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New anti-symmetric random matrix applied to a simple rotor-dynamic system

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Abstract: The Random matrix theory emerged in the mid-1950 in the context of Nuclear physics. Since the early 2000s, it has been used in Computational mechanics for taking into account uncertainties in the operators of the equations of mechanical models. The theory has been used successfully associated to symmetric and positive-definite matrices. On the other hand, applications of anti-symmetric random matrices have been largely ignored in the literature. The present work makes a review of the most important matrix-variate distributions for our study and explores applications of anti-symmetric random matrices in uncertainty quantification. We propose a novel ensemble of anti-symmetric random matrices and apply it to a simple rotor-dynamic system.

Keywords: random matrices, anti-symmetric, global uncertainties, rotor-dynamics

INTRODUCTION

The Random matrix theory was introduced in the context Computational mechanics with the work of Soize [16] in 2000, in which he proposed the use of positive-definite random matrices to address global uncertainties in mechanical systems. In the following years, Soize discussed the method that would be called non-parametric probabilistic approach (NPA). The NPA consists in replacing the uncertain operators (system matrices) of the model by random matrices. The forename *non-parametric* is given according to the capability to deal in principle with other than just parameters uncertainty. In opposition to the parametric probabilistic approach (PPA), which consists in replacing the uncertain parameters of the model by random variables. Therefore, the forename *parametric* is due to the ability to treat parameters uncertainty only. Subsequently, the NPA was developed by [17, 18, 1] and extended to non-linear operators by [8, 13], for instance. Unfortunately, anti-symmetric random matrices did not receive much attention in the literature[1], probably because of their absence of application in Nuclear physics [7]. In spite of its limited application in such context, anti-symmetric operators play a fundamental role in Computational mechanics. Anti-symmetric matrices appear in the modelling of rotor-dynamic systems (gyroscopic effect), fluid forces (Magnus effect), non-inertial forces (Coriolis force), electromagnetic forces (Lorentz force), etc. In this line, we feel that there is a gap in the literature to explore uncertainties in anti-symmetric operators. In addition, anti-symmetric random matrices seem promising in taking into account for global uncertainties in such systems. The contribution of this article is to propose a novel random matrix ensemble suitable to account for uncertainties in anti-symmetric operators and present an example of application of the novel anti-symmetric ensemble.

SIMPLE ROTOR-DYNAMIC MODEL

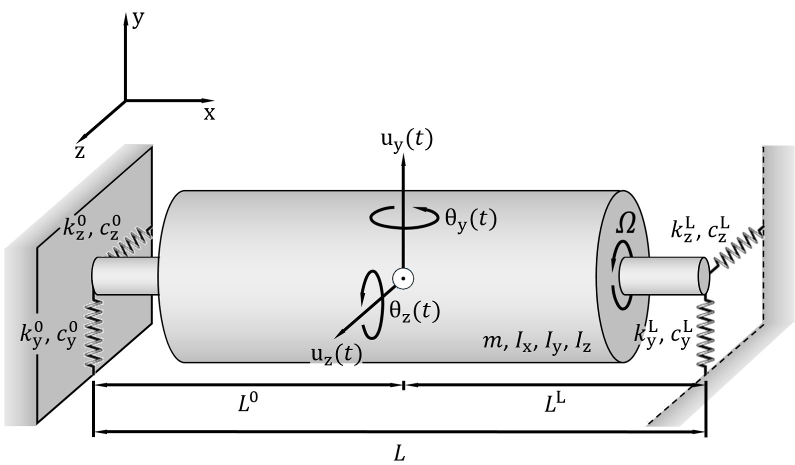

Figure 1 – 4-DOF rotor-dynamic system subject to static and dynamic unbalance

[1]An important exception in Computational mechanics is the article of Mignolet [9].

The simple rotor-dynamic system analyzed here is composed of a rotor of mass m, attached to a rigid shaft spinning with constant rotating-speed Ω, mounted on two spaced flexible bearings (see figure 1). The bearings are assumed to have negligible lengths, such that their geometric positions are regarded to be two distinct points in space. The x-direction coincides with the line connecting these two points. So that, the bearings are located at the positions $x=0$, $x=L$. Initially, the axis-of-rotation lies on the x-axis. The distances from the center-of-mass of the rotor to the bearings locations along the x-axis are L^0, L^{L}, such that the total distance between the bearings is $L = L^0 + L^{\mathrm{L}}$. Usually, due to manufacturing tolerances or defects, the center-of-mass of the rotor does not belong to the axis-of-rotation. The distance between the center-of-mass and the axis-of-rotation is called eccentricity of the rotor (eccentricity e). Also, the polar axis-of-inertia of the rotor (inertia tensor $\mathbf{I} = \mathrm{diag}(I_{\mathrm{x}} \;\; I_{\mathrm{y}} \;\; I_{\mathrm{z}})$) and the axis-of-rotation are not aligned (misalignment-angle a). So that, a static unbalance and a dynamic unbalance are present. The support reactions of the bearings are assumed to lie on the y,z-plane, transversal to the x-axis. So that, the influence of the bearings on the system is represented by the stiffness coefficients k^0_{y}, k^0_{z} and $k^{\mathrm{L}}_{\mathrm{y}}$, $k^{\mathrm{L}}_{\mathrm{z}}$, damping coefficients c^0_{y}, c^0_{z} and $c^{\mathrm{L}}_{\mathrm{y}}$, $c^{\mathrm{L}}_{\mathrm{z}}$ in the y,z-directions. The system has 4-degrees-of-freedom (DOF): 2-DOF for the transversal-displacements $\mathrm{u_y}(t)$, $\mathrm{u_z}(t)$ of the rotor, 2-DOF for the tilting-angles $\theta_{\mathrm{y}}(t)$, $\theta_{\mathrm{z}}(t)$ of the axis-of-rotation. The equations of motion of this 4-DOF rotor-dynamic system can be written as a linear system of 2^{nd}-order coupled differential equations, excited by harmonic generalized forces [2, 21, 20]:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + (\mathbf{W}(\Omega) + \mathbf{C})\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{W}(\Omega) = \Omega\mathbf{G}$ is a dynamic inertia matrix dependent on the rotating-speed Ω through the gyroscopic matrix $\mathbf{G}$, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{f}(t)$ is the generalized forces (force or torque) vector, $\mathbf{u}(t)$ is the displacements vector, in the time t. In their turns, the vectors $\mathbf{u}(t)$, $\mathbf{f}(t)$ are given by [6, 21, 14]:

