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A SINGULAR CONSTITUTIVE RELATION FOR SATURATED/UNSATURATED FLOWS THROUGH POROUS MEDIA

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Abstract. *This work uses a mixture theory framework to describe a porous medium filling up or emptying process by a fluid, able to describe the saturated-unsaturated transition and vice-versa and to use the same mathematical tool for treating unsaturated and saturated flows. The mixture consists of three overlapping continuous constituents: a solid (rigid porous matrix), a liquid (incompressible fluid), and a gas with very low mass density, accounting for the mixture's compressibility. This paper proposes a new constitutive relation for pressure as a function of saturation, ensuring the system remains hyperbolic even when the flow becomes saturated and allowing a closed solution to the Riemann problem associated with the flow. The complete solution to the associated Riemann problem is presented in detail, allowing an explicit function relationship involving the saturation, the eigenvalues, and Riemann invariants, independent of velocity. Some graphs for saturation and velocity for selected time instants highlight the proposed constitutive relation advantages, especially when the states are connected by two shocks.*

Keywords: *nonlinear hyperbolic systems, constitutive relation for pressure, flow through unsaturated porous media, shock waves.*

1. INTRODUCTION

Fluid flows through unsaturated porous media deserve many studies in the literature and can model relevant phenomena such as groundwater flows (including contamination problems) or enhanced oil recovery under a Continuous Mixture Theory viewpoint; the mechanical model considers a mixture of three superimposed continuous constituents occupying the whole mixture volume (Atkin and Craine, 1976; Rajagopal and Tao, 1995): one solid constituent, represents the porous matrix, is supposed to be rigid, isotropic, and homogenous, one liquid, an incompressible fluid, and one gaseous constituent with, by hypothesis, extremely low mass density, being included to account for the mixture compressibility. Therefore, it suffices to solve the equations of motion for the liquid constituent of the mixture. A mathematical description of the problem is obtained by combining the balance equations with appropriate constitutive assumptions, represented by a non-linear hyperbolic system of partial differential equations.

A sequence of previous works used a Mixture Theory approach to describe a porous matrix-filling process by a fluid, accounting for the transition from unsaturated to saturated flow. Saldanha da Gama (2005) proposed a constitutive relation for the pressure to account for the upper bound arising from the rigid porous medium and the fluid incompressibility assumptions preventing physically meaningless solutions. Martins-Costa and Saldanha da Gama (2011) proposed an improvement in the equation that considers the geometrical bound for the fluid fraction in a convenient neighborhood of the porosity. Saldanha da Gama et al. (2012) imposed a unilateral geometrical constraint to ensure that the maximum value reached by the fluid fraction is the porosity identifying the transition from unsaturated to saturated flow – the saturation upper bound (a physical constraint). Martins-Costa et al. (2017) proposed a continuous nondifferentiable function of the fluid fraction relation for the pressure, allowing a very small supersaturation of the porous matrix so that the problem remains hyperbolic when the porous matrix becomes saturated. They use the complete solution of the associated Riemann problem to build a numerical solution employing Glimm's scheme associated with an operator splitting technique. Martins-Costa et al. (2019) proposed a continuous and differentiable equation for pressure, its first derivative being an increasing function so that the fluid fraction can surpass the porosity solely by a (previously chosen) small value. A minimal supersaturation (controlled) is allowed, the problem hyperbolicity is maintained regardless of the fluid fraction value, and they present closed form expressions for the Riemann invariants.

This work studies the transient one-dimensional fluid flow through a rigid porous matrix, with unsaturated/saturated transition, originating a non-linear hyperbolic system of equations, which has as unknowns the velocity and saturation of the fluid constituent. The paper proposes a new constitutive relation for pressure as a function of the saturation due to the strong dependency of fluid motion on saturation in flows through unsaturated porous media, ensuring that the system remains hyperbolic even when saturation is reached, which gives rise to a straightforward associated Riemann problem, with velocity and saturation as unknowns. Its complete (exact) closed solution is presented.

The Riemann invariants associated with the eigenvalues are explicit functions of the eigenvalues, the first derivative of the pressure and the saturation: an exact, closed formula. It is important to note that the velocity is not present in these invariants. The proposed constitutive relation also allows an explicit function for the saturation as a function of the eigenvalues. In summary, the mathematical problem is described by a non-linear hyperbolic system of equations enabling treating the problem with the same mathematical tool for saturated and unsaturated flows.

