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Stick-slip oscillator: a stochastic approach for the run-time

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Abstract: In this article, the run-time required to simulate an oscillator with uncertain dry-friction forces is studied from a stochastic perspective. The oscillator can exhibit the stick-slip phenomenon due to the discontinuos transition between the static and the dynamic friction. The hypothesis that the magnitude of the dry friction force and the duration of the stick phases may have a direct influence on the run-times is explored, as small time-steps are often required to capture the transition between phases when numerical integration is used. The stochastic problem is solved with the Monte Carlo method in combination with two different approximation strategies: a Runge-Kutta numerical integration and an analytical approximation based on the Multiple Scales method. In both cases, the run-time is treated as a random variable. Its dependency on parameters such as the dynamic friction coefficient, and statistics like the mean stick-duration or the number of sticks, are studied. The results, with $5 \cdot 10^5$ *realizations, show that the run-time with the analytical approach is, in mean,* 3.68 *times faster than the numerical one, and it is also less sensitive to variations, considering that the standard deviation is* 2.9 *times smaller than that obtained with the numerical scheme. The improvement observed in the run-time associated with the analytical method can be a determining factor for the feasibility of a stochastic study, given that the Monte Carlo approach requires a large number of realizations, which is time-costly.*

Keywords: Analytical approximation, stick-slip phenomenon, Monte Carlo method, Computation cost

INTRODUCTION

The Monte Carlo (MC) method is an important tool to deal with stochastic problems. It permits the construction of statistical models for random object transformations. An introductory book treating the concepts of probabilities behind the method, its description and some numerical applications in a Matlab environment can be found in de Cursi and Sampaio (2015); Sampaio and Lima (2012). The method is based on the law of large numbers, therefore a large amount of realizations is required to construct a statistical model, as shown in Sobol (1994). To construct the statistical models, the method transforms the stochastic problem into many deterministic ones, where in each, a realization of the random input is used. In particular, for the current application, the procedure employed is, first, to generate a random sample of the input. The next step is to transform each realization of the sample according to some mathematical transformation. This transformation is given by the equations of motion of the oscillator. The results obtained are realizations of the sample of the output, therefore they are also treated as random variables that need to be stored. After collecting a sufficiently large number of realizations, this sample from the output is used to construct a statistical model. A convergence criterion is used to determine whether the sample is sufficiently large. More realizations are added until the criterion is met.

Even though each realization is deterministic in nature, neither the inputs, the outputs, or the computational costs are. The uncertainty associated with the inputs are propagated to the outputs variables, as shown in de Cursi and Sampaio (2012); Lima and Sampaio (2018). Moreover, the elevated number of calculations required to assure an accurate statistical model makes the MC method a big data problem, especially when the transformation is given by a differential equation that is solved by numerical integration, as shown in Lima and Sampaio (2021). Given that the computational resources are limited, the computational costs, such as the total run-time, can be of the utmost importance.

In this paper, the run-time associated with the simulation of a random oscillator is studied from a stochastic perspective. This is achieved by comparing the results of the MC method combined with a Runge-Kutta numerical integration scheme and the MC combined with an analytical approximation based on the multiple scales method. Up to the authors' knowledge, this study is a novelty given that most papers concerning stochastic simulations, such as Wilhelm et al. (2008); Lee et al. (2009), ignore the fact that the run-times are also of stochastic nature. Eventually, the run-time plays a role in determining whether a stochastic analysis is feasible, and its behavior could have an impact regarding the efficient assignment of the available resources.

When numerical integrations are used, the accuracy of the response is related with the time-step employed. Especially, when dealing with a dry friction oscillator that can exhibit stick-slip vibrations. The works of Feeny et al. (1998); Awrejcewicz and Olejnik (2007); Goicoechea et al. (2021) show that this kind of system has two different types of dynamics with an abrupt transition between them. The time-step used influences the accuracy with which the transitions are captured. In other words, an improvement in the accuracy of the numerical integrations is usually linked to a decrease in the integration time-step, but the smaller the step, the higher the amount of data generated and the higher the run-time required. For this reason, the hypothesis that the stick-phase duration and the magnitude of the dry friction have a direct influence on the run-time is explored, especially when numerical integrations are used. Also, MC with analytical approximations is used as a means to reduce the computational costs, and the results of both approaches are compared.

