DINAME 2023 - Proceedings of the XIX International Symposium on Dynamic Problems of Mechanics,
A.S. De Paula, A.T. Fabro and M.R. Machado (Editors), ABCM.
Pirenópolis, Go, Brazil, February 26th to March 3rd, 2023

Different coherent states for lightly supported coupled pendula

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Abstract: We discuss and explain the phenomenon of synchronization in lightly supported mechanical systems. The investigations are focused on the models of self–excited pendula hanged on the horizontally oscillating beam, which is lightly connected with the external support. We uncover and analyze typical dynamical solutions within which the units synchronize and the bifurcations between them, determining the regions and the conditions supporting the synchronization. Our investigations exhibit, that with the increase of the size of the network, the number of co–existing attractors also increases, leading to possible multistability and new types of behaviours. The results obtained numerically match with the analytical ones obtained from the CoM (Centre-of-Mass) Theorem.

Keywords: pendula systems, synchronization, multistability, classical mechanics

INTRODUCTION

Synchronization is one of the most fundamental types of behaviours found in nature “Pikovsky *et al.* (2003)”. The appearance of the coherent motion of dynamical systems has been reported in nonlinear vibrations “Blekhman (1988)”, robotics “Nijmeijer and Rodriguez-Angeles (2003)”, complex networks “Ghosh *et al.* (2022)” or small–world systems “Barahona and Pecora (2002)”, just to mention a few. A large part of synchronization problems refers to fundamental mechanical systems based on coupled pendula. The studies on synchronous dynamics have been performed for various pendula–type models, e.g. rotor-pendula “Fang and Hou (2018)”, chaotic pendula “Baker *et al.* (1998)” or Huygens’ coupling schemes “Ramirez and Alvarez (2015)”, just to mention a few.

In this research we investigate the dynamics and possible synchronous configurations for coupled pendula arranged in a lightly supported system.

RESULTS

The model of interest is schematically shown in Fig. 1.

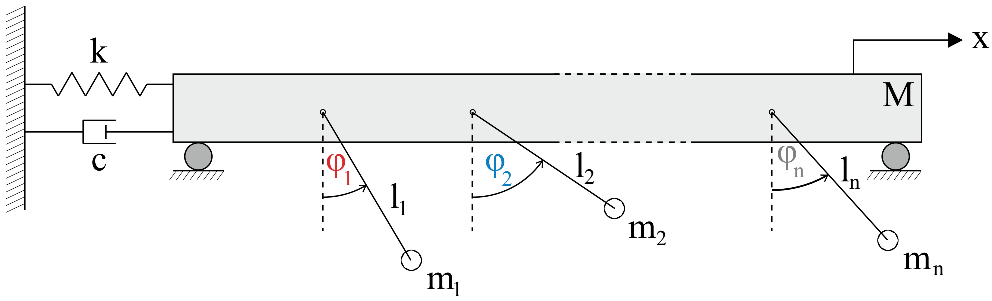

Figure 1 – The model of n oscillating pendula suspended on the horizontally oscillating beam.

The system shown in Fig. 1 consists of the beam of mass M [kg], which is connected with the support by the spring of stiffness k [N/m] and the damper of damping coefficient c [Ns/m]. The beam can oscillate in the horizontal direction and its position is denoted by variable x [m]. The considered structure supports the network of n pendula (nodes) with masses m_i [kg] and lengths l_i [m], as shown in Fig. 1 ($i = 1, \ldots, n$). The pendula are equipped with the van der Pol type drives “Holmes and Rand (1978)” and their self–excited oscillations induce the motion of the beam, which allows to transfer the energy between the pendula. The angular displacement of the i–th node is given by variable $\varphi_i \in (-\pi, \pi]$.

The dynamics of the system presented in Fig. 1 can be investigated using the following equations of motion:

$$\begin{cases} (M + \sum_{i=1}^{n} m_i)\ddot{x} + kx + c\dot{x} + \\ \sum_{i=1}^{n} m_i l_i (\ddot{\varphi}_i \cos\varphi_i - \dot{\varphi}_i^2 \sin\varphi_i) = 0, \\ m_i l_i^2 \ddot{\varphi}_i + m_i l_i \ddot{x} \cos\varphi_i + \\ c_{\varphi_i} \dot{\varphi}_i (\mu \varphi_i^2 - 1) + m_i g l_i \sin\varphi_i = 0, \end{cases} \tag{1}$$

where $i = 1, \ldots, n$.