2. MECHANICAL MODEL

Considering a chemically non-reacting mixture with no phase change of a rigid solid constituent at rest, a liquid constituent – from now on denoted as a fluid constituent (a Newtonian liquid) and a very low mass density gas, included to account for the compressibility of the mixture as a whole – mass and momentum balance equations are to be solved solely for the fluid constituent (Atkin and Craine, 1976; Rajagopal and Tao, 1995) being given by:

$$\begin{aligned} &\frac{\partial \rho_F}{\partial t} + \operatorname{div}\left(\rho_F \mathbf{v}_F\right) = 0 \\ &\rho_F \left[\frac{\partial \mathbf{v}_F}{\partial t} + \left(\nabla \mathbf{v}_F\right)\mathbf{v}_F\right] = \operatorname{div}\mathbf{T}_F + \rho_F \mathbf{b}_F + \mathbf{m}_F \end{aligned} \tag{1}$$

where ρ_F is the fluid constituent mass density (local ratio between fluid constituent mass and corresponding mixture volume), $\mathbf{v}_F$ is the fluid constituent velocity in the mixture, $\mathbf{T}_F$ represents the partial stress tensor associated with the fluid constituent, $\mathbf{b}_F$ is the body force (per unit mass), and $\mathbf{m}_F$ is the momentum supply acting on the fluid constituent due to its interaction with the remaining mixture constituents. The fluid fraction is the ratio between the fluid constituent mass density ρ_F and the fluid mass density ρ_f, ($\phi = \rho_F / \rho_f$), while $\boldsymbol{\varepsilon}$ denotes the porous matrix porosity.

Neglecting the Darcian term (Srinivasan and Rajagopal, 2014), the momentum supply to the fluid constituent is:

$$\mathbf{m}_F = -\frac{\mu_f D}{K}\nabla\phi \tag{2}$$

where μ_f is the fluid viscosity and K represents the specific permeability of the porous matrix (both in a Continuum Mechanics approach), and D is a diffusion coefficient.

The partial stress tensor is analogous to the Cauchy tensor in Continuum Mechanics, in this work, assumed proportional to the pressure acting on the fluid constituent (Williams, 1978; Saldanha da Gama, 2005):

$$\mathbf{T}_F = -\phi p \mathbf{1} \tag{3}$$

where $p = p(\phi)$ is a pressure and $\mathbf{1}$ is the identity tensor.

Assuming a one-dimensional description, all the quantities depend solely on time t and on position x and the only nonvanishing component of the fluid constituent velocity, $\mathbf{v}_F$, denoted by v. Defining a pressure $\bar{p} = \hat{p}(\phi)$

$$\bar{p} = \frac{1}{\phi_f}\left(p + \frac{\mu_f D}{K}\right) \tag{4}$$

and assuming that body forces can be neglected, equations (1)-(4) give rise to the following system:

$$\begin{aligned} &\frac{\partial}{\partial t}\phi + \frac{\partial}{\partial x}(\phi v) = 0 \\ &\frac{\partial}{\partial t}(\phi v) + \frac{\partial}{\partial x}\left(\phi v^2 + \bar{p}\right) = 0 \end{aligned} \tag{5}$$

3. CONSTITUTIVE RELATION

The problem presented in Eq. (5) may be rewritten as a function of the saturation $\psi \equiv \phi / \varepsilon$, conveniently renaming the pressure as p, and presenting the proposed constitutive relation for the pressure as a function of the saturation. This gives rise to the following nonlinear hyperbolic system of partial differential equations

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(\psi v) = 0 \\ &\frac{\partial}{\partial t}(\psi v) + \frac{\partial}{\partial x}(\psi v^2 + p) = 0 \end{aligned} \qquad \text{Proposed Constitutive Relation:} \quad p = \hat{p}(\psi) = \frac{\psi^3}{3(1-\psi)^3} \tag{6}$$

in which ψ must always be positively valued.

Problem (6) may be rewritten in a more appropriate form, as

$$\frac{\partial}{\partial t}\begin{bmatrix} \psi \\ \psi v \end{bmatrix} + \mathbf{A}\frac{\partial}{\partial x}\begin{bmatrix} \psi \\ \psi v \end{bmatrix} = 0, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ p' - v^2 & 2v \end{bmatrix} \tag{7}$$