SYSTEM DYNAMICS AND DEFINITION OF THE STICK AND SLIP PHASES

The system analyzed in this paper is a mass-spring-damper system where the mass moves over a constant-speed moving belt, as shown in Fig. 1. The dry friction between the two surfaces in contact can induce stick-slip vibrations. When this occurs, the system's response is characterized by two different behaviors, called stick and slip phases. These phases alternate with an abrupt transition and have a non-zero duration, that is, they are not instantaneous.

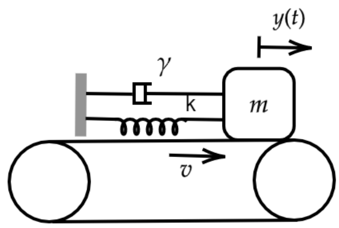

Figure 1 – Mass-spring-damper system with dry friction.

The initial value problem (IVP) for the system is given by

$$m\ddot{y}(t) + \gamma\dot{y}(t) + k\,y(t) = f_{at}\,, \tag{1}$$

with initial conditions $y(0)$ and $\dot{y}(0)$. In the previous equation, y is the position of the mass m, γ the damping coefficient, k the spring stiffness, f_{at} the friction force between the mass and the belt and V the relative velocity between them. For $V \neq 0$, the friction force is modeled as

$$f_{at}(V) = \frac{1}{3}\,a\,V\,(V^2 - 3) + f_d\,sign(V) \tag{2}$$

where $V = (v - \dot{y})$, a is constant, v is the speed of the belt, and f_d is the dynamic friction force. For $V = 0$, the dry friction force can assume values between $[-f_e, f_e]$, where f_e is the magnitude of the static friction. Fig. 2 illustrates the behavior of V.

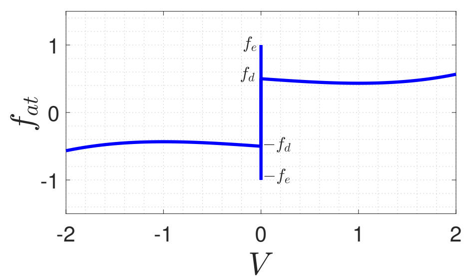

Figure 2 – Friction force model with $a = 0.1$ and $f_d = 0.5$.

Due to the discontinuity in the transition between static and dynamic friction, the problem is discontinuous and the solution is expressed with a piece-wise switch as

$$y(t) = \begin{cases} y_j(t) & \text{, if stick conditions are met} \\ y_p(t) & \text{, otherwise} \end{cases} \tag{3}$$

where $y_j(t)$ is the solution corresponding to the stick phases, $y_p(t)$ is the solution to the slip phases, with

$$j \in [1,..,N_{sticks}],\;\; p \in [1,..,N_{slips}],\;\; l \in [1,..,N_{sticks} + N_{slips}] \tag{4}$$

In the previous, N_{sticks} and N_{slips} are the total number of phases of sticks and slips, respectively. Figure 3 shows an example sequence of a stick-slip oscillator's response that begins with a stick phase during a interval $[0, t_n]$. More details on how to construct this model can be found in Lima and Sampaio (2017a,b). The conditions that allow to predict whether the current phase is a stick or a slip will be obtained in what follows. In all cases, the notation $t_{0,l} \leq t \leq t_{e,l}$ is used to refer to the beginning and ending instant of each phase, l, whether a stick or slip.