$$\mathbf{u}(t) = \begin{bmatrix} \mathrm{u_y}(t) \\ \mathrm{u_z}(t) \\ \theta_{\mathrm{y}}(t) \\ \theta_{\mathrm{z}}(t) \end{bmatrix} \qquad \mathbf{f}(t) = \begin{bmatrix} me\Omega^2\cos(\Omega t) \\ me\Omega^2\sin(\Omega t) \\ (I_{\mathrm{z}} - I_{\mathrm{x}})a\Omega^2\cos(\Omega t) \\ (I_{\mathrm{y}} - I_{\mathrm{x}})a\Omega^2\sin(\Omega t) \end{bmatrix} \tag{2}$$

furthermore, the system matrices $\mathbf{M}$, $\mathbf{G}$, $\mathbf{C}$, $\mathbf{K}$ are written by [21, 14]:

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I_{\mathrm{y}} & 0 \\ 0 & 0 & 0 & I_{\mathrm{z}} \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\mathrm{x}} \\ 0 & 0 & -I_{\mathrm{x}} & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} C_{\mathrm{u_y u_y}} & 0 & 0 & C_{\mathrm{u_y}\theta_{\mathrm{z}}} \\ 0 & C_{\mathrm{u_z u_z}} & C_{\mathrm{u_z}\theta_{\mathrm{y}}} & 0 \\ 0 & C_{\theta_{\mathrm{y}}\mathrm{u_z}} & C_{\theta_{\mathrm{y}}\theta_{\mathrm{y}}} & 0 \\ C_{\theta_{\mathrm{z}}\mathrm{u_y}} & 0 & 0 & C_{\theta_{\mathrm{z}}\theta_{\mathrm{z}}} \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} K_{\mathrm{u_y u_y}} & 0 & 0 & K_{\mathrm{u_y}\theta_{\mathrm{z}}} \\ 0 & K_{\mathrm{u_z u_z}} & K_{\mathrm{u_z}\theta_{\mathrm{y}}} & 0 \\ 0 & K_{\theta_{\mathrm{y}}\mathrm{u_z}} & K_{\theta_{\mathrm{y}}\theta_{\mathrm{y}}} & 0 \\ K_{\theta_{\mathrm{z}}\mathrm{u_y}} & 0 & 0 & K_{\theta_{\mathrm{z}}\theta_{\mathrm{z}}} \end{bmatrix} \tag{3}$$

where the entries of the matrices $\mathbf{C}$, $\mathbf{K}$ are expressed by the damping and stiffness coefficients of the bearings (see appendix B). In addition, the matrices $\mathbf{M} \in \mathbb{S}\mathrm{YM}^*_+(4)$, $\mathbf{C} \in \mathbb{S}\mathrm{YM}^*_+(4)$, $\mathbf{K} \in \mathbb{S}\mathrm{YM}^*_+(4)$ are positive-definite. In contrast, the matrix $\mathbf{W}(\Omega) \in \mathbb{so}(4)$ or $\mathbf{G} \in \mathbb{so}(4)$ is anti-symmetric (it couples the 2-DOF for tilting-angles as consequence of the gyroscopic effect). Suppose that we want to take into account for global uncertainties in the dynamic inertia matrix $\mathbf{W}(\Omega)$. The general idea of the NPA consists in replacing this matrix by a random matrix $\mathcal{W}(\Omega)$. Moreover, to construct its probability distribution, first we need to look at the information available for the system matrices. As a preliminary analysis, let us consider two cases:

(i) The dynamic inertia matrix $\mathbf{W}(\Omega)$ is replaced by an anti-symmetric random matrix $\mathcal{W}(\Omega) \in \mathbb{so}(4)$. This leads to an error additive to Ω.

(ii) The gyroscopic matrix $\mathbf{G}$ is replaced by an anti-symmetric random matrix $\mathcal{G} \in \mathbb{so}(4)$. Then $\mathcal{W}(\Omega)$ is constructed from $\mathcal{G}$ through $\mathcal{W}(\Omega) = \Omega\mathcal{G}$. This leads to an error proportional to Ω.

Note that cases (i), (ii) yield probability distributions with different parameters for $\mathcal{W}(\Omega)$, as will be seen in the next sections. The idea of introducing global uncertainties through extra-coupling between the DOF can be found in [16], and has been applied extensively since the 2000s. In our example of application extra-coupling can happen, for instance, if the principal axes-of-inertia are not aligned with the rotor axi-symmetric geometry (non-homogeneous mass distribution), or if there is another physical phenomenon not encompassed in the simple model hypotheses. In any case, the consideration of uncertainties in the dynamic inertia matrix makes the equation (1) to become:

$$\mathbf{M}\ddot{\mathbb{U}}(t) + (\mathcal{W}(\Omega) + \mathbf{C})\dot{\mathbb{U}}(t) + \mathbf{K}\mathbb{U}(t) = \mathbf{f}(t) \tag{4}$$

where $\mathbb{U}(t)$ is the random response of the system.

