



Bayesian Operational Modal Analysis with Likelihood Free Methods

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Abstract: This work presents the Bayesian Operational Modal Analysis (BAYOMA) used to estimate the modal parameters of a structure under operational conditions and the Approximate Bayesian Computation (ABC) used to approximate the posterior distribution when the likelihood function is not available in explicit form. A numerical example using BAYOMA in conjunction with the Laplace Approximation and the ABC to estimate the modal parameters is presented.

Keywords: OMA, Approximate Bayesian Computation, Rejecting Sampling, Uncertainty Quantification

INTRODUCTION

Modal identification has the objective of identifying the modal properties of a structure using vibration time history data. Structural system identification aims at identifying the structural properties. Modal and structural system identification are inter-related objectives, the general aim is to provide data that can be used to increase the accuracy of predictive models leading to optimized scenarios for control, optimization and decision making model accuracy (Au, 2017).

It is possible to obtain vibration data using a free vibration test, a forced vibration test or an ambient vibration test (Au, 2017). The advantage of the ambient vibration test is that the excitation is not measured and can be used in a structure under operational conditions (Reynders, 2012).

Operational Modal Analysis (OMA) uses ambient vibration data to identify the modal properties of a structure under the assumption that the excitation is broadband random (Au, 2017). Identification of modal properties of a structure is the process of correlating the dynamic characteristics of a mathematical model with the physical properties of the system derived from experimental measurements (Ventura and Brincker, 2015).

Commonly referred as 'non-Bayesian' approach, operational modal analysis based on the classical statistical approach is currently the conventional perspective on the field. The theory behind the 'non-Bayesian' or frequentist approach as well as multiple methods can be found in multiple works such as in Reynders (2012) and Ventura and Brincker (2015). In this work however, we are going to address the Bayesian approach, introduced by Yuen and Katafygiotis (2001) and (2003) and later developed throughout multiple works, for example (Au, 2017).

The Bayesian approach is based on the evaluation of a likelihood function, and it is one of the most popular to deal with model selection and parameter identification problems. The Bayesian approach requires the definition of a likelihood function to measure the level of agreement between the observed and the simulated data (Abdessalem et al, 2019). However in some circumstances the likelihood function is not available or is too costly to be evaluated. The Approximate Bayesian Computation (ABC) was conceived aiming at inferring posterior densities in these situations (Toni et al, 2009).

The main goal of this work is to propose a novel strategy for Operational Modal Analysis which is based on ABC. To the best of the author's knowledge, this strategy is novel and may deal with data variability stemming from different sources as well as can naturally deal with any violations of the premise that the excitation would be perfectly white. Our goal is to consider measurements of the structure under different conditions, similar approach used by Sedehi et al (2020), and use the ABC to compute a posterior distribution using all the measurements under different conditions.

MATHEMATICAL FORMULATION

In this section we will review the Bayesian Fast Fourier Transform (FFT) modal inference method for well-separated modes established by (Yuen and Katafygiotis 2003) and enhanced later by (Au, 2017), as well as the Approximate Bayesian Computation (ABC) rejecting sampling algorithm developed by Pritchard et al. (1999).

Bayesian Operational Modal Analysis

Let $\{\hat{\mathbf{y}}_j\}_{j=0}^{N-1}$ ($n \times 1$) be the ambient vibration time history data measured at n Degrees of Freedom, where N is the number of samples per data channel. Its scaled FFT at frequency $f_k = \frac{k}{N\Delta t}$ is

$$\hat{\mathbf{F}}_k = \sqrt{\frac{\Delta t}{N}} \sum_{j=0}^{N-1} \hat{\mathbf{y}}_j e^{-2\pi i j k / N} \quad (1)$$

Where $\mathbf{i}^2 = -1$, and Δt (s) is the sampling time interval. The FFTs on a selected frequency band, denoted by $\hat{\mathbf{F}}_k$, are used to identify the modal properties of the structure. Typically the selected frequency band covers the resonance band of modes of interest.