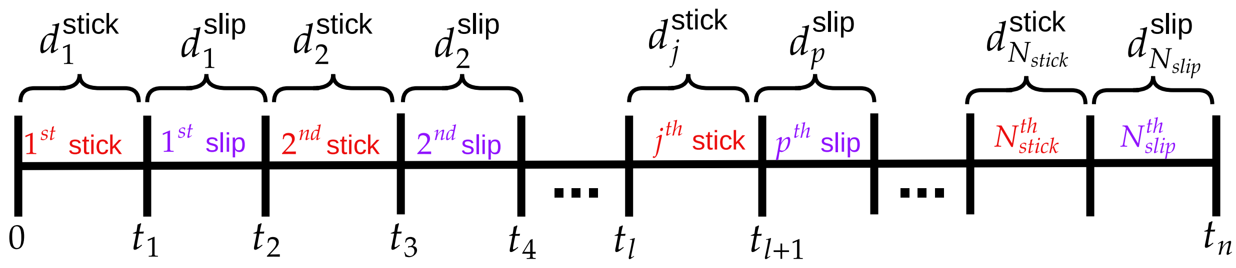

Figure 3 – Sequence of a stick-slip oscillator response that begin with a stick phase.

The stick phase exhibits the following properties: a null relative velocity, $V = 0$, and a friction force that lies within the interval $-f_e \leq f_{at} \leq f_e$. During a stick phase the velocity of the mass equals the belt's constant velocity, $\dot{y}_j(t) = v$. Therefore the mass' acceleration is zero, $\ddot{y}_j(t) = 0$. Using these observations in combination with Eq. (1), one obtains

$$\gamma v + k y_j = f_{at}\,, \tag{5}$$

that, together with $-f_e \leq f_{at} \leq f_e$, provide a limit on the values that y can take for stick phases to occur. These limits are given by

$$\frac{-f_e - \gamma v}{k} \leq y_j \leq \frac{f_e - \gamma v,}{k}\,. \tag{6}$$

During this phase the mass exhibits a uniform motion with an analytical solution given by

$$y_j(t) = y_j(t = t_{0,l}) + v(t - t_{0,l}) \text{ for } t_{0,l} \leq t \leq t_{e,l}, \tag{7}$$

where $t_{0,l}$ and $t_{e,l}$ are the instants associated to the start and the end of the j-th phase, which in this case corresponds to a stick. Thus, $y_j(t = t_{0,j})$ represents the position of the mass at the instants of initiation of the stick phases. These values also coincide with the position of the mass at the end of the immediately previous slip phase.

Now, the general solution of the oscillator problem can be written as a piece-wise response in the form of

$$y(t) = \begin{cases} y_j(t) = y_j(t = t_{0,l}) + (t - t_{0,l})v & \text{, if } \frac{-f_e - \gamma v}{k} \leq y \leq \frac{f_e - \gamma v,}{k} \text{ and } \dot{y} = v \\ y_p(t) & \text{, otherwise} \end{cases} \tag{8}$$

where the condition to check if the current state corresponds to a stick or a slip phase have been is now stated explicitly, and where $y_p(t)$ represent the solution of the slip phases, yet to be found.

If the necessary conditions for a stick phase to occur are not met, then the system is in a slip phase. In that case, the relative velocity between the mass and the belt is not null and the friction forces are defined by Eq. (2), so Eq. (1) becomes

$$m\ddot{y}_p(t) + \gamma \dot{y}_p(t) + k y_p(t) = \frac{1}{3} a V (V^2 - 3) + f_d\, sign(V)\,. \tag{9}$$

In the previous equation, the slip phase is governed by a nonlinear IVP with an unknown analytical solution. In what follows, an analytical approximation for $y_p(t)$, valid during a slip phase, will be obtained. The analytical approximation is based on the Multiple Scale method. A detailed description of the whole procedure can be found in Gomes, Lima, and Sampaio (2021).