RANDOM MATRIX THEORY

Let $\mathbb{M}(n,p,\mathbb{F})$ be the non-commutative ring of $(n \times p)$-matrices under the usual matrix addition and multiplication with entries in the field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, etc.), null-element $\mathbf{0}_{n,p}$ and identity-element $\boldsymbol{\delta}_{n,p}$. A random matrix is a matrix $\mathcal{X} \in \mathbb{M}(n,p,\mathbb{F})$ defined on a probability-space. We turn our attention to the most important matrix distributions for our study.

GAUSSIAN RANDOM MATRIX ENSEMBLE

The matrix-variate normal distribution or Gaussian ensemble is a generalization of the multi-variate normal distribution to matrix-valued random variables. It serves as building blocks of more elaborate random matrix ensembles. Suppose a random matrix $\mathcal{X}$ which satisfies the following constraints:

$$\begin{aligned} \mathrm{supp}(\mathcal{X}) &= \mathbb{M}(n,p,\mathbb{R}) \\ \mathrm{E}(\mathcal{X}) &= \boldsymbol{\mu} \\ \mathrm{E}((\mathcal{X}-\boldsymbol{\mu})^{\mathrm{T}}(\mathcal{X}-\boldsymbol{\mu})) &= \mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{\Psi} \\ \mathrm{E}((\mathcal{X}-\boldsymbol{\mu})(\mathcal{X}-\boldsymbol{\mu})^{\mathrm{T}}) &= \mathrm{tr}(\boldsymbol{\Psi})\boldsymbol{\sigma} \end{aligned} \tag{5}$$

with the parameter matrices $\boldsymbol{\mu} \in \mathbb{M}(n,p,\mathbb{R})$, $\boldsymbol{\sigma} \in \mathbb{S}\mathrm{YM}^*_+(n)$ and $\boldsymbol{\Psi} \in \mathbb{S}\mathrm{YM}^*_+(p)$. We say that a random matrix $\mathcal{X}$ which satisfies the equation (5) follows a matrix-variate normal distribution $\mathcal{X} \sim \mathcal{MN}_{n,p}(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\Psi})$ if the joint probability density function (PDF) of its entries is given by [5, 15]:

$$\mathrm{f}_{\mathcal{X}}(\mathrm{vec}(\mathbf{X})) = \frac{1}{(2\pi)^{np/2}(\det(\boldsymbol{\sigma}))^{p/2}(\det(\boldsymbol{\Psi}))^{n/2}}\,\mathrm{e}^{-\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Psi}^{-1}(\mathbf{X}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right)} \tag{6}$$

where $\boldsymbol{\mu}$ is the mean parameter matrix, $\boldsymbol{\sigma}$ is the scale parameter matrix for the rows and $\boldsymbol{\Psi}$ is the scale parameter matrix for the columns. While the parameter $\boldsymbol{\mu}$ is easily identifiable with the expected-value of the distribution by: $\mathrm{E}(\mathcal{X}) = \boldsymbol{\mu}$. The isolated identification of the scale matrices $\boldsymbol{\sigma}$, $\boldsymbol{\Psi}$ is not possible [11]. Because, their dependence enters the distribution only through the covariance matrix: $\mathrm{Cov}(\mathrm{vec}(\mathcal{X}),\mathrm{vec}(\mathcal{X})) = \boldsymbol{\sigma} \otimes \boldsymbol{\Psi}$; where $\otimes$ denotes the direct product (see appendix C). It is interesting to note that $\dim(\boldsymbol{\sigma}) = n^2$, $\dim(\boldsymbol{\Psi}) = p^2$ and the exponents of the determinants in the denominator of the PDF are in reversed order with the dimensions of the matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\Psi}$. Note that the expression in the trace of the exponent of the PDF formula is just the Frobenius inner product of the matrices $(\mathbf{X}-\boldsymbol{\mu})\boldsymbol{\Psi}^{-1}$, $\boldsymbol{\sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})$, which is known as the Mahalanobis distance from Statistics. For notational clarity, we shall denote the random matrix by $\mathcal{X}$, and its occurrence by $\mathbf{X}$. Also note that we do not demand any particular structure for such matrix, besides having real entries [12].

ANTI-SYMMETRIC RANDOM MATRIX ENSEMBLE

The novel ensemble proposed here is called matrix-variate anti-symmetric normal distribution or Gaussian infinitesimal rotation ensemble. The name *infinitesimal rotation* comes from the fact that the derivative (Lie algebra) of a rotation matrix is given by an anti-symmetric matrix[2]. It serves as a model of random anti-symmetric matrices and skew-operators (cross product[3], curl of a vector field, exterior product, etc.). Suppose that we are interested in a random matrix $\mathcal{X}$ (case (i): $\mathcal{X} = \mathcal{W}(\Omega)$, case (ii): $\mathcal{X} = \mathcal{G}$) which satisfies the following constraints:

[2]Let $\mathbf{x} \in \mathbb{R}^n$ be an invariant vector and $\mathbf{y} = \mathbf{Q}\mathbf{x}$; where $\mathbf{Q} \in \mathbb{O}(n)$ is an orthogonal matrix. Thus: $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \boldsymbol{\delta}_n$ and $\mathbf{x} = \mathbf{Q}^{\mathrm{T}}\mathbf{y}$. Differentiating: $\mathrm{d}\mathbf{y} = \mathrm{d}\mathbf{Q}\mathbf{Q}^{\mathrm{T}}\mathbf{y}$ and $\mathrm{d}(\mathbf{Q}\mathbf{Q}^{\mathrm{T}}) = \mathbf{0}_n$. Yielding: $\mathrm{d}\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = -(\mathrm{d}\mathbf{Q}\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}}$. Therefore: $\mathrm{d}\mathbf{y} = \mathbf{W}\mathbf{y}$; where $\mathbf{W} \in \mathbb{so}(n)$ is an anti-symmetric matrix.