4. RIEMANN PROBLEM SOLUTION

Consider the Cauchy problem

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(\psi v) = 0 \\ &\frac{\partial}{\partial t}(\psi v) + \frac{\partial}{\partial x}(\psi v^2 + p) = 0 \end{aligned} \qquad (\psi, v) = \begin{cases} (\psi_L, v_L), & -\infty < x < 0, \quad t = 0 \\ (\psi_R, v_R), & 0 < x < \infty, \quad t = 0 \end{cases} \tag{8}$$

in which ψ_L and ψ_R are positively valued constants. Problem (8) is called a Riemann Problem, and its solution (in a generalized sense) depends only on the ratio x/t (Smoller, 1983). In other words, (6) may be represented as

$$\begin{aligned} &\frac{d\psi}{d\eta}\frac{\partial \eta}{\partial t} + \frac{d}{d\eta}(\psi v)\frac{\partial \eta}{\partial x} = 0 \\ &\frac{d}{d\eta}(\psi v)\frac{\partial \eta}{\partial t} + \frac{d}{d\eta}(\psi v^2 + p)\frac{\partial \eta}{\partial x} = 0 \end{aligned} \tag{9}$$

in which the similarity variable is defined as

$$\eta = x/t \quad \Rightarrow \quad \frac{\partial \eta}{\partial t} = -\frac{1}{t}\eta; \quad \frac{\partial \eta}{\partial x} = \frac{1}{t} \tag{10}$$

In this way (7) becomes

$$\begin{bmatrix} -\eta & 1 \\ p' - v^2 & 2v - \eta \end{bmatrix} \frac{d}{d\eta}\begin{bmatrix} \psi \\ \psi v \end{bmatrix} = 0 \tag{11}$$

To satisfy the system (7), either the saturation ψ and the velocity v must be constants or $\eta = \lambda$ ($\lambda \rightarrow$ eigenvalue of $\mathbf{A}$ and $\left(d\psi, d(\psi v)\right) \rightarrow$ eigenvector).

The eigenvalues of the matrix $\mathbf{A}$ are given by (in increasing order)

$$\lambda_1 = v - \sqrt{p'} \quad \text{and} \quad \lambda_2 = v + \sqrt{p'} \tag{12}$$

while the associated eigenvectors are obtained from

$$-\lambda_1 d\psi + d(\psi v) = 0 \quad \text{and} \quad -\lambda_2 d\psi + d(\psi v) = 0 \tag{13}$$

Taking into account Eqs. (12) and (13) the following results are obtained

$$-\lambda_1 d\psi + d(\psi v) = 0 \;\Rightarrow\; \int \frac{\sqrt{p'}}{\psi} d\psi + v = constant \quad (associated\ to\ \lambda_1) \tag{14}$$

$$-\lambda_2 d\psi + d(\psi v) = 0 \;\Rightarrow\; -\int \frac{\sqrt{p'}}{\psi} d\psi + v = constant \quad (associated\ to\ \lambda_2) \tag{15}$$

or, in an alternative form, in which the following Riemann invariants are explicit functions of the eigenvalues, the first derivative of the pressure and the saturation ψ. It is important to note that the velocity v is absent in these invariants.

$$\begin{aligned} &\left(\int \frac{\sqrt{p'}}{\psi} d\psi + \sqrt{p'}\right) + \lambda_1 = constant \quad (associated\ to\ \lambda_1) \\ -&\left(\int \frac{\sqrt{p'}}{\psi} d\psi + \sqrt{p'}\right) + \lambda_2 = constant \quad (associated\ to\ \lambda_2) \end{aligned} \tag{16}$$

The considered relationship between the pressure and the saturation ψ ensures that $2p' + \psi p'' > 0$. Since $\eta = \lambda$, two states (ψ_1, v_1) and (ψ_2, v_2) are connected by a continuous function of η satisfying (14) if, and only if, ψ decreases as η increases (this connection is called a 1-rarefaction). On the other hand, two states (ψ_1, v_1) and (ψ_2, v_2) are connected by a continuous function of η satisfying (15) if, and only if, ψ increases as η increases (this connection is called a 2-rarefaction).

Now, returning to the problem (8), to connect the constant state (ψ_L, v_L) to the constant state (ψ_*, v_*), employing a continuous function of η, the following relation must be verified

$$\int_a^{\psi_L} \frac{\sqrt{p'}}{\psi} d\psi + v_L = \int_a^{\psi_*} \frac{\sqrt{p'}}{\psi} d\psi + v_*, \quad with \quad \psi_L > \psi_*, \quad 0 < a < 1 \tag{17}$$

On the other hand, to connect the constant state (ψ_R, v_R) to the constant state (ψ_*, v_*), using a continuous function of η, the following relation must be verified

$$-\int_a^{\psi_R} \frac{\sqrt{p'}}{\psi} d\psi + v_R = -\int_a^{\psi_*} \frac{\sqrt{p'}}{\psi} d\psi + v_*, \quad with \quad \psi_R > \psi_*, \quad 0 < a < 1 \tag{18}$$

