



# Multiple Frozen Models $\mathcal{H}_2$ Dynamic Controller for a Rotor-Blade System

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*Abstract: Rotating flexible structures are part of machines such as wind turbines, aircraft engines, compressors and many other industrial appliances. For high efficiency, the rotor-blade system must comply to rigorous low vibration levels. Passive control can be considered, but it is only effective at specific operating conditions, providing limited damping when compared to active control. Since rotor-blade systems exhibit periodic dynamics, linear time-invariant strategies usually do not guarantee closed-loop stability nor acceptable performance. Periodic controllers with guaranteed closed-loop stability and performance still feature numeric problems when the period is large or when the coefficients of the system equations exhibit large variations over the period. Further difficulties arise when it is impractical to measure, in real-time, the displacement and velocity of fast-moving machine parts. This work presents the design of a multiple frozen models  $\mathcal{H}_2$  controller for a rotor-blade system. Numerical results show the benefits of the proposed approach.*

**Keywords:** vibration control,  $\mathcal{H}_2$  control, rotor blade, periodic systems, gain scheduling

## INTRODUCTION

Periodic dynamics can model many engineering systems (Sinha, 1997; Bittanti, 2001). For instance, wind turbines and aeronautical engines have as main component a rotor-blade system, which consists of a rotating flexible structure that exhibits periodic dynamics when rotating at constant angular speed (Christensen and Santos, 2005).

Applications of optimal control problems to rotor-blade vibration control have been proposed in Viganò et al. (2010); Ulker and Nitzsche (2012); Jakobsen et al. (2013); Camino and Santos (2018, 2019b,a). Solutions to optimal control problems usually rely on solving an algebraic Riccati equation, for the linear time invariant case (Doyle et al., 1989; Bittanti et al., 2012), and a differential Riccati equation, for the time-varying case (Kalman, 1960; Bittanti, 2001). Procedures to solve the periodic Riccati equations have been proposed in Gusev et al. (2010) and references therein.

Techniques to solve periodic Riccati equations feature numeric implementation problems when the period is large or the coefficients of the system equations present large variation over the period (Varga, 2013). The work Hansen (2013) reported numeric failure when trying to compute a periodic  $\mathcal{H}_2$  dynamic output feedback controller using the method proposed in Gusev et al. (2007). As discussed in Gusev et al. (2010); Varga (2013), even when it is possible to compute periodic controllers, the underlying numerical methods feature high computational complexity.

Control design techniques that are computationally intensive are not practical in real situations due to limitation of memory and processing power in embedded systems (Albertos et al., 2005). To avoid high complexity, a divide-and-conquer control based on the design of local linear time invariant (LTI) controllers can be used. The multiple frozen models control strategy consists in successively alternating between LTI controllers designed for LTI plants that are time-frozen instances of the periodic system. For instance, the LQR problem is used in Jakobsen et al. (2013) for the design of local LTI controllers, in the context of the multiple frozen models method.

This paper proposes for the rotor-blade system a multiple frozen models controller in which the local LTI controllers are designed using the  $\mathcal{H}_2$  control problem. Full-state feedback controllers as well as dynamic output feedback controllers are designed. Numerical results show that the proposed multiple frozen models  $\mathcal{H}_2$  strategy is able to attenuate rotor-blade vibrations.

## EQUATION OF MOTION

The equation of motion (Christensen and Santos, 2005) of the rotor-blade system, when rotating at constant angular speed  $\Omega(t) \equiv \Omega$ , is given by

$$M(t)\ddot{q}(t) + D(t)\dot{q}(t) + S(t)q(t) = Q_u u(t) + w(t)$$

where  $q(t)$  is the vector of generalized coordinates,  $u(t)$  is the control input,  $w(t)$  is the exogenous input,  $M(t)$  is the mass matrix,  $D(t)$  is the damping matrix,  $S(t)$  is the stiffness matrix,  $Q_u$  is the matrix of control inputs. As shown in Jakobsen et al. (2013), internal forces due to unbalance can be handled using a feedforward control law. The system matrices  $M(t)$ ,