[3]Let $\mathbf{w} = [w_{\mathrm{x}}\ \ w_{\mathrm{y}}\ \ w_{\mathrm{z}}]^{\mathrm{T}} \in \mathbb{R}^3$ and $\mathbf{u} = [u_{\mathrm{x}}\ \ u_{\mathrm{y}}\ \ u_{\mathrm{z}}]^{\mathrm{T}} \in \mathbb{R}^3$ be two space vectors. Thus, the cross product: $\mathbf{w} \times \mathbf{u} = [(w_{\mathrm{y}}u_{\mathrm{z}} - w_{\mathrm{z}}u_{\mathrm{y}})\ \ (w_{\mathrm{z}}u_{\mathrm{x}} - w_{\mathrm{x}}u_{\mathrm{z}})\ \ (w_{\mathrm{x}}u_{\mathrm{y}} - w_{\mathrm{y}}u_{\mathrm{x}})]^{\mathrm{T}}$. Yielding: $\mathbf{w} \times \mathbf{u} = \begin{bmatrix} 0 & -w_{\mathrm{z}} & w_{\mathrm{y}} \\ w_{\mathrm{z}} & 0 & -w_{\mathrm{x}} \\ -w_{\mathrm{y}} & w_{\mathrm{x}} & 0 \end{bmatrix} \begin{bmatrix} u_{\mathrm{x}} \\ u_{\mathrm{y}} \\ u_{\mathrm{z}} \end{bmatrix}$. Therefore: $\mathbf{w} \times \mathbf{u} = \mathbf{W}\mathbf{u}$; where $\mathbf{W} \in \mathbb{so}(3)$ is an anti-symmetric matrix.

$$\begin{aligned}\mathrm{supp}(\mathcal{X}) &= \mathbb{so}(n)\\ \mathrm{E}(\mathcal{X}) &= \boldsymbol{\mu}\\ \mathrm{E}((\mathcal{X}-\boldsymbol{\mu})^{\mathrm{T}}(\mathcal{X}-\boldsymbol{\mu})) &= \frac{1}{4}(\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{\Psi}+\mathrm{tr}(\boldsymbol{\Psi})\boldsymbol{\sigma}-\boldsymbol{\sigma}\boldsymbol{\Psi}-\boldsymbol{\Psi}\boldsymbol{\sigma})\end{aligned} \tag{7}$$

with the parameter matrices $\boldsymbol{\mu} \in \mathbb{so}(n)$, $\boldsymbol{\sigma} \in \mathbb{SYM}^*_+(n)$, $\boldsymbol{\Psi} \in \mathbb{SYM}^*_+(n)$, such that $\boldsymbol{\sigma}\boldsymbol{\Psi} = \boldsymbol{\Psi}\boldsymbol{\sigma}$. We say that a random matrix $\mathcal{X}$ which satisfies the equation (7) follows a matrix-variate anti-symmetric normal distribution $\mathcal{X} \sim \mathcal{MA}_n(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\Psi})$ if the PDF of its entries is given by:

$$\mathrm{f}_{\mathcal{X}}(\mathrm{vecq}(\mathbf{X})) = \frac{1}{(2\pi)^{n(n-1)/4}(\det\big(\mathbf{A}_n^+(\boldsymbol{\sigma}\otimes\boldsymbol{\Psi})(\mathbf{A}_n^+)^{\mathrm{T}}\big))^{1/2}}\,\mathrm{e}^{-\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Psi}^{-1}(\mathbf{X}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right)} \tag{8}$$

where $\boldsymbol{\mu}$ is the mean matrix (case (i): $\boldsymbol{\mu} = \mathbf{W}(\Omega)$, case (ii): $\boldsymbol{\mu} = \mathbf{G}$), $\boldsymbol{\sigma}$ and $\boldsymbol{\Psi}$ are scale matrices, $\mathbf{A}_n^+$ is the pseudo-inverse of the mirroring matrix, named cancellation matrix or transition to anti-symmetric matrix (see appendix C). The parameter $\boldsymbol{\mu}$ is identified with the expected-value of the distribution by: $\mathrm{E}(\mathcal{X}) = \boldsymbol{\mu}$. Moreover, the scale matrices $\boldsymbol{\sigma}$, $\boldsymbol{\Psi}$ are related to the covariance of the distribution by: $\mathrm{Cov}(\mathrm{vecq}(\mathcal{X}),\mathrm{vecq}(\mathcal{X})) = \mathbf{A}_n^+(\boldsymbol{\sigma}\otimes\boldsymbol{\Psi})(\mathbf{A}_n^+)^{\mathrm{T}}$. Besides the apparent complication of the PDF, the construction of an anti-symmetric normal random matrix is not complicated, as can be seen below.

GENERATION OF THE RANDOM MATRIX ENSEMBLES

We will present procedures for generating samples from the random matrix ensembles suitable for uncertainty quantification in computational mechanical models. The standard method for uncertainty quantification consists in reproducing a large sample of the random objects (random variables, random vectors, random matrices, etc.) by a pseudo-random number generator. After ensured convergence of the sample measures to the population measures, the model equations are solved for each set of values in the sample. This method is known as Monte Carlo simulations. There are some different criteria for stochastic convergence in the literature. To see how these different criteria are related, the reader is referred to [10, 3, 19, 4].

