Table 1. Distribution of stochastic design parameters for the Jeffcott rotor with a non-central disc.

Parameter	**Distribution**	**Mean value**	**Standard deviation**
Elastic modulus of the rotor material E, GPa	gaussian	200	10
Density of the rotor material ρ, kg/m³	gamma	7850	392.5
Center of mass eccentricity e, mm	gamma	1	0.02
Bearing stiffness k_{A_1}, kN/m	gaussian	100	5
Bearing stiffness k_{A_2}, kN/m	gaussian	20	1
Bearing stiffness k_{B_1}, kN/m	gaussian	100	5
Bearing stiffness k_{B_2}, kN/m	gaussian	20	1
Disc inclination γ, º	uniform	0	$\frac{180}{\sqrt{3}}$
Center of mass inclination β, º	uniform	0	$\frac{180}{\sqrt{3}}$

Table 2. Values of deterministic parameters for the Jeffcott rotor with a non-central disc.

Parameter	**Value**
Disc radius R, mm	45
Disc thickness w, mm	20
Shaft length l, mm	800
Damping d, N·s/m	1.5
Rotational damping $\hat{d}$, Nm·s/rad	1.5
Angular velocity Ω, Hz	10

In this paper, only the particular solution of Eq. (16) is considered, assuming that the damping of the system causes its transient response to rapidly decrease until it reaches steady state. Therefore, solutions similar to those shown in Eq. (17) can be considered. Their coefficients are obtained using symbolic computation in MATLAB$^{\text{TM}}$.

$$z(t) = Z_0 + Z_1 e^{j\Omega t} + Z_2 e^{-j\Omega t} \qquad \psi(t) = \Psi_0 + \Psi_1 e^{j\Omega t} + \Psi_2 e^{-j\Omega t}, \tag{17}$$

The rotor's vibration in directions 1 and 2 is given by Eq. (18), while the global vibration is given by Eq. (19).

$$x_1(t) = Re\left[z(t)\right] \qquad x_2(t) = Im\left[z(t)\right], \tag{18}$$

$$x(t) = \sqrt{x_1(t)^2 + x_2(t)^2}, \tag{19}$$

4. FORMULATION OF AN RBDO PROBLEM FOR THE JEFFCOTT ROTOR

The RBDO problem for the Jeffcott rotor involves optimizing the diameter D of its shaft to achieve the smallest possible value while maintaining a 99% reliability for the design, ensuring that the rotor's maximum vibration does not exceed a limit of 1 mm. Thus, the RBDO problem for the Jeffcott rotor can be formulated by Eq. (20), where $D_{min} = 10$ mm and $D_{max} = 20$ mm are the minimum and maximum values for the shaft's diameter range, respectively, x is the maximum rotor vibration, x_{lim} is the vibration limit, $\mathbf{d} = \{D\}$ is the deterministic design parameter, and $\mathbf{Z}$ are the random environmental variables defined by the model. Since there are no geometric constraints on this rotor, its diameter can achieve any positive value that is consistent with the model, and the flexible constraints $f_j(\mathbf{d})$ can be ignored.

$$\mathbf{d}^* = \underset{\mathbf{d}\in[D_{min},D_{max}]}{\arg\min} \quad \mathbf{c}(\mathbf{d}) = \frac{\pi D^2}{4} \; subject\ to\ \mathbb{P}(g(\mathbf{d},\mathbf{Z}) = x_{lim} - x) \leq \bar{P}_f = 0.01, \tag{20}$$

5. RESULTS AND DISCUSSIONS

This section presents the RBDO results for the Jeffcott rotor with a non-central disc and flexible bearings. Table 3 displays the optimized rotor shaft's diameter values for each approach along with the corresponding optimization convergence criterion that was satisfied.

Table 3. RBDO results for the Jeffcott rotor.