$D(t)$ ,  $S(t)$  are periodic, with period  $P = 2\pi/\Omega$ , that is:

$$M(t) = M(t + P), \quad D(t) = D(t + P), \quad S(t) = S(t + P)$$

Defining the state vector  $x(t) = [q(t)^T \quad \dot{q}(t)^T]^T$ , the equation of motion can be written in the state-space form

$$\dot{x}(t) = A(t)x(t) + B_u(t)u(t) + B_w(t)w(t) \quad (1)$$

with

$$A(t) = \begin{bmatrix} 0 & I \\ -M(t)^{-1}S(t) & -M(t)^{-1}D(t) \end{bmatrix}, \quad B_u(t) = \begin{bmatrix} 0 \\ M(t)^{-1}Q_u \end{bmatrix}, \quad B_w(t) = \begin{bmatrix} 0 \\ M(t)^{-1} \end{bmatrix}$$

## CONTROL STRATEGY

This section presents the multiple frozen models control strategy for a periodic system with state-space representation:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_w(t)w(t) + B_u(t)u(t) \\ z(t) &= C_z(t)x(t) + D_{zu}(t)u(t) \\ y(t) &= C_y(t)x(t) + D_{yw}(t)w(t) \end{aligned} \quad (2)$$

where  $x(t)$  is the state vector,  $u(t)$  is the control input,  $w(t)$  is the exogenous input,  $y(t)$  is the measured output, and  $z(t)$  is the performance output. For this system, it will be designed both full-state feedback and dynamic output feedback multiple-frozen-models controllers.

### The Closed-Loop System

This section summarizes the formulas for the closed-loop system using either the static or the dynamic controllers. For the full state feedback control law

$$u(t) = Kx(t),$$

the closed-loop system, denoted by  $T_{SF}$ , is given by

$$\begin{aligned} \dot{x}(t) &= (A + B_u K)x(t) + B_w w(t) \\ z(t) &= (C_z + D_{zu} K)x(t) \end{aligned}$$

On the other hand, for the dynamic output feedback control law

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t) \end{aligned}$$

the closed-loop system, denoted by  $T_{OF}$ , is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw} \end{bmatrix} w$$

$$z(t) = [C_z + D_{zu} D_c C_y \quad D_{zu} C_c] \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + [D_{zu} D_c D_{yw}] w(t)$$

### The Multiple Frozen Models Control Strategy

The multiple frozen models control strategy consists in closing the loop of the periodic system (2) with a control law that activates multiple LTI controllers. The steps to design a multiple frozen models controller are as follows:

- Step 1 Suppose the periodic system has a state-space representation as in (2).
- Step 2 Choose  $N$  time instants  $t_1, \dots, t_i, \dots, t_N$ . For each time instant  $t_i$ , compute a local LTI model from the periodic system by evaluating the system matrices at the frozen time  $t_i$ , i.e.,  $\hat{A}_i = A(t_i)$ ,  $\hat{B}_{u_i} = B(t_i)$ , and so on. This step will provide  $N$  LTI models.
- Step 3 Design an LTI controller for each of the local LTI models. All local controllers have to be designed using the same technique for the same objective function.
- Step 4 Verify the performance of each LTI system in closed-loop with its respective local LTI controller.
- Step 5 The multiple frozen models controller is generated by adequately activating the designed local LTI controllers.

It is desirable to choose enough points such that the transition from one local LTI system to another is smooth. Instead of alternating among the local controller in real time, as proposed in Step 5, it is also possible to generate a time-dependent controller that interpolates the local LTI controllers.

The  $\mathcal{H}_2$  full state feedback control problem and the  $\mathcal{H}_2$  dynamic output feedback control problem (Doyle et al., 1989; Chen and Saberi, 1993; Zhou et al., 1996) are presented for LTI systems in the next sections.

## The $\mathcal{H}_2$ Full State Feedback Control Problem

The solution of the  $\mathcal{H}_2$  full state feedback control problem for linear time invariant systems is presented in Theorem 1 and the underlying hypotheses are described in Hypothesis 1.