GENERATION OF THE GAUSSIAN ENSEMBLE

Suppose that we want to generate a sample following $\mathcal{X} \sim \mathcal{MN}_{n,p}(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\Psi})$ with mean $\boldsymbol{\mu} \in \mathbb{M}(n,p,\mathbb{R})$, positive-definite scale matrices $\boldsymbol{\sigma} \in \mathbb{SYM}^*_+(n)$, $\boldsymbol{\Psi} \in \mathbb{SYM}^*_+(p)$. We resort to an auxiliary simpler random matrix $\mathcal{Z} \sim \mathcal{MN}_{n,p}(\mathbf{0}_{n,p},\boldsymbol{\delta}_n,\boldsymbol{\delta}_p)$ with mean $\mathbf{0}_{n,p}$, rows scale $\boldsymbol{\delta}_n$, columns scale $\boldsymbol{\delta}_p$, called standard matrix-variate normal distribution. The entries of this auxiliary matrix are generated by: $\mathcal{Z}_{ij} \sim \mathcal{N}(0,1)$. Note that the entries of $\mathcal{Z}$ are independent and identically distributed (uni-variate) standard normal random variables. Thus, the desired normal random matrix $\mathcal{X}$ is composed by performing an affine[4] transformation of $\mathcal{Z}$: $\mathcal{X} := \boldsymbol{\mu}+\mathbf{L}_{\boldsymbol{\sigma}}\mathcal{Z}\mathbf{L}_{\boldsymbol{\Psi}}^{\mathrm{T}}$; where $\boldsymbol{\sigma} = \mathbf{L}_{\boldsymbol{\sigma}}\mathbf{L}_{\boldsymbol{\sigma}}^{\mathrm{T}}$ and $\boldsymbol{\Psi} = \mathbf{L}_{\boldsymbol{\Psi}}\mathbf{L}_{\boldsymbol{\Psi}}^{\mathrm{T}}$ are the Cholesky decompositions of the scale matrices.

GENERATION OF THE ANTI-SYMMETRIC ENSEMBLE

To generate a sample following $\mathcal{X}^{\mathrm{AS}} = -(\mathcal{X}^{\mathrm{AS}})^{\mathrm{T}} \sim \mathcal{MA}_n(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\Psi})$ with anti-symmetric mean matrix $\boldsymbol{\mu} \in \mathbb{so}(n)$, positive-definite scale matrices $\boldsymbol{\sigma} \in \mathbb{SYM}^*_+(n)$, $\boldsymbol{\Psi} \in \mathbb{SYM}^*_+(n)$, such that $\boldsymbol{\sigma}\boldsymbol{\Psi} = \boldsymbol{\Psi}\boldsymbol{\sigma}$. We resort to an auxiliary normal random matrix $\mathcal{X} \sim \mathcal{MN}_{n,n}(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\Psi})$ which is square, but not necessarily anti-symmetric. Thus, the desired anti-symmetric random matrix $\mathcal{X}^{\mathrm{AS}}$ is constructed by taking the anti-symmetric part of $\mathcal{X}$: $\mathcal{X}^{\mathrm{AS}} := \frac{1}{2}(\mathcal{X}-\mathcal{X}^{\mathrm{T}})$. Note that both matrices $\mathcal{X}^{\mathrm{AS}}$ and $\mathcal{X}$ are composed by normal distributed random variables, but the matrix $\mathcal{X}^{\mathrm{AS}}$ is forced to be anti-symmetric by construction.

NUMERICAL RESULTS

Figures 2 and 3 show the stochastic Campbell diagrams for the cases (i), (ii). In the vertical-axis are presented the values of the stochastic natural-frequencies $\mathbb{w}$ of the rotor-dynamic system, while in the horizontal-axis are presented the values of rotating-speeds Ω. The deterministic curve (DET) is solid black and corresponds to the response of the mean model. The confidence intervals obtained with the non-parametric probabilistic approach (NPA) using the new anti-symmetric ensemble are painted with different color hues, rated: 75%, 90%, 95%, 99%, 99.8% probability. The stochastic results were obtained with number of Monte Carlo simulations: 10000, sufficient to guarantee stochastic convergence of the sample measures of the random matrices. The reference line $\mathbb{w}(\Omega) = \Omega$ is dotted black. Figures 4 and 5 show the corresponding stochastic frequency response functions (FRF) of the transversal-displacement $|\hat{\mathrm{u}}_{\mathrm{y}}(\mathrm{i}\Omega)|$ of the rotor for the cases (i), (ii). Anyway, it can be noted that the effect of uncertainties is very small in case (i). On the other hand, just the opposite is observed in case

[4] An affine transformation can be thought as a deformation followed by a shift. In mathematical terms, an affine transformation is the composition of a linear transformation with a translation.

(ii). This is due to the fact that uncertainties grow linearly with Ω for case (ii); hence higher rotating-speeds are predominant. It is worth commenting that simulating case (i) took from 2 to 6-times the computational time for simulating case (ii). This might be a consequence that in case (i) for every call of Ω in the iterative procedures for tracing the plots, the mean matrix $\mathbf{W}(\Omega)$ changes. So that, another pseudo-random generator call is necessary to obtain the random matrix $\mathcal{W}(\Omega)$. On the contrary, in case (ii) the random matrix $\mathcal{G}$ is the same for all Ω in the plots, since the mean matrix $\mathbf{G}$ does not depend on the rotating-speed. This last feature saves a lot of computational time.