During the analysis we have fixed the following parameters of system (1): $M = 10$ [kg] (the mass of the beam), $k = 4$ [N/m] (the stiffness of the spring), $c = 1.53$ [Ns/m] (the damping coefficient of the damper), $g = 9.81$ [m/s^2] (the standard gravity term) and $\mu = 32.88$ (the van der Pol type drive parameter). The damping of the i–th pendulum c_{φ_i} has been selected depending on the pendulum's mass m_i and its length l_i to preserve fixed logarithmic decrement for each node.

When the beam in Fig. 1 is lightly supported, i.e. $k \approx 0$ [N/m] and $c \approx 0$ [Ns/m], we can approximate system (1) as an isolated one. In such a case, the centre of mass of the model (the beam and the suspended pendula) is not moving, according to the Centre-of-Mass (CoM) Theorem. Assuming similar pendula lengths $l_i \approx l = \text{const}$, $i = 1, \ldots, n$ and investigating the horizontal position of the centre of mass of the system with small oscillations of the nodes, one can show, that:

$$\hat{M}\dot{x} + \sum_{i=1}^{n} m_i l \dot{\varphi}_i = 0, \tag{2}$$

where $\hat{M} = M + \sum_{i=1}^{n} m_i$ is the total mass of the system (the beam and the pendula).

The solutions of equation (2) depend on the number of pendula n and possible synchronous configurations. The latter ones can include typical in–phase and anti–phase solutions (for $n = 2$ oscillators), or phase–locking scenarios ($n = 3$). With the increased number of pendula, more complex and new types of behaviours can be observed, which has been presented in Fig. 2 for $n = 4$ nodes and lengths $l_1 = 0.24848$ [m], $l_2 = 0.24849$ [m], $l_3 = 0.2485$ [m] and $l_4 = 0.24851$ [m].

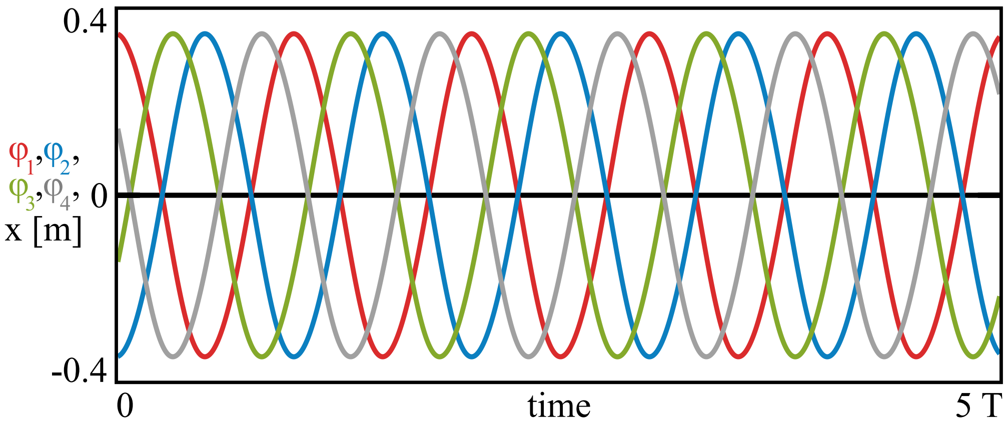

Figure 2 – Possible behaviours of the pendula for $m_1 = m_2 = m_3 = m_4 = 1.0$ [kg] – identical masses and the traveling phase state. System (1) with $n = 4$ nodes.

If the pendula synchronize with equal amplitudes and phases locked, i.e. $\varphi_i = A\sin(\alpha t + \beta_i)$, $i = 1, 2, 3, 4$, where $\beta_1 = 0$ and $\beta_2, \beta_3, \beta_4 \in [0, 360)$ degrees, then for $\dot{x} \approx 0$ [m/s] (the beam slightly oscillating) equation (2) transforms into the following system:

$$\begin{cases} m_1 + m_2\cos\beta_2 + m_3\cos\beta_3 + m_4\cos\beta_4 = 0, \\ m_2\sin\beta_2 + m_3\sin\beta_3 + m_4\sin\beta_4 = 0. \end{cases} \tag{3}$$