**Hypothesis 1.** The pair  $(A, B_u)$  is stabilizable. Matrix  $D_{zu}$  has full column rank. Matrix  $\begin{bmatrix} A - j\omega I & B_u \\ C_z & D_{zu} \end{bmatrix}$  has full column rank for all  $\omega$ .

**Theorem 1.** Assume Hypothesis 1 holds. There exists a gain matrix  $K$  such that the full state feedback control law  $u(t) = Kx(t)$  guarantees asymptotic stability of  $T_{SF}$  and minimizes  $\|T_{SF}\|_2$  if and only if there exists a unique solution  $X = X^T \geq 0$  to the algebraic Riccati equation

$$A^T X + XA - (XB_u + C_z^T D_{zu}) (D_{zu}^T D_{zu})^{-1} (D_{zu}^T C_z + B_u^T X) + C_z^T C_z = 0$$

The matrix  $K$  that provides the control law  $u(t) = Kx(t)$  is given by

$$K = - (D_{zu}^T D_{zu})^{-1} (B_u^T X + D_{zu}^T C_z) \quad (3)$$

## The $\mathcal{H}_2$ Output Feedback Control Problem

The solution of the  $\mathcal{H}_2$  dynamic output feedback control problem for linear time invariant systems is presented in Theorem 2 and the underlying hypotheses are described in Hypothesis 2.

**Hypothesis 2.** The pair  $(A, B_u)$  is stabilizable and the pair  $(A, C_y)$  is detectable. Matrices  $R_1 = D_{zu}^T D_{zu} > 0$  and  $R_2 = D_{yw} D_{yw}^T > 0$ . Matrix  $\begin{bmatrix} A - j\omega I & B_u \\ C_z & D_{zu} \end{bmatrix}$  has full column rank for all  $\omega$  and matrix  $\begin{bmatrix} A - j\omega I & B_w \\ C_y & D_{yw} \end{bmatrix}$  has full row rank for all  $\omega$ .

**Theorem 2.** Assume Hypothesis 2 holds. There exists a dynamic output feedback controller  $(A_c, B_c, C_c, D_c)$  that guarantees asymptotic stability of  $T_{OF}$  and minimizes  $\|T_{OF}\|_2$  if and only if there are unique solutions  $X_2 = X_2^T \geq 0$  and  $Y_2 = Y_2^T \geq 0$  to the algebraic Riccati equations

$$\begin{aligned} (A - B_u R_1^{-1} D_{zu}^T C_z)^T X_2 + X_2 (A - B_u R_1^{-1} D_{zu}^T C_z) - X_2 B_u R_1^{-1} B_u^T X_2 + C_z^T (I - D_{zu} R_1^{-1} D_{zu}^T) C_z &= 0 \\ (A - B_w D_{yw}^T R_2^{-1} C_y) Y_2 + Y_2 (A - B_w D_{yw}^T R_2^{-1} C_y)^T - Y_2 C_y^T R_2^{-1} C_y Y_2 + B_w (I - D_{yw}^T R_2^{-1} D_{yw}) B_w^T &= 0 \end{aligned}$$

The matrices of the dynamic output feedback controller are given by

$$A_c = A + B_u F_2 + L_2 C_y, \quad B_c = -L_2, \quad C_c = F_2, \quad D_c = 0 \quad (4)$$

with

$$F_2 = -R_1^{-1} (B_u^T X_2 + D_{zu}^T C_z) \quad \text{and} \quad L_2 = -(Y_2 C_y^T + B_w D_{yw}^T) R_2^{-1}$$

## NUMERICAL RESULTS

All numerical simulations are performed using the following initial condition for the rotor-blade system:

$$x(0) = [1 \quad 1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 0_{1 \times 6}]^T \quad (5)$$

The initial conditions  $x_c(0)$  for the dynamic controller are generated randomly. The angular velocity of the rotor-blade system is constant and set to  $\Omega = 300$  rpm, which gives a period of  $T = 0.2$  seconds. The data for the system matrices in the equation of motion (1) can be found in Camino and Santos (2018, 2019a). The matrices for the performance output channel  $z(t)$  and the measurement output  $y(t)$ , used in (2), are as follows:

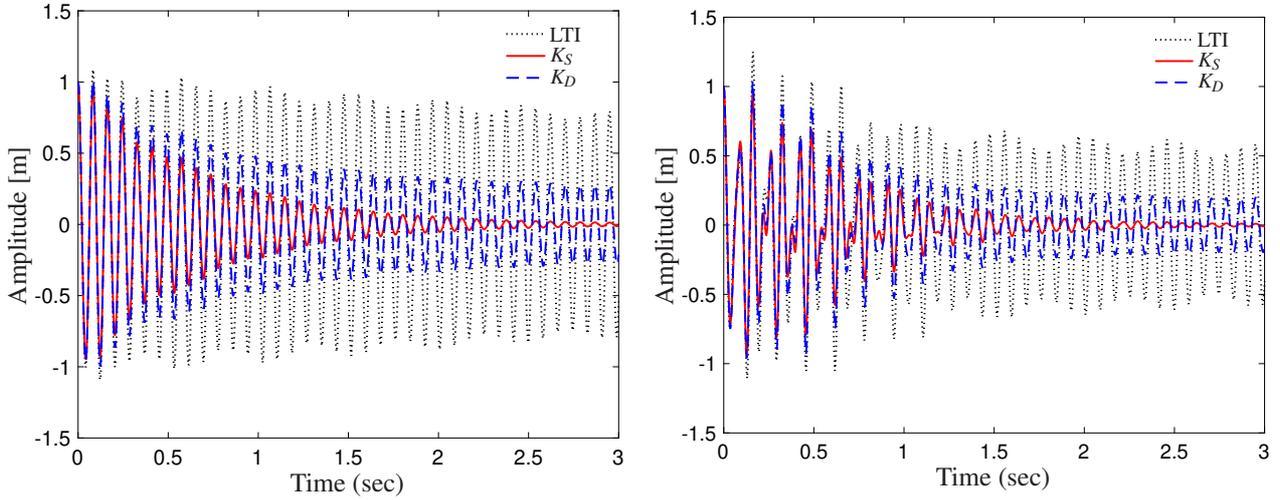
$$C_z = \begin{bmatrix} C_z^1 \\ 0_{6 \times 12} \end{bmatrix}, \quad D_{zu} = \frac{\sqrt{10}}{10} \begin{bmatrix} 0_{12 \times 6} \\ I_6 \end{bmatrix}, \quad C_y = [Q_u^T \quad 0], \quad D_{yw} = I_6$$

with  $C_z^1 = 10^{-1} \text{blkdiag}(10^2 I_2, I_4, 10^2 I_2, I_4)$ , where  $\text{blkdiag}(\cdot)$  denotes the block diagonal matrix concatenation of its arguments. Note that the measured output channel  $y(t)$  is used as input signal to the dynamic controller. For this choice of matrix  $C_y$ , only the displacements are been used as feedback. The performance output  $z(t)$  is the cost function for both (static and dynamic)  $\mathcal{H}_2$  designs. This cost penalizes all states (displacements and velocities) of the rotor blade system, as well as the control effort.

It is interesting to investigate how a single LTI controller is able to attenuate the vibration of the periodic rotor-blade system (1)-(2). In order to design this single LTI controller, an LTI model  $(\hat{A}, \hat{B}_u, \dots)$  must be obtained from the periodic