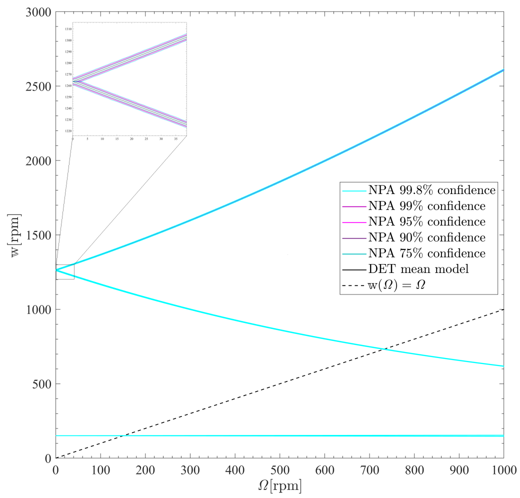

Figure 2 – Campbell diagram for the 4-DOF rotor system showing the confidence intervals using the novel anti-symmetric ensemble applied to $\mathcal{W}(\Omega)$: case (i)

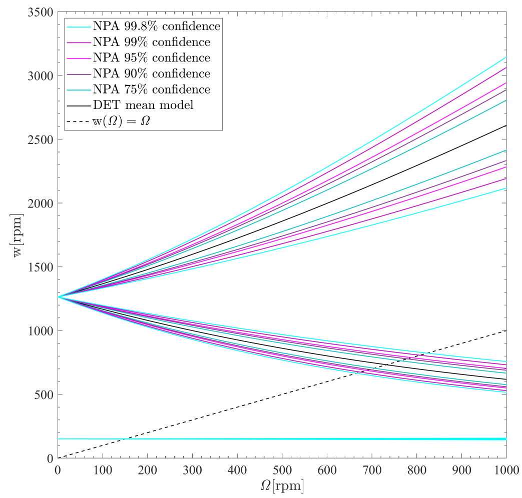

Figure 3 – Campbell diagram for the 4-DOF rotor system showing the confidence intervals using the novel anti-symmetric ensemble applied to $\mathcal{G}$: case (ii)

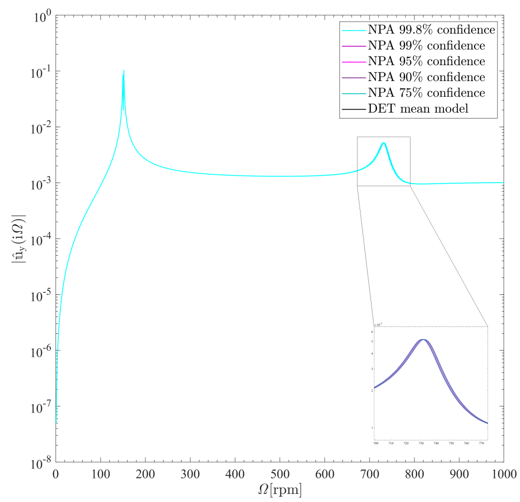

Figure 4 – FRF for the 4-DOF rotor system showing the confidence intervals using the novel anti-symmetric ensemble applied to $\mathcal{W}(\Omega)$: case (i)

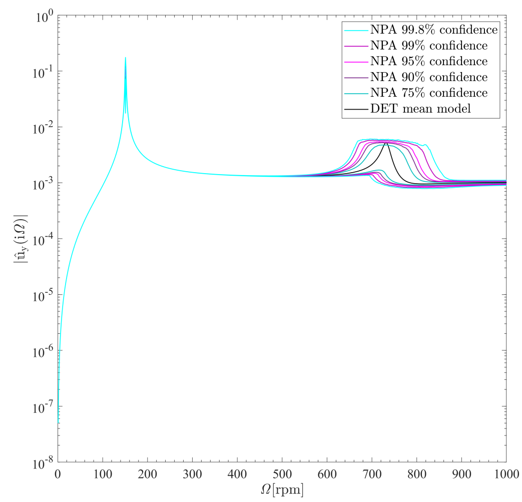

Figure 5 – FRF for the 4-DOF rotor system showing the confidence intervals using the novel anti-symmetric ensemble applied to $\mathcal{G}$: case (ii)

CONCLUSIONS

A new anti-symmetric random matrix ensemble is proposed and applied to a 4-DOF rotor-dynamic system. It is based on the matrix-variate normal distribution and it is anti-symmetric by construction. Two different strategies for uncertainty quantification were considered. One in which the dynamic inertia matrix $\mathcal{W}(\Omega)$ is assumed random, and other in which the gyroscopic matrix $\mathcal{G}$ is random. The strategy in which the gyroscopic matrix $\mathcal{G}$ is assumed random has a greater impact on the response: Campbell and FRF. Because the error is proportional to the rotating-speed Ω. Apart from the fact that, the strategy in which the dynamic inertia matrix $\mathcal{W}(\Omega)$ is considered random took a significantly longer time to obtain the stochastic results than the other strategy. Of course, experimental data are necessary to verify which strategy proposed in the present paper is more appropriate for application to the rotor-dynamic system considered. This should definitely be investigated in the future.

A VALUES OF THE DETERMINISTIC VARIABLES

Table 1 – Values of the deterministic variables used in the computational simulations

Variable	Value	Unit	Description
m	60	kg	Mass of the rotor
I_x	0.6750	$\mathrm{kg{\cdot}m^2}$	Polar moment-of-inertia
I_y, I_z	0.3375	$\mathrm{kg{\cdot}m^2}$	Transversal moments-of-inertia
L^0	0.75	m	Distance to the left bearing
L^L	0.25	m	Distance to the right bearing
Ω	200	rpm	Rotating-speed of the shaft
e	0.001	m	Eccentricity of the rotor
a	0.0314	rad	Misalignment-angle of the polar axis
k_y^0, k_z^0, k_y^L, k_z^L	9425	$\mathrm{kg/s^2}$	Stiffness coefficients of the bearings
c_y^0, c_z^0, c_y^L, c_y^L	3	kg/s	Damping coefficients of the bearings
$\boldsymbol{\sigma}$, $\boldsymbol{\Psi}$	$0.10\,\boldsymbol{\delta}_4$	–	Scale parameter matrices