Let $\boldsymbol{\theta}$ be the set of modal parameters to be identified. Using Bayes Theorem we can write the posterior Probability Density Function (PDF) of $\boldsymbol{\theta}$ given $\hat{\mathbf{F}}_k$ as

$$\pi(\boldsymbol{\theta}|\hat{\mathbf{F}}_k) = \pi(\hat{\mathbf{F}}_k)^{-1} \pi(\hat{\mathbf{F}}|\boldsymbol{\theta}) \pi(\boldsymbol{\theta}) \quad (2)$$

The term $\pi(\hat{\mathbf{F}}_k)^{-1}$ is a normalizing constant, $\pi(\hat{\mathbf{F}}|\boldsymbol{\theta})$ is the likelihood function, and $\pi(\boldsymbol{\theta})$ is the prior distribution of the parameters $\boldsymbol{\theta}$. Assuming long data, (Au, 2017) shows that $\hat{\mathbf{F}}_k$ are complex Gaussian and independent at different frequencies, therefore we can write the likelihood function as

$$\pi(\hat{\mathbf{F}}_k|\boldsymbol{\theta}) = \frac{\pi^{-n}}{|\mathbf{E}_k(\boldsymbol{\theta})|} \exp[-\hat{\mathbf{F}}_k^* \mathbf{E}_k(\boldsymbol{\theta})^{-1} \hat{\mathbf{F}}_k] \quad (3)$$

Where $\hat{\mathbf{E}}_k(\boldsymbol{\theta}) = E[\hat{\mathbf{F}}_k \hat{\mathbf{F}}_k^*]$ is the theoretical Power Spectral Density (PSD) of data given $\boldsymbol{\theta}$, and the symbol $(*)$ denotes the conjugate transpose.

Resonance Band Modeling

Within the selected frequency band it is assumed that

$$\hat{\mathbf{F}}_k = \mathbf{F}_k + \boldsymbol{\varepsilon}_k \quad (4)$$

Where \mathbf{F}_k is the scaled FFT at frequency k of the theoretical structural dynamic response, and $\boldsymbol{\varepsilon}_k$ is the prediction error (data noise). Suppose the selected band is dominated by m vibration modes, we can write Equation 4 as

$$\hat{\mathbf{F}}_k = \sum_{i=1}^m \phi_i h_{ik} p_{ik} + \boldsymbol{\varepsilon}_k \quad (5)$$

Where for the mode i , ϕ_i ($n \times 1$) is the partial mode shape (related to the measured DOFs), h_{ik} is the transfer function as in Equation 6, p_{ik} is the scaled FFT of the i th modal force at frequency f_k .

$$h_{ik} = \frac{(2\pi \mathbf{i} f_k)^{-q}}{1 - \beta_{ik}^2 - 2\zeta_i \beta_{ik} \mathbf{i}} \quad (6)$$

q takes on 0, 1 and 2 for acceleration, velocity and displacement response measurements respectively, $\beta_{ik} = \frac{f_i}{f_k}$, f_i (Hz) and ζ_i are respectively the natural frequency and damping ratio for mode i .

Assuming that the modal forces have a constant PSD S within the selected band, and the prediction errors at different measured DOFs assumed to be i.i.d with a constant PSD S_e in the band, we can write the PSD of scaled FFT of measured data, $\mathbf{E}_k(\boldsymbol{\theta})$, as

$$\mathbf{E}_k(\boldsymbol{\theta}) = E[\mathbf{F}_k \mathbf{F}_k^*|\boldsymbol{\theta}] + E[\boldsymbol{\varepsilon}_k \boldsymbol{\varepsilon}_k^*|\boldsymbol{\theta}] = \sum_{i=1}^m \sum_{j=1}^m h_{ik} h_{jk}^* S_{ij} \phi_i \phi_j^T + S_e \mathbf{I}_n \quad (7)$$

Assuming that only 1 mode contributes to the response within the selected band, we can simplify Eq.(7) taking $i = j = 1$. The structural parameters $\boldsymbol{\theta}$ are

$$\boldsymbol{\theta} = \begin{Bmatrix} f \\ \zeta \\ S \\ S_e \\ \boldsymbol{\phi} \end{Bmatrix} \quad (8)$$

Substituting Eq.(7) and Eq. (3) into Eq.(2), and modeling the prior distribution $\pi(\boldsymbol{\theta})$ as an uniform density, we can write the posterior PDF as

$$\pi(\boldsymbol{\theta}|\hat{\mathbf{F}}_k) \propto \pi(\hat{\mathbf{F}}_k|\boldsymbol{\theta}) = e^{-L(\boldsymbol{\theta})} \quad (9)$$

Where $L(\boldsymbol{\theta})$ is the negative log likelihood function (NLLF).

$$L(\boldsymbol{\theta}) = nN_f \ln \pi + \sum_k |\mathbf{E}_k(\boldsymbol{\theta})| + \sum_k \hat{\mathbf{F}}_k^* \mathbf{E}_k(\boldsymbol{\theta})^{-1} \hat{\mathbf{F}}_k \quad (10)$$

Minimization of Eq.(10) gives us the Maximum a Posteriori values for $\boldsymbol{\theta}$. One alternative to the posterior PDF is to use the Laplace Approximation to calculate the marginal PDFs to each parameter (Au, 2017).