Approach	Diameter, mm	Evaluations	Convergence criterion
RIA	12.0000	14460	No feasible solution found
PMA	15.6612	25000	Change in value less than threshold
SLA	11.0191	45	No feasible solution found
SORA	16.5064	9630	Change in value less than threshold
TL	12.7066	4140000	Value of global step size below tolerance
QMC	12.6677	3450000	Value of global step size below tolerance

After analyzing Tab. 3, it is evident that all RBDO approaches, except for the RIA and SLA approaches, successfully obtained optimized diameters of the Jeffcott rotor with a non-central disc and flexible bearings, meeting the optimization algorithm's convergence criterion, which indicates that conventional RBDO approaches, as implemented in UQLab^{TM}, can be applied for optimizing simpler stochastic rotor models while ensuring their reliability. However, there is a significant variation in the results obtained by each approach, suggesting that some may have found local optima while others found global optima for the problem. So, to verify each case accurately, it is important to analyze the convergence curves of the cost and constraint functions for each approach, which are presented between Fig. 2 and Fig. 7.

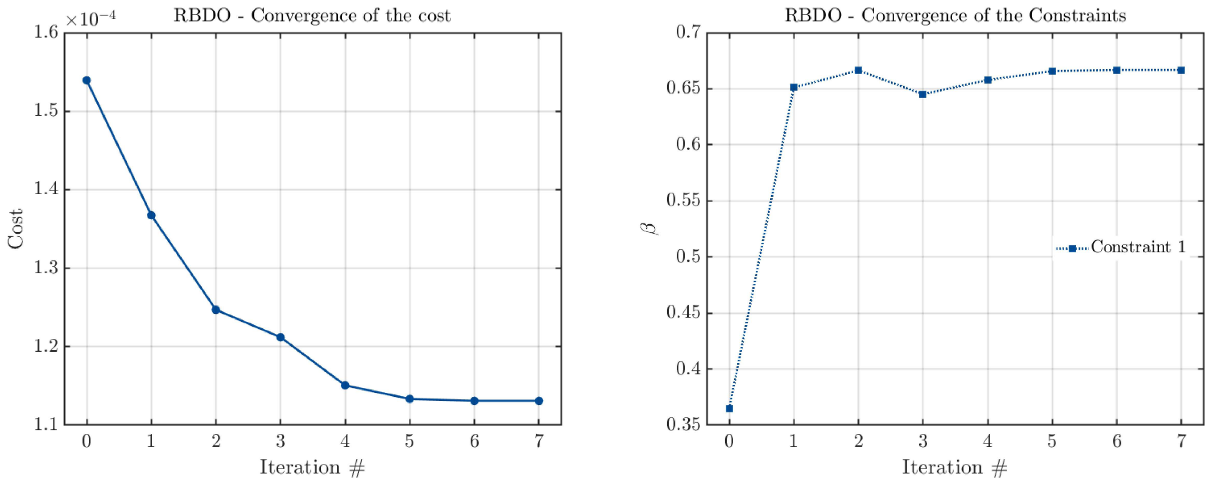

Figure 2. Convergence curves for the reliability index approach (RIA).