B STIFFNESS AND DAMPING COEFFICIENTS

The entries $C_{\mathrm{u}_i\mathrm{u}_j}$, $C_{\theta_i\theta_j}$, $C_{\mathrm{u}_i\theta_j}$, $C_{\theta_i\mathrm{u}_j}$ and $K_{\mathrm{u}_i\mathrm{u}_j}$, $K_{\theta_i\theta_j}$, $K_{\mathrm{u}_i\theta_j}$, $K_{\theta_i\mathrm{u}_j}$ are related to the damping and stiffness coefficients of the bearings by [21]:

$$C_{\mathrm{u_y u_y}} = c_\mathrm{y}^0 + c_\mathrm{y}^\mathrm{L}$$
$$C_{\mathrm{u_z u_z}} = c_\mathrm{z}^0 + c_\mathrm{z}^\mathrm{L}$$
$$C_{\theta_\mathrm{y}\theta_\mathrm{y}} = c_\mathrm{z}^0 (L^0)^2 + c_\mathrm{z}^\mathrm{L} (L^\mathrm{L})^2$$
$$C_{\theta_\mathrm{z}\theta_\mathrm{z}} = c_\mathrm{y}^0 (L^0)^2 + c_\mathrm{y}^\mathrm{L} (L^\mathrm{L})^2$$
$$C_{\mathrm{u_y}\theta_\mathrm{z}} = +c_\mathrm{y}^0 L^0 - c_\mathrm{y}^\mathrm{L} L^\mathrm{L}$$
$$C_{\mathrm{u_z}\theta_\mathrm{y}} = -c_\mathrm{z}^0 L^0 + c_\mathrm{z}^\mathrm{L} L^\mathrm{L}$$
$$C_{\theta_\mathrm{y}\mathrm{u_z}} = -c_\mathrm{z}^0 L^0 + c_\mathrm{z}^\mathrm{L} L^\mathrm{L}$$
$$C_{\theta_\mathrm{z}\mathrm{u_y}} = +c_\mathrm{y}^0 L^0 - c_\mathrm{y}^\mathrm{L} L^\mathrm{L}$$

$$K_{\mathrm{u_y u_y}} = k_\mathrm{y}^0 + k_\mathrm{y}^\mathrm{L}$$
$$K_{\mathrm{u_z u_z}} = k_\mathrm{z}^0 + k_\mathrm{z}^\mathrm{L}$$
$$K_{\theta_\mathrm{y}\theta_\mathrm{y}} = k_\mathrm{z}^0 (L^0)^2 + k_\mathrm{z}^\mathrm{L} (L^\mathrm{L})^2$$
$$K_{\theta_\mathrm{z}\theta_\mathrm{z}} = k_\mathrm{y}^0 (L^0)^2 + k_\mathrm{y}^\mathrm{L} (L^\mathrm{L})^2$$
$$K_{\mathrm{u_y}\theta_\mathrm{z}} = +k_\mathrm{y}^0 L^0 - k_\mathrm{y}^\mathrm{L} L^\mathrm{L}$$
$$K_{\mathrm{u_z}\theta_\mathrm{y}} = -k_\mathrm{z}^0 L^0 + k_\mathrm{z}^\mathrm{L} L^\mathrm{L}$$
$$K_{\theta_\mathrm{y}\mathrm{u_z}} = -k_\mathrm{z}^0 L^0 + k_\mathrm{z}^\mathrm{L} L^\mathrm{L}$$
$$K_{\theta_\mathrm{z}\mathrm{u_y}} = +k_\mathrm{y}^0 L^0 - k_\mathrm{y}^\mathrm{L} L^\mathrm{L}$$

The convention adopted here is as follows:

$C_{\mathrm{u}_i\mathrm{u}_j}$, $K_{\mathrm{u}_i\mathrm{u}_j}$: related to the reaction force in the equation for $\ddot{\mathrm{u}}_i(t)$ given a virtual-displacement $\partial\mathrm{u}_j$.

$C_{\theta_i\theta_j}$, $K_{\theta_i\theta_j}$: related to the reaction torque in the equation for $\ddot{\theta}_i(t)$ given a virtual-angle $\partial\theta_j$.

$C_{\mathrm{u}_i\theta_j}$, $K_{\mathrm{u}_i\theta_j}$: related to the reaction force in the equation for $\ddot{\mathrm{u}}_i(t)$ given a virtual-angle $\partial\theta_j$.

$C_{\theta_i\mathrm{u}_j}$, $K_{\theta_i\mathrm{u}_j}$: related to the reaction torque in the equation for $\ddot{\theta}_i(t)$ given a virtual-displacement $\partial\mathrm{u}_j$.

C NOTATION

Although the system matrices are of (4×4)-size, we made the choice to present the formulas in (3×3)-matrices due to didactic and space saving purposes. However, the reader shall have no difficulty in extending the reasoning to $(n\times p)$-matrices. The notation used here is as follows:

x, $\mathbf{x}$, x_i, $\mathbf{X}$, X_{ij}: deterministic variable, deterministic vector, deterministic component, deterministic matrix, deterministic entry.

$\mathbb{x}$, $\mathbb{X}$, $\mathbb{x}_i$, $\mathcal{X}$, $\mathcal{X}_{ij}$: random variable, random vector, random component, random matrix, random entry.

$^\mathrm{T}$: non-conjugate transpose.

dim( ): dimension.

det( ): determinant.

tr( ): trace.

supp( ): support.

E( ): expected-value.

Cov(,): covariance matrix.

$\mathcal{N}(\mu,\sigma^2)$: uni-variate normal distribution with mean μ, variance σ^2.

$\mathcal{MN}_{n,p}(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\Psi})$: $(n\times p)$-matrix-variate normal distribution with mean $\boldsymbol{\mu}$, rows scale $\boldsymbol{\sigma}$, columns scale $\boldsymbol{\Psi}$.