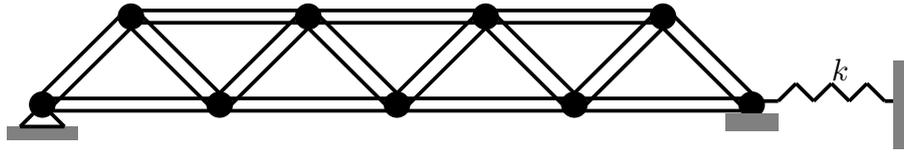


Figure 1 – 15 DOF truss frame.

Approximate Bayesian Computation

The ABC methods replaces the likelihood by evaluating the discrepancy between the observed data and the data generated by a simulation using a given model as stated by Ben Abdesslem et al. (2019). We can write the Bayes Theorem, Eq. (2) as:

$$\pi(\boldsymbol{\theta}|\hat{\mathbf{F}}_k) \propto \pi(\Delta(\hat{\mathbf{F}}_k, \bar{\mathbf{F}}_k) < \varepsilon|\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \quad (11)$$

Where $\hat{\mathbf{F}}_k$ is the scaled FFT of measured data, $\bar{\mathbf{F}}_k$ is the simulated data given a set of candidate parameters $\bar{\boldsymbol{\theta}}$, Δ is a discrepancy metric that measures how similar $\hat{\mathbf{F}}_k$ and $\bar{\mathbf{F}}_k$ are, $\pi(\boldsymbol{\theta}|\hat{\mathbf{F}}_k)$ is an approximation for the posterior density and $\varepsilon > 0$ is a tolerance threshold.

The most simple algorithm for ABC is the ABC rejection sampling (ABC - RS) proposed by Pritchard et al. (1999). It consists in generating a vector candidate $\bar{\boldsymbol{\theta}}$ from the prior distribution, simulate $\bar{\mathbf{F}}_k$ using a given model $\mathcal{M}(\cdot)$, comparing the simulated values $\bar{\mathbf{F}}_k$ with the measured values $\hat{\mathbf{F}}_k$, and accepting the values of $\bar{\boldsymbol{\theta}}$ that produces $\Delta(\hat{\mathbf{F}}_k, \bar{\mathbf{F}}_k) < \varepsilon$. The output of an ABC algorithm is a sample of parameters from a distribution $\pi(\boldsymbol{\theta}|\Delta(\hat{\mathbf{F}}_k, \bar{\mathbf{F}}_k) < \varepsilon)$. If ε is sufficiently small, the distribution $\pi(\boldsymbol{\theta}|\Delta(\hat{\mathbf{F}}_k, \bar{\mathbf{F}}_k) < \varepsilon)$ is a good approximation for the posterior distribution $\pi(\boldsymbol{\theta}|\hat{\mathbf{F}}_k)$.

Although the ABC-RS is easy to implement it has a low acceptance rate. Other methods of ABC have been developed in the literature, for example the ABC Sequential Monte Carlo (ABC SMC) by Toni et al (2009), and the ABC Nested Sampling (NS) by Ben Abdesslem et al. (2019).

NUMERICAL RESULTS

We are going to present a numerical example of the identification of the natural frequencies and damping ratios of the truss frame shown in Fig. 1, using the BAYOMA approach with the Laplace Approximation, and the ABC approach. The numerical results were calculated using a Finite Element model for the truss frame, and the Newmark Method was used to solve the governing equations of the system. More details can be found in (Gerardin and Rixen, 2015).

Consider a $n = 9$ Degree of Freedom (DOF) truss frame, under white noise excitation in all it's DOF. Synthetic noise was added to the system response aiming at simulating noise-polluted data, and only data of 3 DOF of the structure were used. To try to simulate different conditions of the structure, simulations using a different value for the spring stiffness were performed. Each simulation using a different value for the spring stiffness was considered as a measurement from a specific day.