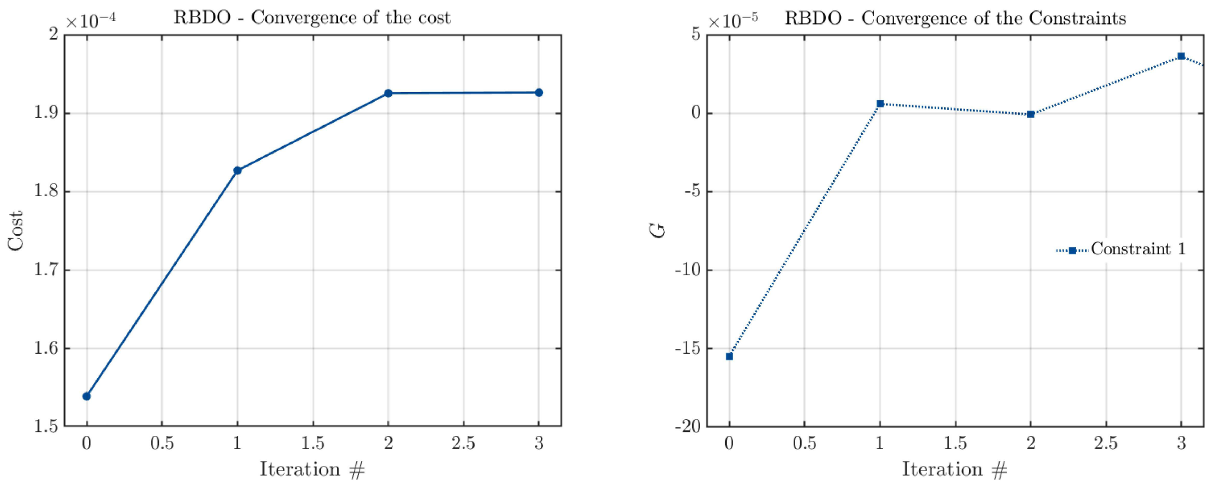

Figure 3. Convergence curves for the performance measure approach (PMA).

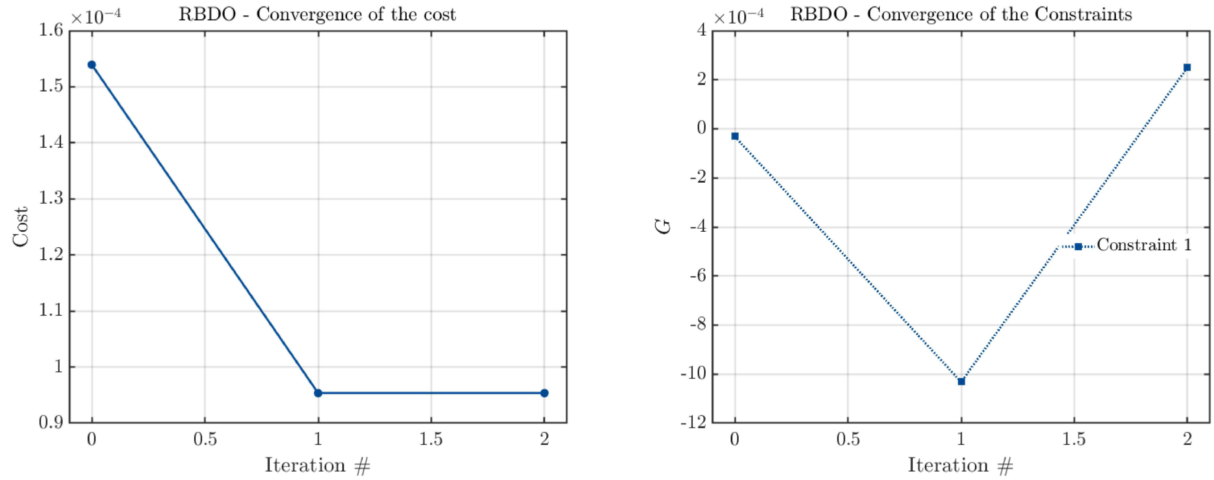

Figure 4. Convergence curves for the single loop approach (SLA).