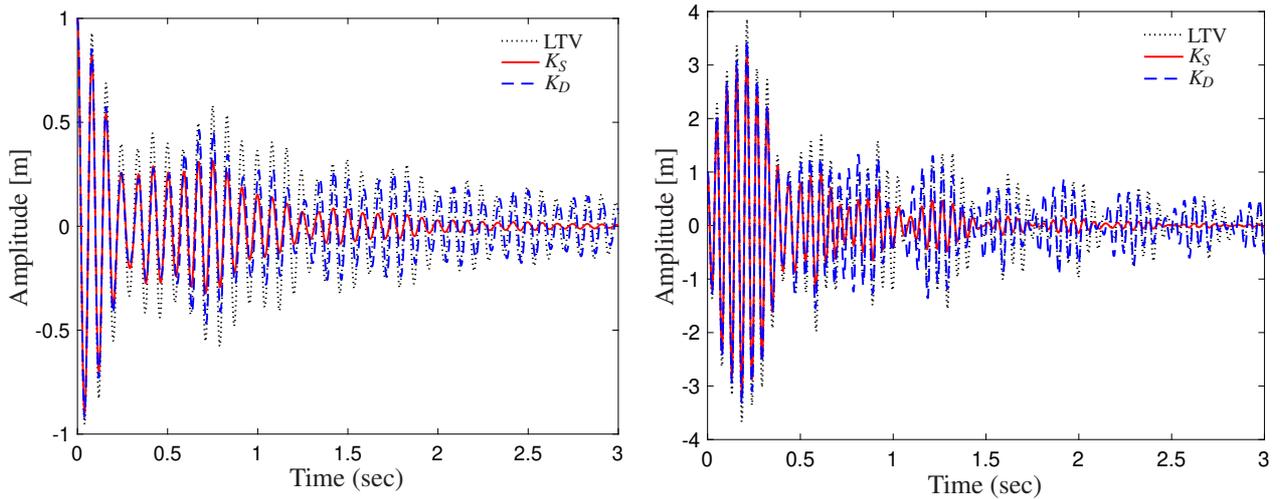
system equation by *freezing* the time-dependent matrices at a chosen time instant  $t = \hat{t}$ , i.e.,  $\hat{A} = A(\hat{t}), \hat{B}_u = B_u(\hat{t}), \dots$ . This approach was denoted in Jakobsen et al. (2013) as *single frozen model method*. The time instant is chosen as  $\hat{t} = P/2 = 0.1$ . For the frozen LTI model at this time instant  $\hat{t}$ , two LTI  $\mathcal{H}_2$  controllers, one static and one dynamic, respectively denoted by  $K_S$  and  $K_D$ , are designed using the formulas provided in the previous sections.

Figure 1 shows the homogeneous responses for the initial conditions  $x(0)$  and  $x_c(0)$  for the LTI plant and the closed-loop systems using both  $K_S$  and  $K_D$  LTI controllers. As can be seen, both  $\mathcal{H}_2$  controllers reduced the system vibration. The full state feedback controller  $K_S$  provides higher performance than the dynamic output feedback controller  $K_D$ . This is expected, since the  $K_D$  controller uses less information than the  $K_S$  controller.



**Figure 1 – Homogeneous responses of the LTI open-loop system (dotted line) and the LTI closed-loop systems with the  $K_S$  (solid line) and  $K_D$  (dashed line) LTI controllers: hub-x position (left) and blade 2 tip position (right).**

Since the results shown in Figure 1 considered only the frozen LTI model, it is important to check how these LTI controllers perform on the original periodic system. Figure 2 shows the homogeneous response of the open-loop periodic LTV rotor-blade system given by (2) and the periodic LTV system in closed-loop with the static  $K_S$  and dynamic  $K_D$  LTI controllers. Clearly, the performance significantly degraded, mainly the performance using the dynamic controller  $K_D$ , which is now barely able to mitigate the vibration of the periodic rotor-blade system.