First the values in the MAP point, \hat{f} , $\hat{\zeta}$, \hat{S} , \hat{S}_e and $\hat{\phi}$ were calculated by minimizing Eq. (10), then the Laplace approximation was used to calculate the Covariance Matrix of the parameters $\boldsymbol{\theta}$, \mathbf{C}_θ . Therefore the posterior density of model parameters is approximated by $\mathcal{N}(\hat{\boldsymbol{\theta}}, \mathbf{C}_\theta)$, a normal distribution with mean $\hat{\boldsymbol{\theta}}$, and covariance \mathbf{C}_θ .

For the ABC, the prior distribution for each parameter adopted was $\pi(\theta_i) = \mathcal{U}(\hat{\theta}_i - \lambda_i \sqrt{C_{\theta_{ii}}}, \hat{\theta}_i + \lambda_i \sqrt{C_{\theta_{ii}}})$, where a different value of λ_i was adopted for each parameter. The discrepancy metric adopted Δ is:

$$\Delta(\hat{\mathbf{E}}, \bar{\mathbf{E}}) = \sum_{r=1}^n \sum_{s=1}^n \frac{(\hat{E}_{rs} - \bar{E}_{rs})^T (\hat{E}_{rs} - \bar{E}_{rs})}{\hat{E}_{rs} \bar{E}_{rs}} < \varepsilon \quad (12)$$

Where \mathbf{E} is calculated using Eq. (7), and E_{rs} is one of the components of \mathbf{E} . The frequency band adopted was the same used in the BAYOMA computation, where only one mode contributes to the frequency response. For this example we used a similar approach to the one presented in (Castello and Ritto, 2022) for the choice of ε , where different values $\varepsilon_1 > \varepsilon_2 > \dots \varepsilon_G$ were made, and updated in the next population using the median of the previous population.

It is important to notice that because of the white noise excitation used to generate the measurements, as well as the generated measurement errors, the lowest value for ε occurs at the MAP point. The measured values used in the ABC metric were the averaged PSD using $M = 10$ segments of the signal.

ABC using data from 1 day

The prior distribution for each parameter was chosen as

$$\pi(\theta_i) = \mathcal{U}(\hat{\theta}_i - 8\sqrt{C_{\theta_{ii}}}, \hat{\theta}_i + 8\sqrt{C_{\theta_{ii}}}) \quad (13)$$

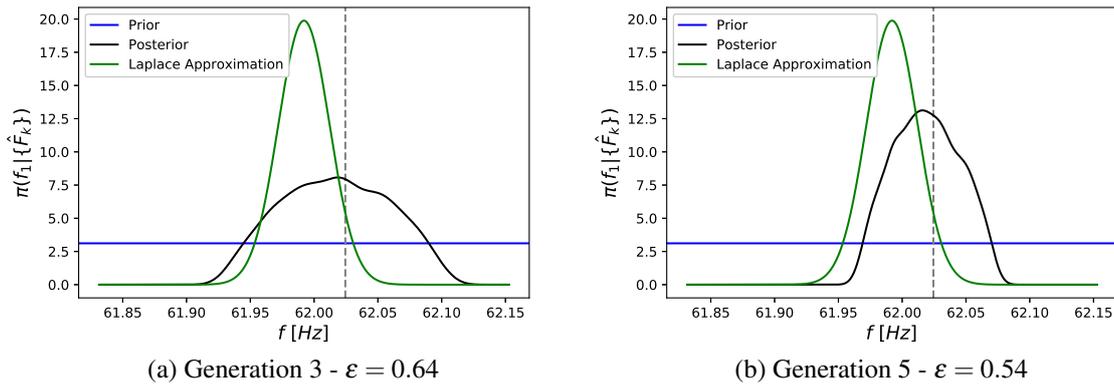


Figure 2 – Posterior PDF for f_1 . Vertical line indicates the real value used in the simulation.

The number of particles accepted was 2000, and the samples obtained were approximated by a kernel density estimation.

Figure 2 presents the marginal posterior densities of the first natural frequency provided by the Laplace Approximation Conventional OMA and by the ABC. We can notice that as the value of ϵ decreases, the posterior PDF obtained using the ABC approaches a Gaussian distribution, however with a different value in the MAP point than the one obtained using the Bayesian OMA.