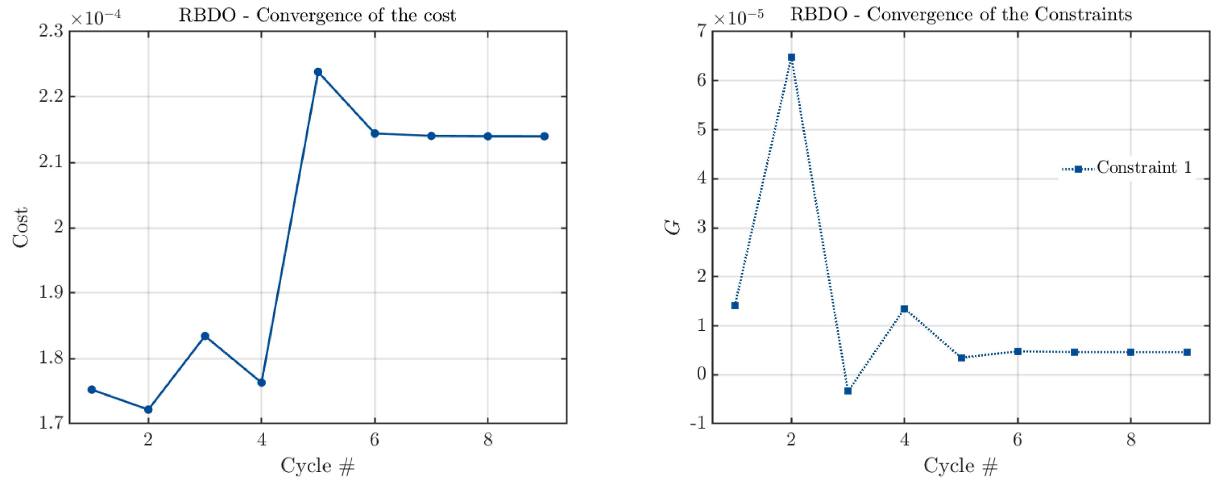

Figure 5. Convergence curves for the sequential optimization and reliability assessment approach (SORA).

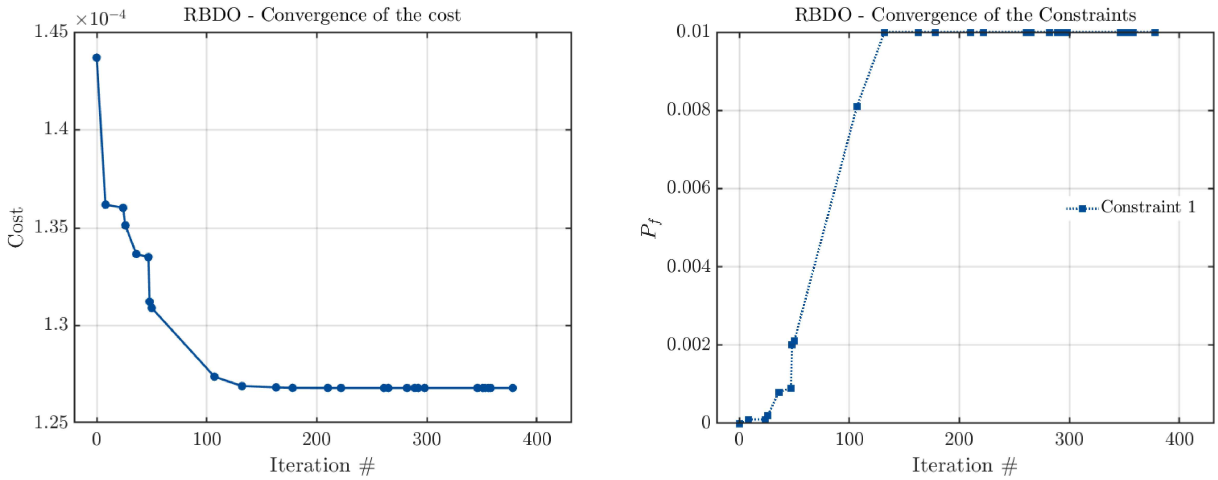

Figure 6. Convergence curves for the two-loop approach (TL).

Considering a 1% failure probability for the model, it is expected that the RIA approach will achieve convergence when the constraints function reaches a reliability index value of approximately $\beta = 2.33$. However, Fig. 2 shows that the constraints function has reached a much smaller value than this threshold, indicating that the approach failed to reach a global optimum. Consequently, the results of this approach are less accurate, and the model's reliability in this case will be lower than 99%. This inconsistency is further supported by Tab. 3, which shows that this approach did not find a feasible solution for the optimization problem, which justifies its incoherent result.