So, in this case, ψ_* and v_* are obtained from

$$\begin{aligned} v_* &= \frac{v_L + v_R}{2} + \int_{\psi_R}^{\psi_L} \frac{\sqrt{p'}}{\psi} d\psi \\ \int_{\psi_L}^{\psi_*} \frac{\sqrt{p'}}{\psi} d\psi &= \frac{1}{2}\left(v_L - v_R + \int_{\psi_L}^{\psi_R} \frac{\sqrt{p'}}{\psi} d\psi\right) \end{aligned} \tag{19}$$

provided that the following necessary and sufficient condition (for ensuring the inequalities $\psi_L > \psi_*$ and $\psi_R > \psi_*$) is satisfied

$$v_R - v_L \geq \left| \int_{\psi_L}^{\psi_R} \frac{\sqrt{p'}}{\psi} d\psi \right| \tag{20}$$

Since $2p'+\psi p''>0$ and ψ must be positively valued, it is convenient to define the (invertible) functions G and F as

$$G=\hat{G}(\psi)=\int_a^{\psi}\frac{\sqrt{\hat{p}'(\xi)}}{\xi}d\xi+\frac{1}{1-a}=\frac{1}{1-\psi},\quad 0<a<1$$
$$F=\hat{F}(\psi)=\int_a^{\psi}\frac{\sqrt{\hat{p}'(\xi)}}{\xi}d\xi+\sqrt{\hat{p}'(\psi)}+\frac{1}{1-a}=\frac{1}{(1-\psi)^2},\quad 0<a<1 \tag{21}$$

Clearly

$$\psi=1-\frac{1}{\sqrt{F}},\quad \psi=1-\frac{1}{G} \tag{22}$$

If (20) holds, the solution is given by

$$\psi=\begin{cases}\psi_L, & -\infty<\eta\le\lambda_{1L}\\ 1-\left\{(1-\psi_L)^{-2}+\lambda_{1L}-\eta\right\}^{-1/2}, & \lambda_{1L}<\eta<\lambda_{1*}\\ \psi_*, & \lambda_{1*}\le\eta\le\lambda_{2*}\\ 1-\left\{(1-\psi_R)^{-2}-\lambda_{2R}+\eta\right\}^{-1/2}, & \lambda_{2*}<\eta<\lambda_{2R}\\ \psi_R, & \lambda_{2R}\le\eta<\infty\end{cases}$$
$$v=\begin{cases}v_L, & -\infty<\eta\le\lambda_{1L}\\ (1-\psi_L)^{-1}+v_L-\left\{(1-\psi_L)^{-2}+\lambda_{1L}-\eta\right\}^{1/2}, & \lambda_{1L}<\eta<\lambda_{1*}\\ v_*, & \lambda_{1*}\le\eta\le\lambda_{2*}\\ -(1-\psi_R)^{-1}+v_R+\left\{(1-\psi_R)^{-2}-\lambda_{2R}+\eta\right\}^{1/2}, & \lambda_{2*}<\eta<\lambda_{2R}\\ v_R, & \lambda_{2R}\le\eta<\infty\end{cases} \tag{23}$$

in which

$$\lambda_{1L}=v_L-\left[\sqrt{p'}\right]_L;\quad \lambda_{1*}=v_*-\left[\sqrt{p'}\right]_*$$
$$\lambda_{2*}=v_*+\left[\sqrt{p'}\right]_*;\quad \lambda_{2R}=v_R+\left[\sqrt{p'}\right]_R \tag{24}$$

The problem is not restricted to the initial data satisfying (20). So, the space of admissible solutions must be enlarged, allowing not only continuous solutions but also discontinuous ones.

These discontinuous solutions must satisfy the Rankine-Hugoniot jump conditions and the entropy conditions (Smoller, 1983). The jump conditions associated with (6) are given by

$$\frac{[\psi v]}{[\psi]}=\frac{[\psi v^2+p]}{[\psi v]}=s \tag{25}$$

in which $[\bullet]$ denotes the "jump" of "$\bullet$".

So, two states (ψ_1,v_1) and (ψ_2,v_2), connected by a shock, must satisfy

$$\frac{\psi_2v_2-\psi_1v_1}{\psi_2-\psi_1}=\frac{\psi_2v_2^2-\psi_1v_1^2+p_2-p_1}{\psi_2v_2-\psi_1v_1}=s \tag{26}$$

in which s is the shock speed, and $\eta = s$ is the shock position. To satisfy the entropy conditions, two states will be connected by a shock if, and only if, they cannot be connected by a continuous function (called a rarefaction).