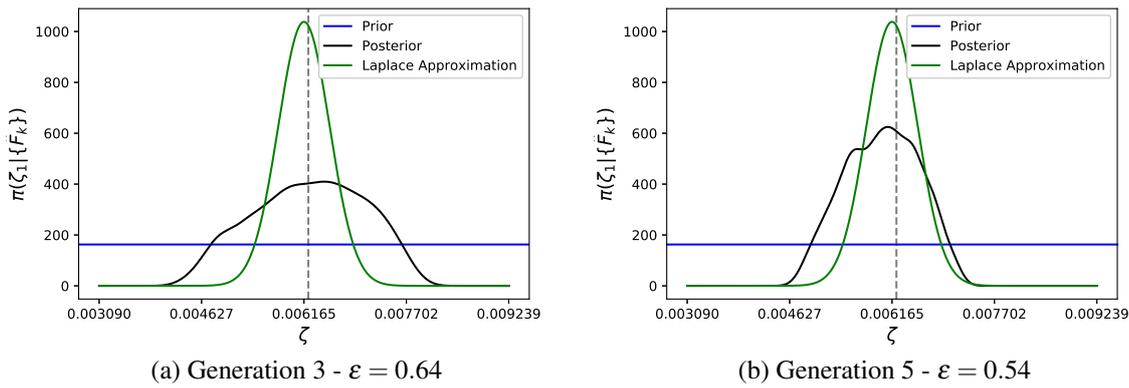


Figure 3 – Posterior PDF for ζ_1 . Vertical line indicates the real value used in the simulations.

Figure 3 presents the marginal posterior densities of the damping ratio associated with the first mode. We can notice that for the damping ratio as the tolerance threshold decreases the posterior distribution obtained using the ABC gets closer to the one obtained in the Bayesian OMA.

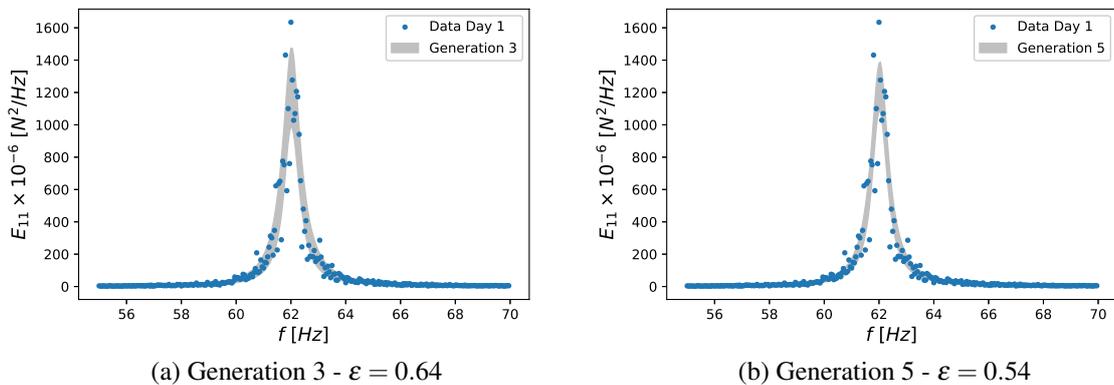


Figure 4 – PSD of measured data E_1 in blue dots and 95% stochastic envelope of model predictions in grayed areas.

Figure 4 presents the PSD of measured data in blue dots and the 95% stochastic envelope provided by the posterior density of model parameters at different generations/populations of the BAYOMA-ABC. As the tolerance decreases the envelopes becomes thinner, and the number of data points inside the envelope decreases.

ABC using data from 2 days

In the next example the data of the system under 2 different conditions was considered in an attempt to capture a posterior PDF using the ABC. To simulate a different condition of the system, the value of the spring stiffness was altered.

To take in account the values of the measurements under different conditions, the values of each component of the PSD matrix, equation 7, were written in the form:

$$\mathbf{E}_{rs} = \begin{bmatrix} E_{rs}^1 \\ E_{rs}^2 \end{bmatrix} \quad (14)$$

Where the superscript indicates the day of the measurement.

To take into account the different values of the system natural frequency due to the change in the spring stiffness, for this example the prior distribution of the natural frequency adopted is $\mathcal{U}(61.5, 64)$. The prior distribution for the other parameters were equal to the ones presented in Equation 13.