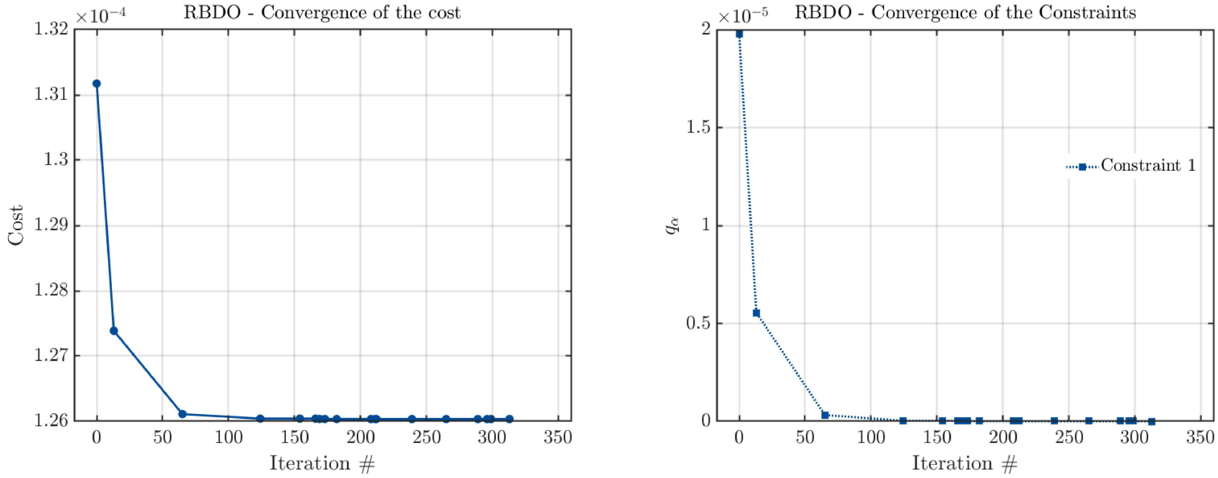

Figure 7. Convergence curves for the quantile Monte Carlo approach (QMC).

Analyzing the convergence curve of the constraints function for the PMA approach, as shown in Fig. 3, it can be seen that its convergence criterion was not fulfilled. The G value of the limit state function at the MPTP is greater than zero, indicating that this approach did not achieve a global optimum. Consequently, the results obtained through this approach will exhibit lower precision when compared to the results of other approaches that have fully converged, and the model's reliability in this case will be higher than 99%.

Tab. 3 reveals that the SLA approach also failed to produce a feasible result for the optimization problem, indicating that its results are inaccurate. Analyzing the convergence curve of the constraints function, as shown in Fig. 4, it becomes evident that convergence was not achieved, as the G value of the limit state function at the MPTP is not zero, justifying the inadequate result obtained by this approach.

Regarding the SORA approach, it is observed that its constraints function, as shown in Fig. 5, did not converge. The G value of the limit state function at the MPTP is greater than zero, suggesting that a local optimum for the rotor parameter was reached and, consequently, its optimized diameter value is not accurate enough. Given that the G value is greater than zero, it is expected that the reliability of the model in this case will also be higher than 99%.

Regarding the TL approach, its constraints function, as shown in Fig. 6, successfully converged. The final failure probability P_f matches the predefined acceptable value, indicating that the optimized rotor shaft's diameter obtained through this approach represents a global optimum and, consequently, a suitable value.

Regarding the QMC approach, the constraints function, as shown in Fig. 7, also successfully achieved convergence. As the algorithm execution progressed, the estimated quantile q_α tended to zero, indicating the attainment of a global optimum. This convergence is further corroborated by comparing the results obtained by the QMC approach with those of the TL approach, as both methods reached full convergence and yielded similar diameter's values.

These results are consistent with the characteristics of the approaches used. The RIA, PMA, SLA, and SORA approaches are only compatible with MATLAB™ local optimization algorithms, while the TL and QMC approaches allow the use of global optimization algorithms (UQLab, 2022). Consequently, the TL and QMC approaches are more suitable for working with rotating machines and can be applied for the optimization of more complex rotor models, as selecting approaches with worse convergence may lead to unsatisfactory results when dealing with more complex models.

Despite achieving satisfactory results, both TL and QMC approaches exhibited a significantly high number of evaluations for the model's cost function. These evaluations correspond to the iterations of the model during the optimization process. This observation is particularly noteworthy given the simplicity of the Jeffcott rotor, so that, for more complex models, such as those found in real-world rotors, the computational cost may become prohibitively high. Thus, it is crucial to develop customized implementations of these approaches to reduce their computational cost in future works. Additionally, it is important to explore advanced RBDO methods, particularly those that use surrogate models, as they are supposed to be better suited for more complex models and offer better convergence and faster execution (UQLab, 2022).