The left state (ψ_L, v_L) will be connected to the constant state (ψ_*, v_*), through a shock if $\psi_L < \psi_*$. This shock is called a 1-shock. The relationship between these two states is given by

$$v_* - v_L = -\sqrt{\left(\frac{1}{\psi_L} - \frac{1}{\psi_*}\right)(p_* - p_L)}, \quad \psi_L < \psi_* \tag{27}$$

while the speed of the 1-shock is given by

$$s_1 = \frac{\psi_* v_* - \psi_L v_L}{\psi_* - \psi_L} \tag{28}$$

The same protocol may be used to connect the constant state (ψ_*, v_*) to the right state through a shock called a 2-shock. If $\psi_R < \psi_*$, then the states (ψ_*, v_*) and (ψ_R, v_R) will be such that

$$v_* - v_R = \sqrt{\left(\frac{1}{\psi_*} - \frac{1}{\psi_R}\right)(p_R - p_*)}, \quad \psi_R < \psi_* \tag{29}$$

while the speed of the 2-shock is given by

$$s_2 = \frac{\psi_R v_R - \psi_* v_*}{\psi_R - \psi_*} \tag{30}$$

The possible solutions in a generalized sense for (8) are shown in Table 1.

Table 1. Riemann problem possible solutions: conditions for saturation and velocity.

$\psi_L > \psi_* < \psi_R$	$\rightarrow$	$1-rarefaction\,/\,2-rarefaction$	$\rightarrow$	$v_L < v_* < v_R$
$\psi_L < \psi_* > \psi_R$	$\rightarrow$	$1-shock\,/\,2-shock$	$\rightarrow$	$v_L > v_* > v_R$
$\psi_L > \psi_* > \psi_R$	$\rightarrow$	$1-rarefaction\,/\,2-shock$	$\rightarrow$	$v_L < v_* > v_R$
$\psi_L < \psi_* < \psi_R$	$\rightarrow$	$1-shock\,/\,2-rarefaction$	$\rightarrow$	$v_L > v_* < v_R$

that means

Table 2. An a priori choice for the Riemann problem solution.

$v_R - v_L \geq \left\lvert \int_{\psi_L}^{\psi_R} \frac{\sqrt{p'}}{\phi} d\phi \right\rvert \Leftrightarrow 1-Rarefaction\,/\,2-Rarefaction$
$v_R - v_L \leq -\sqrt{\left(\frac{1}{\psi_R} - \frac{1}{\psi_L}\right)(p_L - p_R)} \Leftrightarrow 1-Shock\,/\,2-Shock$
$-\sqrt{\left(\frac{1}{\psi_R} - \frac{1}{\psi_L}\right)(p_L - p_R)} < v_R - v_L < \left\lvert \int_{\psi_L}^{\psi_R} \frac{\sqrt{p'}}{\phi} d\phi \right\rvert, \psi_L > \psi_R \Leftrightarrow 1-Rarefaction\,/\,2-Shock$
$-\sqrt{\left(\frac{1}{\psi_R} - \frac{1}{\psi_L}\right)(p_L - p_R)} < v_R - v_L < \left\lvert \int_{\psi_L}^{\psi_R} \frac{\sqrt{p'}}{\phi} d\phi \right\rvert, \psi_L < \psi_R \Leftrightarrow 1-Shock\,/\,2-Rarefaction$

It should be noticed that when the intermediate state (ψ_*, v_*) coincides with the left state (ψ_L, v_L) the solution will be 2-rarefaction or 2-shock. If the intermediate state (ψ_*, v_*) coincides with the right state, the solution will be 1-rarefaction or 1-shock.

The complete solution for 1-rarefaction/2-rarefaction is presented in (21). For 1-shock/2-shock, the solution is given by

$$\psi = \begin{cases} \psi_L, & -\infty < \eta < s_1 \\ \psi_*, & s_1 < \eta < s_2 \\ \psi_R, & s_2 < \eta < \infty \end{cases} \qquad v = \begin{cases} v_L, & -\infty < \eta < s_1 \\ v_*, & s_1 < \eta < s_2 \\ v_R, & s_2 < \eta < \infty \end{cases} \tag{31}$$

in which ψ_* and v_* are obtained from

$$v_* - v_L = -\sqrt{\left(\frac{1}{\psi_L} - \frac{1}{\psi_*}\right)(p_* - p_L)}, \quad \psi_L < \psi_* \qquad v_* - v_R = \sqrt{\left(\frac{1}{\psi_*} - \frac{1}{\psi_R}\right)(p_R - p_*)}, \quad \psi_R < \psi_* \tag{32}$$