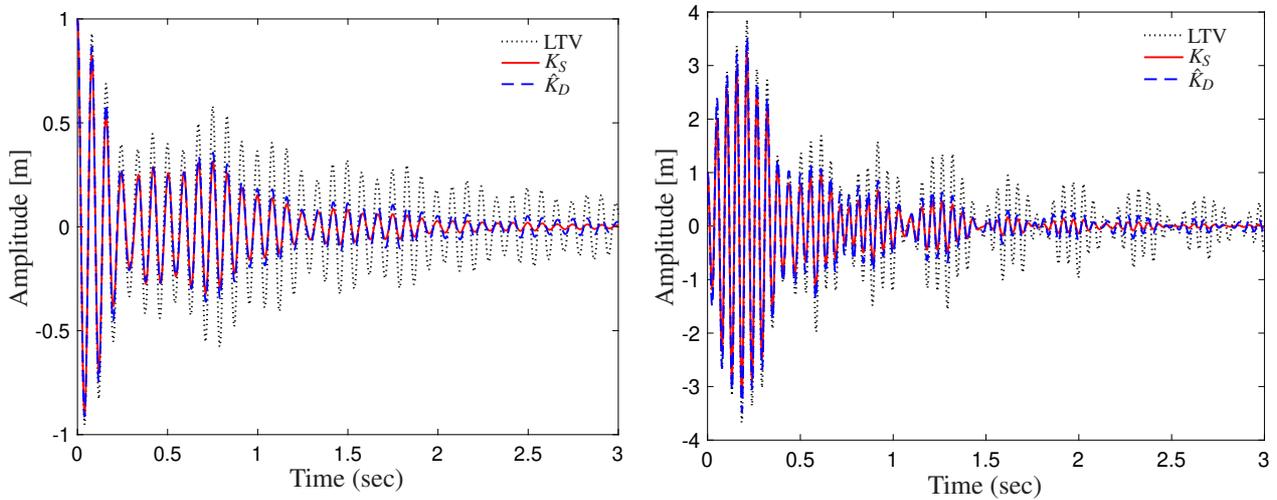


**Figure 2 – Homogeneous responses of the LTV open-loop system (dotted line) and the LTV closed-loop systems with the  $K_S$  (solid line) and  $K_D$  (dashed line) LTI controllers: hub-x position (left) and blade 2 tip position (right).**

To overcome the lack of performance given by the single frozen model method, as shown in Figure 2, a multiple frozen models (MFM)  $\mathcal{H}_2$  dynamic controller, denoted by  $\hat{K}_D$  is now designed. This MFM activates 10 local LTI dynamic output feedback controllers designed from 10 *frozen* LTI models obtained by evaluating the periodic system at the following time instants:

$$\hat{t} = \{0, 0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18\}$$

The results using the MFM dynamic  $\mathcal{H}_2$  controller  $\hat{K}_D$  are shown in Figure 3, which presents the response to initial conditions for the original periodic system (dotted line), the periodic rotor-blade system in closed-loop with the LTI full state feedback controller (solid line), and the MFM dynamic  $\mathcal{H}_2$  controller (dashed line). As can be seen, the performance of the MFM dynamic controller was significantly better than the single frozen dynamic controller, computed for the LTI model at time instant  $\hat{t} = 0.1$ . Actually, the performance of the MFM dynamic controller  $\hat{K}_D$  was similar to the full state feedback controller  $K_S$ .



**Figure 3 – Homogeneous responses of the LTV open-loop system (dotted line) and the LTV closed-loop systems with the  $K_S$  (solid line) and  $\hat{K}_D$  (dashed line) controllers: hub-x position (left) and blade 2 tip position (right).**

## CONCLUSION

This work presented the design of a multiple frozen models (MFM)  $\mathcal{H}_2$  dynamic controller for a periodic rotor-blade system, aiming to mitigate its vibration. First, two  $\mathcal{H}_2$  LTI controllers, one static  $K_S$  and the other dynamic  $K_D$ , were designed for an LTI plant obtained at a frozen time instant of the periodic system. The performance output channel (cost function) for both projects was the same. Notice that the static controller  $K_S$  is a full state feedback controller, which assumes all states are available. While this is not a realistic assumption for most practical applications, it does serve as an indicator of the best possible performance that can be achieved, as shown in Figure 1, in which the dynamic controller  $K_D$ , based only on the system position, provided lower performance than the full state feedback controller  $K_S$ . The behavior of these two LTI controllers was investigated in the original time-varying periodic system. As expected, the performances of both LTI controllers degraded. In particular, the dynamic controller  $K_D$  was no longer able to mitigate system vibration, as shown in Figure 2. Subsequently, an MFM  $\mathcal{H}_2$  dynamic controller was designed using 10 LTI plants, obtained at 10 equally spaced time instants over the period of the system. This MFM controller was able to provide an adequate level of performance, as shown in Figure 3. Although the MFM approach is quite simple and versatile as it is based on standard LTI control design techniques, it does not guarantee stability nor performance for the closed-loop system. Therefore, an a posteriori analysis must be carried out.

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