The mean value and dispersion of the Jeffcott rotor's orbits are presented in Fig. 8 and Fig. 9, respectively, for each optimized diameter value, enabling the examination of the diameter's influence on the vibration amplitude of the system. Figure 8 was plotted with $\beta = 60^o$ and $\gamma = 90^o$ to enhance the visibility of the orbits. Additionally, a circumference with a diameter of 1 mm is plotted in each case, representing the vibration limit defined for the project and allowing to verify if the calculated diameters will result in amplitudes exceeding this limit, ensuring the reliability of the model.

Figure 8 validates the modeling presented, as the circular or elliptical orbits align with the expected movement toward the rotor shaft's center. Circular orbits are expected for isotropic bearings, and elliptical orbits are expected for anisotropic bearings (Krämer, 1993), which is the case for this rotor.

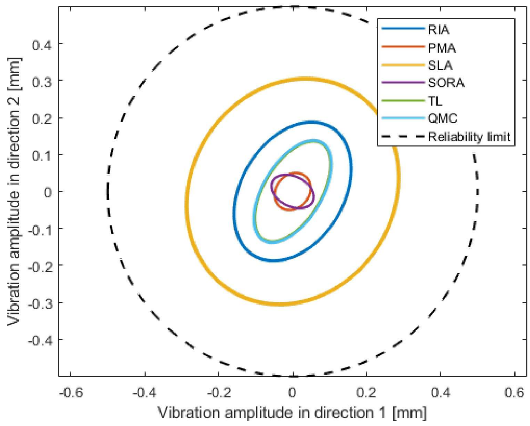

Figure 8. Jeffcott rotor orbits representing shaft displacement in the radial plane.

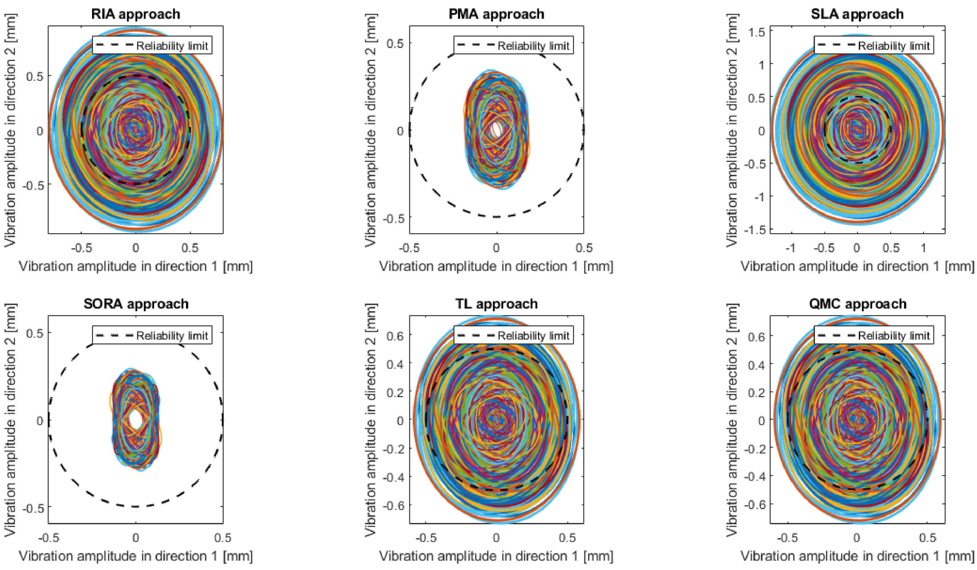

Figure 9. Dispersion of the Jeffcott rotor orbits for each RBDO approach considering 1000 samples generated.

From this figure, it can be seen that increasing the shaft's diameter led to reduction in the size of the resulting orbit, which is consistent with the mathematical relationship between vibration amplitude and shaft diameter. If there are cases where the diameter becomes excessively small and surpasses the limit defined for the model's failure probability, the resulting orbits will become excessively large and extend beyond the circumference, indicating that the model is not safe.

A consistent behavior can be observed in Fig. 9. The RIA and SLA approaches yielded very small diameters, leading their constraints functions to values within the system's failure state region. As a result, the majority of their orbits surpassed the reliability limit curve of the model. As the PMA and SORA approaches did not attain a global optimum for the rotor axis diameter but converged to a region within the system's safe state region, it is natural for none or only a small number of their orbits to exceed the reliability limit curve - since the calculated diameter is larger than what is necessary to ensure a 99% reliability level for the project. For the TL and QMC approaches, which converged, it is still expected that some of their orbits exceed the reliability limit of the model, since, as specified by the project, there remains a 1% probability of rotor failure under these operating conditions.