For 1-rarefaction/2-shock, the solution is given by

$$\psi = \begin{cases} \psi_L, & -\infty < \eta \le \lambda_{1L} \\ 1 - \left\{(1-\psi_L)^{-2} + \lambda_{1L} - \eta\right\}^{-1/2}, & \lambda_{1L} < \eta < \lambda_{1*} \\ \psi_*, & \lambda_{1*} \le \eta < s_2 \\ \psi_R, & s_2 < \eta < \infty \end{cases} \qquad v = \begin{cases} v_L, & -\infty < \eta \le \lambda_{1L} \\ (1-\psi_L)^{-1} + v_L - \left\{(1-\psi_L)^{-2} + \lambda_{1L} - \eta\right\}^{1/2}, & \lambda_{1L} < \eta < \lambda_{1*} \\ v_*, & \lambda_{1*} \le \eta < s_2 \\ v_R, & s_2 < \eta < \infty \end{cases} \tag{33}$$

in which ψ_* and v_* are obtained from

$$v_* - v_L = \hat{G}(\psi_L) - \hat{G}(\psi_*), \quad \psi_L > \psi_* \qquad v_* - v_R = \sqrt{\left(\frac{1}{\psi_*} - \frac{1}{\psi_R}\right)(p_R - p_*)}, \quad \psi_R < \psi_* \tag{34}$$

For 1-shock/2-rarefaction, the solution is given by

$$\psi = \begin{cases} \psi_L, & -\infty < \eta < s_1 \\ \psi_*, & s_1 < \eta \le \lambda_{2*} \\ 1 - \left\{(1-\psi_R)^{-2} - \lambda_{2R} + \eta\right\}^{-1/2}, & \lambda_{2*} < \eta < \lambda_{2R} \\ \psi_R, & \lambda_{2R} \le \eta < \infty \end{cases} \qquad v = \begin{cases} v_L, & -\infty < \eta < s_1 \\ v_*, & s_1 < \eta \le \lambda_{2*} \\ -(1-\psi_R)^{-1} + v_R + \left\{(1-\psi_R)^{-2} - \lambda_{2R} + \eta\right\}^{1/2}, & \lambda_{2*} < \eta < \lambda_{2R} \\ v_R, & \lambda_{2R} \le \eta < \infty \end{cases} \tag{35}$$

in which ψ_* and v_* are obtained from

$$v_* - v_L = -\sqrt{\left(\frac{1}{\psi_L} - \frac{1}{\psi_*}\right)(p_* - p_L)}, \quad \psi_L < \psi_*$$
$$v_* - v_R = -\hat{G}(\psi_R) + \hat{G}(\psi_*), \quad \psi_R > \psi_* \tag{36}$$

5. RESULTS

The results presented in this section for saturation and velocity along position focus on the connection of left and right states by 1-shock/2-shock, when unrealistic results could occur with classical constitutive relations, to show the robustness of the proposed model for the pressure.

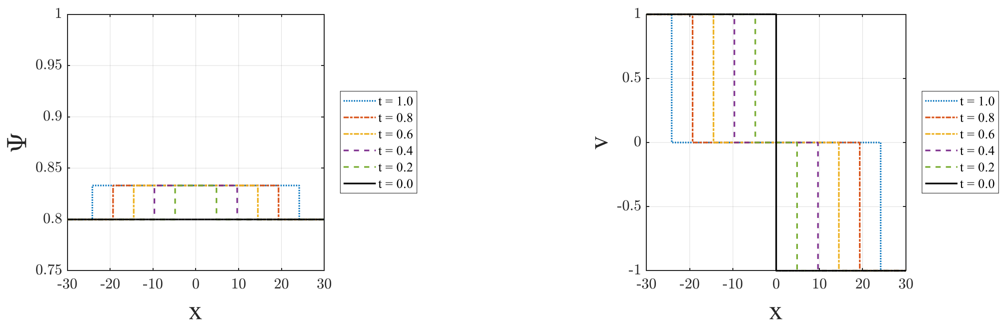


Figure 1. Saturation and velocity evolution for $v_L = 1$, $v_R = -1$, and $\psi_L = \psi_R = 0.80$.