If the pendula are identical, i.e. $m_i = \text{const}$, $i = 1, 2, 3, 4$, then system (3) can be simplified by dividing the equations by the masses. It can be easily shown, that in such scenario, the solutions of (3) are given as follows: $\beta_2 = 180$, $\beta_3 = \hat{\beta}$ and $\beta_4 = \hat{\beta} + 180$, where $\hat{\beta} \in [0, 360)$ becomes the parameter of system (3) (the order of the values of phases β_2, β_3 and β_4 can be arbitrarily chosen since the pendula are identical). In this case, system (3) indicates infinitely many phase–locked synchronous states along parameter $\hat{\beta} \in [0, 360)$, within which pendula form two clusters: (i) the cluster of the 1st and the 2nd pendulum oscillating in the anti–phase and (ii) the cluster of the 3rd and the 4th pendulum also oscillating in the anti–phase. The phase shift between the clusters is given by parameter $\hat{\beta}$, which also determines the phase between the 1st and the 3rd pendulum.

The example of the described scenario is presented in Fig. 2 for $m_1 = m_2 = m_3 = m_4 = 1.0$ [kg], where the time plots of the nodes are marked in red (1st), blue (2nd), green (3rd) and grey (4th), while the beam is shown in black. Since the oscillations within the clusters are anti–phase (red vs. blue and green vs. grey), the beam is not moving. As we have observed, the phase between the clusters (parameter $\hat{\beta}$) changes continuously with time, leading to the solution, which can be called the 'traveling phase' state. The traveling character of the observed configuration is shown in Fig. 3.

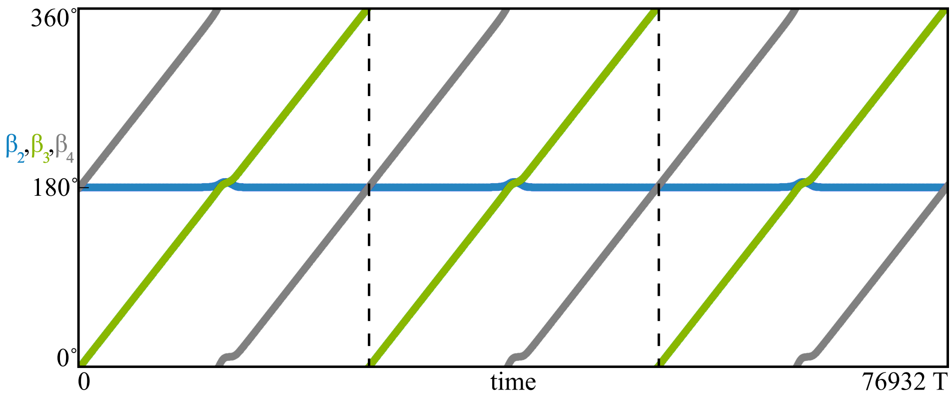

Figure 3 – The traveling of phases β_2, β_3 and β_4 for identical pendula $m_i = 1.0$ [kg], $i = 1,2,3,4$. The arise of the traveling phase state is related to the similarities in the pendula properties (the masses).

The diagrams included in Fig. 3 present the values of β_2, β_3 and β_4 phases in a long time interval of model (1) simulations. As one can see, the numerical solution matches with the analytical one ($\beta_2 = 180$, $\beta_3 = \hat{\beta}$, $\beta_4 = \hat{\beta} + 180$) and the configuration of the clusters (the 1st and the 2nd pendulum clustered; the 3rd and the 4th pendulum clustered) is preserved, except for short disruptions around $\beta_3 = 180$, when the clusters overlap in the phase space. The results shown in Fig. 3 include three cycles of phase traveling, which has been indicated by the vertical, dashed lines; the approximated length of one full cycle equals 25 644 T.

CONCLUSIONS

In this research we have investigated the dynamics of coupled self–excited pendula, arranged in a lightly supported system. Depending on the network's size (the number of the nodes suspended on the beam), one can observe different types of synchronous configurations and behaviours. We have described typical dynamical structures, including the in–phase and the anti–phase synchronization, as well as more complex patterns like the clustering of the pendula or the phase–locked solutions. During the research we have uncovered the 'traveling phase' state, which is characterized by the continuous change of the phases between the synchronized oscillators. The analysis of the considered models has also exhibited possible high multistability with many co–existing attractors, that can be observed for slightly different nodes. The analytical results obtained from the CoM Theorem match with the numerical ones calculated during simulations. The former ones explain and allow to understand the dynamics that is observed within the models, especially the synchronous configurations between the pendula.

ACKNOWLEDGMENTS

This work has been supported by the National Science Centre, Poland, SONATA Programme (Project No 2019/35/D/ST8/00412) and OPUS Programme (Project No 2018/29/B/ST8/00457).

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