6. CONCLUSIONS

This paper presents a detailed analysis of the Reliability-Based Design Optimization (RBDO) method applied to the Jeffcott rotor with a non-central disc and flexible bearings. The RBDO methodology employed in this study follows the guidelines presented in the UQLab$^{\rm TM}$ user manual (UQLab, 2022). By validating this method on the Jeffcott rotor, which is the simplest existing rotor model, it can be applied in future research on more complex models that better represent the real-world operation of rotating machines.

The study showed that the traditional RBDO techniques can effectively perform stochastic optimization of the Jeffcott rotor, with the TL and QMC approaches providing excellent results, and the other approaches, except for the RIA and SLA approaches, finding at least local optimum points for the rotor diameter. As the TL and QMC approaches achieved good convergence for a simpler model, they can be applied in future works that will study more complex rotor models. However, these approaches had a high computational cost considering the simplicity of the Jeffcott rotor. This raises questions regarding the feasibility of using these approaches to optimize more complex rotor models that require numerical modeling using methods such as Finite Element, as the computational cost for these cases may become too high to be practical.

Therefore, further research is needed, and future studies should aim to develop customized implementations of this method to reduce its computational cost. Additionally, studies should explore the application of surrogate models, such as Kriging and polynomial chaos expansion metamodels, which can be used in conjunction with traditional RBDO techniques to obtain better convergence of results and lower computational costs when studying more complex models of rotating machines, making it possible to apply the RBDO method to real-world problems.

7. ACKNOWLEDGEMENTS

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

8. REFERENCES

Cafeo, J.A. and Thomas, R.S., 1997. “The use of statistical tools to contend with variability in engineering”. *Experimental Techniques*, Vol. 21, pp. 26–28.

Cho, T.M. and Lee, B.C., 2011. “Reliability-based design optimization using convex linearization and sequential optimization and reliability assessment method”. *Structural Safety*, Vol. 33, pp. 42–50.

Du, X. and Chen, W., 2004. “Sequential optimization and reliability assessment method for efficient probabilistic design”. *Journal of Mechanical Design*, Vol. 126, pp. 225–233.

Krämer, E., 1993. *Dynamics of rotors and foundations*. Editora Springer Berlin, Heidelberg.

Liang, J., Mourelatos, Z. and Tu, J., 2008. “A single-loop method for reliability-based design optimization”. *International Journal of Product Development*, Vol. 5, pp. 76–92.

Marelli, S. and Sudret, B., 2014. “Uqlab: A framework for uncertainty quantification in matlab”. In *Proceedings of the 2nd International Conference on Vulnerability and Risk Analysis and Management - ICVRAM 2014*. ASCE - American Society of Civil Engineers, Liverpool, United Kingdom.

Natarajan, K. and Santhakumar, A.R., 1995. “Reliability-based optimization of transmission line towers”. *Computers & Structures*, Vol. 55, pp. 387–403.

Rackwitz, R. and Flessler, B., 1978. “Structural reliability under combined random load sequences”. *Computers & Structures*, Vol. 9, pp. 489–494.

UQLab, 2022. “Uqlab user manual – reliability-based design optimization, report uqlab-v2.0-115”. Chair of Risk, Safety and Uncertainty Quantification, ETH Zurich, Switzerland, https://www.uqlab.com/rbdo-user-manual. Accessed 02 March 2023.

Visnadi, L.B., 2018. *Uncertainty analysis in the stability limit in rotors supported by cylindrical hydrodynamic bearings (in Portuguese)*. Master’s thesis, Graduate Program in Mechanical Engineering, State University of Campinas, Campinas, Brasil.

Youn, B.D., Choi, K.K. and Du, L., 2005. “Enriched performance measure approach for reliability-based design optimization”. *AIAA Journal*, Vol. 43, pp. 874–884.

9. RESPONSIBILITY NOTICE

The authors are solely responsible for the printed material included in this paper.