$\mathcal{MA}_n(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\Psi})$: $(n\times n)$-matrix-variate anti-symmetric normal distribution with mean $\boldsymbol{\mu}$, scales $\boldsymbol{\sigma}\boldsymbol{\Psi}=\boldsymbol{\Psi}\boldsymbol{\sigma}$.

$\mathbb{M}(n,p,\mathbb{F})$: ring of $(n\times p)$-matrices with entries in $\mathbb{F}$, null-element $\mathbf{0}_{n,p}$, identity-element $\boldsymbol{\delta}_{n,p}$.

$\mathbb{S}\mathrm{YM}(n)$: set of $(n\times n)$-symmetric matrices.

$\mathbb{S}\mathrm{YM}_+^*(n)$: set of $(n\times n)$-positive-definite matrices.

$\mathbb{O}(n)$: group of $(n\times n)$-orthogonal matrices, identity-element $\boldsymbol{\delta}_n$.

$\mathbb{so}(n)$: set of $(n\times n)$-anti-symmetric matrices.

Vectorization operator:

$$\mathrm{vec}(\mathbf{X})=\mathrm{vec}\left(\begin{bmatrix} X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\end{bmatrix}\right):=[X_{11}\ \ X_{21}\ \ X_{31}\ \ X_{12}\ \ X_{22}\ \ X_{32}\ \ X_{13}\ \ X_{23}\ \ X_{33}]^\mathrm{T}$$

Semi-half vectorization operator:

$$\mathrm{vecq}(\mathbf{X})=\mathrm{vecq}\left(\begin{bmatrix} X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\end{bmatrix}\right):=[X_{21}\ \ X_{31}\ \ X_{32}]^\mathrm{T}$$

Direct product or Kronecker product:

$$\mathbf{X}\otimes\mathbf{Y}=\begin{bmatrix} X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\end{bmatrix}\otimes\begin{bmatrix} Y_{11} & Y_{12} & Y_{13}\\ Y_{21} & Y_{22} & Y_{23}\\ Y_{31} & Y_{32} & Y_{33}\end{bmatrix}:=\begin{bmatrix} X_{11}\mathbf{Y} & X_{12}\mathbf{Y} & X_{13}\mathbf{Y}\\ X_{21}\mathbf{Y} & X_{22}\mathbf{Y} & X_{23}\mathbf{Y}\\ X_{31}\mathbf{Y} & X_{32}\mathbf{Y} & X_{33}\mathbf{Y}\end{bmatrix}=$$

$$=\begin{bmatrix}
X_{11}Y_{11} & X_{11}Y_{12} & X_{11}Y_{13} & X_{12}Y_{11} & X_{12}Y_{12} & X_{12}Y_{13} & X_{13}Y_{11} & X_{13}Y_{12} & X_{13}Y_{13}\\
X_{11}Y_{21} & X_{11}Y_{22} & X_{11}Y_{23} & X_{12}Y_{21} & X_{12}Y_{22} & X_{12}Y_{23} & X_{13}Y_{21} & X_{13}Y_{22} & X_{13}Y_{23}\\
X_{11}Y_{31} & X_{11}Y_{32} & X_{11}Y_{33} & X_{12}Y_{31} & X_{12}Y_{32} & X_{12}Y_{33} & X_{13}Y_{31} & X_{13}Y_{32} & X_{13}Y_{33}\\
\\
X_{21}Y_{11} & X_{21}Y_{12} & X_{21}Y_{13} & X_{22}Y_{11} & X_{22}Y_{12} & X_{22}Y_{13} & X_{23}Y_{11} & X_{23}Y_{12} & X_{23}Y_{13}\\
X_{21}Y_{21} & X_{21}Y_{22} & X_{21}Y_{23} & X_{22}Y_{21} & X_{22}Y_{22} & X_{22}Y_{23} & X_{23}Y_{21} & X_{23}Y_{22} & X_{23}Y_{23}\\
X_{21}Y_{31} & X_{21}Y_{32} & X_{21}Y_{33} & X_{22}Y_{31} & X_{22}Y_{32} & X_{22}Y_{33} & X_{23}Y_{31} & X_{23}Y_{32} & X_{23}Y_{33}\\
\\
X_{31}Y_{11} & X_{31}Y_{12} & X_{31}Y_{13} & X_{32}Y_{11} & X_{32}Y_{12} & X_{32}Y_{13} & X_{33}Y_{11} & X_{33}Y_{12} & X_{33}Y_{13}\\
X_{31}Y_{21} & X_{31}Y_{22} & X_{31}Y_{23} & X_{32}Y_{21} & X_{32}Y_{22} & X_{32}Y_{23} & X_{33}Y_{21} & X_{33}Y_{22} & X_{33}Y_{23}\\
X_{31}Y_{31} & X_{31}Y_{32} & X_{31}Y_{33} & X_{32}Y_{31} & X_{32}Y_{32} & X_{32}Y_{33} & X_{33}Y_{31} & X_{33}Y_{32} & X_{33}Y_{33}
\end{bmatrix}$$

Mirroring matrix is the $(n^2\times n(n-1)/2)$-matrix with $\{-1\ \ 0\ \ 1\}$-entries which transforms:

$$\mathrm{vec}(\mathbf{X}^{\mathrm{AS}})=\mathbf{A}_n\,\mathrm{vecq}(\mathbf{X}^{\mathrm{AS}});\qquad\qquad \text{for all anti-symmetric matrices } \mathbf{X}^{\mathrm{AS}}\in\mathbb{so}(n).$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Pseudo-inverse or Moore-Penrose inverse:

$$\mathbf{X}^{+} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}$$

Cancellation matrix or transition to anti-symmetric matrix:

$$\mathbf{A}_3^{+} = \begin{bmatrix} 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \end{bmatrix}$$

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