Figures 1 to 3 impose a high initial saturation ($\psi_L = \psi_R = 0.80$) and a large range among the velocities. Figure 1 depicts the evolution of saturation and velocity with initial conditions given by a high saturation and low-velocity modulus. There is a very discrete increase in the intermediate saturation ($\psi_* = 0.83$) and zero intermediate velocity, as expected. Computing the pressure before the shock, a moderate increase is detected in the intermediate pressure ($p_* / p_L = p_* / p_R = 1.94$). In Figure 2, the same saturation is considered, with the velocity modulus ten times higher. The intermediate saturation reached $\psi_* = 0.92$, while for the same pressure before the shock, a higher intermediate pressure reached: $p_* / p_L = p_* / p_R = 28.83$.

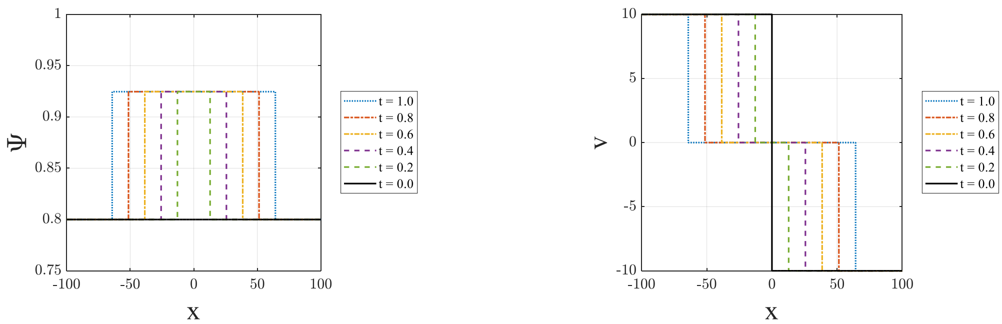


Figure 2. Saturation and velocity evolution for $v_L = 10$, $v_R = -10$, and $\psi_L = \psi_R = 0.80$.

Figure 3 depicts the evolution of saturation and velocity for a very high velocity modulus, giving rise to a very high intermediate saturation ($\psi_* = 0.98$) and an enormous intermediate pressure reached with $p_* / p_L = p_* / p_R = 2.03 \times 10^3$. This pressure may be sufficient to provoke fractures in the porous matrix. If the velocity modulus increases even more (for $v_L = 1000$ and $v_R = -1000$), the intermediate saturation reaches $\psi_* = 0.9957$, while the intermediate pressure is such that $p_* / p_L = p_* / p_R = 1.9 \times 10^5$. While this pressure is not attainable in a real porous matrix, the flow may be clearly considered a saturated flow.

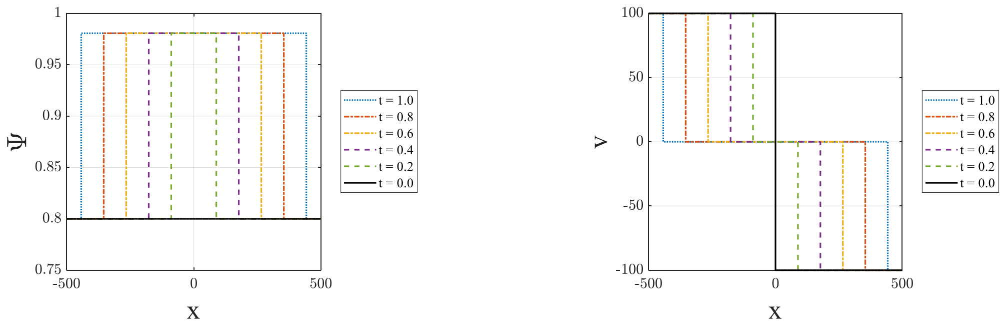


Figure 3. Saturation and velocity evolution for $v_L = 100$, $v_R = -100$, and $\psi_L = \psi_R = 0.80$.

Figures 4 to 6 consider a smaller saturation ($\psi_L = \psi_R = 0.40$) and, again a large range of velocities. The evolution of saturation and velocity shown in Figure 4 considers a moderate initial saturation, with half the value used in Figures 1 to 3 and the same low-velocity modulus used to obtain Figure 1 as initial velocities. In this case, the intermediate saturation increase is more substantial than in Figure 1, with $\psi_* = 0.61$. There is a meaningful increase in the intermediate pressure $p_* / p_L = p_* / p_R = 12.89$. Figure 5 depicts the behavior for the same initial saturation, while the initial velocities are ten times greater than in the previous figure. The velocity increase provoked an increase in intermediate saturation ($\psi_* = 0.85$) and intermediate pressure such that $p_* / p_L = p_* / p_R = 765$.