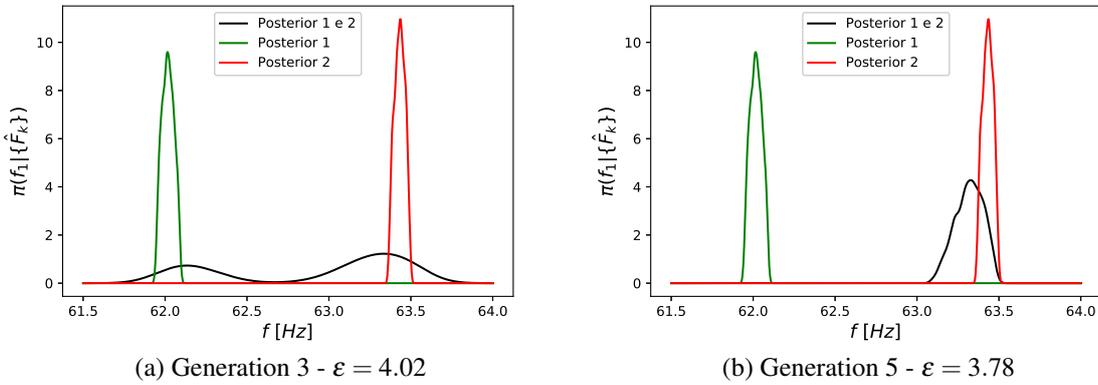


Figure 5 – Posterior PDF for f_1 .

Figure 5 shows the posterior PDF for the first natural frequency f_1 using data from day 1, day 2, and both days combined. For larger values of ϵ the posterior PDF of the 2 days combined is a PDF with a higher probability around the areas of the parameters values of the individual days. However as we decrease the tolerance value, the posterior PDF of the combined days approaches the posterior PDF of day 2.

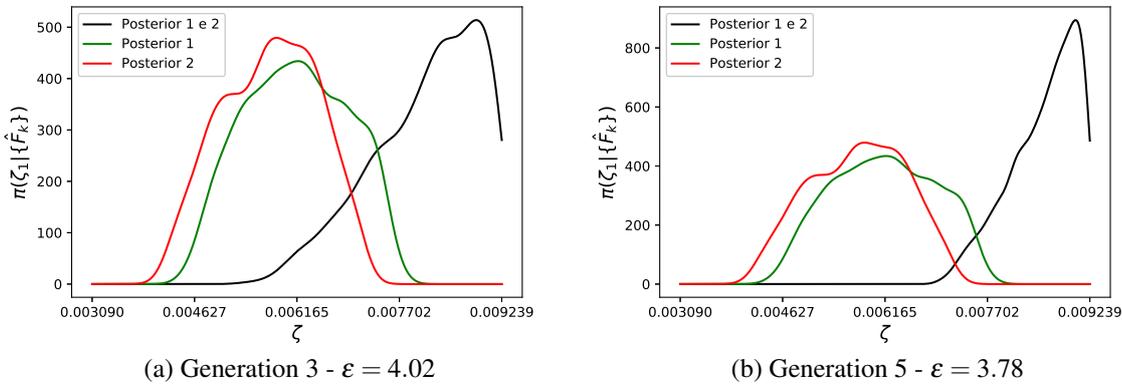


Figure 6 – Posterior PDF for ζ_1 .

Figure 6 shows the posterior PDF for the damping ratio ζ_1 using data from day 1, day 2, and both days combined. The posterior PDF using data from both days combined approaches the posterior of each individual day as the tolerance value

ε decreases. Using data from both days combined had higher values accepted for the damping ratio than the values for each individual day.

CONCLUSION

A strategy utilizing the Approximate Bayesian Computation (ABC) to identify the modal parameters of a structure under operational conditions was proposed. The Bayesian Operational Modal Analysis (OMA) formulation that is commonly used in the literature was introduced, and so was the ABC, used to obtain an approximation of the posterior Probability Density Function when the Likelihood Function is not available.

A numerical example using the Bayesian Operational Modal Analysis and the proposed approach was presented, showing similar results between the 2 approaches. However when trying to use the ABC approach using data from the structure under different operational conditions, the results from the numerical example indicates that a different approach than the one proposed, simply using the data from different days together in the ABC, might be necessary. In an attempt to solve this problem, one possibility might be to make use of a hierarchical approach using the ABC to account for data of multiple operational conditions.

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