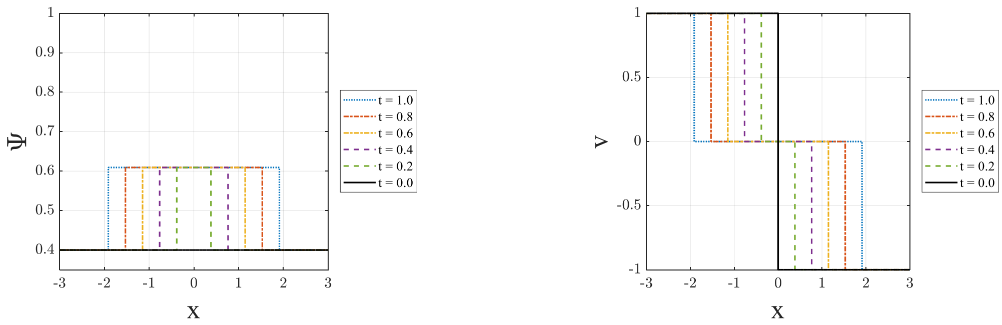


Figure 4. Saturation and velocity evolution for $v_L = 1$, $v_R = -1$, and $\psi_L = \psi_R = 0.40$.

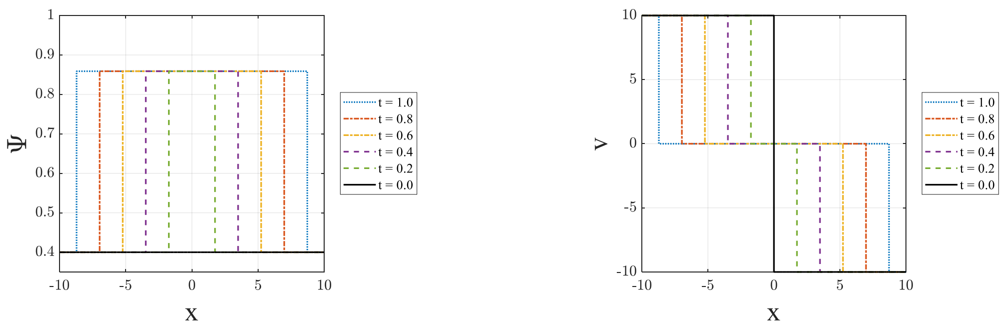
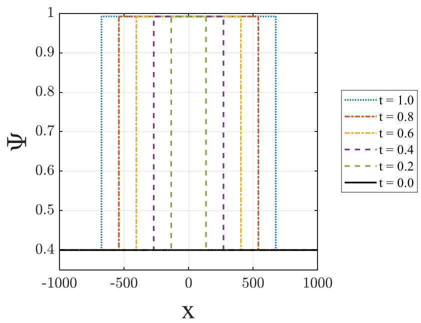


Figure 5. Saturation and velocity evolution for $v_L = 10$, $v_R = -10$, and $\psi_L = \psi_R = 0.40$.

Increasing ten times the velocities ($v_L = 100$ and $v_R = -100$) the intermediate saturation is $\psi_* = 0.96$ with a very high intermediate pressure $p_* / p_L = p_* / p_R = 6.9694 \times 10^4$. When the velocities are increased ten times again, as shown in Figure 6, saturated flow is reached ($\psi_* = 0.9921$) with $p_* / p_L = p_* / p_R = 6.837 \times 10^6$, a huge intermediate pressure.

All the considered problems reach very high intermediate saturation and very high intermediate pressure for very high initial velocity moduli.



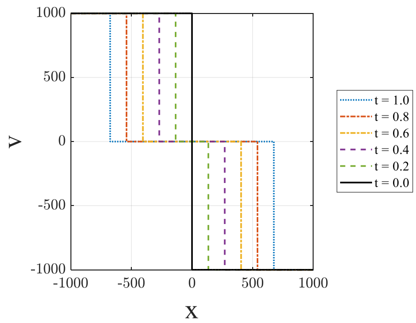


Figure 6. Saturation and velocity evolution for $v_L = 1000$, $v_R = -1000$, and $\psi_L = \psi_R = 0.40$.

6. FINAL REMARKS

This paper proposes a new constitutive relation for pressure as a function of saturation which maintains the problem of hyperbolicity and the porous matrix rigidity regardless of the saturation value. This proposed relation for pressure allows the simple, exact solution to the Riemann problem associated with the flow presented in the article. The Riemann invariants associated with the eigenvalues are explicit functions of the eigenvalues, the first derivative of the pressure and the saturation: an exact, closed formula. It is important to note that the velocity is not present in these invariants. The proposed constitutive relation also allows an explicit function for the saturation as a function of the eigenvalues.

7. ACKNOWLEDGEMENTS

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