THE ANALYTICAL APPROXIMATION

To construct an approximation for $y_p(t)$, the Multiple Scales method, which is a perturbation method, is used in this paper. The central idea of this technique is to transform the often complex IVP with an unknown analytical solution into a family of linear IVPs with known analytical solutions, by introducing a perturbation parameter, see Kevorkian and Cole (1996); Gomes, Lima, and Sampaio (2019). To do this, a solution in the form of a power series of the perturbation parameter is proposed. When the proposal is introduced into the equation, by application of the Fundamental Theorem of Perturbation Theory, a new set of IVPs is obtained, Simmonds and Mann (1988). To compute a solution, all these linear IVPs should be solved hierarchically, but this is an impossible task. For this reason, we produce an N-term approximation by truncating the series. The number of terms is related to the quality of the approximation, i.e. the domain of validity. The details to computing an analytical approximation to the solution of Eq. (9) are explained in Gomes, Lima, and Sampaio (2021). A one-term approximation is given by

$$y_p(t) \approx e^{z\,(t-t_{0,l})}\,[C_{1,p}\cos(\sqrt{k}\,(t-t_{0,l})) + C_{2,p}\sin(\sqrt{k}\,(t-t_{0,l}))] + \frac{f_d}{k}\,, \tag{10}$$

where ε is a perturbation parameter that was included in Eq. (9) as a factor multiplying $\dot{y}_p$, $z = \varepsilon(a-\gamma)/2$, $C_{1,p} = y_p(t = t_{0,l}) - f_d/k$ e $C_{2,p} = [\dot{y}_p(t = t_{0,l}) - zd]/\sqrt{k}$. The values of $y_p(t = t_{0,l})$ and $\dot{y}_p(t = t_{0,l})$, the initial conditions of the phase, vary. At the beginning of the simulation, they match the initial conditions $y(0)$ and $\dot{y}(0)$, respectively. After that, they coincide with the value at the ending of the previous phase, that is, if the current phase is a slip: $y_p(t = t_{0,l}) = y_j(t = t_{e,l-1})$ and $\dot{y}_p(t = t_{0,l}) = \dot{y}_j(t = t_{e,l-1})$. Otherwise, $y_j(t = t_{0,l}) = y_p(t = t_{e,l-1})$ and $\dot{y}_j(t = t_{0,l}) = \dot{y}_p(t = t_{e,l-1})$

One of the features of the Perturbation Theory is that the perturbation parameter should assume small values, i.e. $\varepsilon << 1$, see Simmonds and Mann (1988). Thus, once the value of z depends on ε, the value of z is also small, i.e $z \to 0$ and consequently $e^{zt} \to 1$. With these assumptions and using trigonometric transformations, Eq. (10) can be rewritten as

$$y_p(t) \approx C_{3,p}\,cos(\sqrt{k}\,(t-t_{0,l}) - C_{4,p}) + \frac{f_d}{k} \tag{11}$$

where $C_{3,p} = \sqrt{C_{1,p}^2 + C_{2,p}^2}$ and $C_{4,p} = atan(C_{2,p}/C_{1,p})$. This rewriting of Eq. (11) is used as an aid to calculating an expression for the transition instants between phases.

TRANSITION INSTANTS BETWEEN STICK-SLIP PHASES

As already described, the oscillator shows dynamics characterized by two highly distinctive phases. The mass during stick phases exhibits a uniform motion given by Eq. (5). While during slip phases, the movement is given by a nonlinear expression as in Eq. (9).

The transition instants between phases are amongst the most relevant information required to characterize the dynamics of the stick-slip behavior. To detect these instants with accuracy, if a numerical integration is used, a small time-step is needed. This can be a determining factor for the feasibility of the problem. Since numerical methods calculate the information of the current instant using the data of the immediately previous instant, the smaller the time-step, the more calculations are required and the more costly the problem is. In addition, the conditions to identify if the numerical integrator is in a stick or slip phase must be tested every single step. All these details influence the run-time of the numerical integration. In contrast, an analytical approximation allows to predict when the transitions will occur without the need of a time-step smaller than the duration of the phase itself, reducing the computational cost.

To calculate the transition instants analytically, the parameter t is isolated in the right side of Eq. (7) and (11) according to the condition of each phase.

First, we calculate the transition instants from stick-to-slip using Eq. (7). The stick phase is defined by a null relative velocity between the mass and the belt, and the mass position being in the range of $[y_{min}\,,\ y_{max}]$. The sign of the belt velocity determines if the mass position is either y_{min} or y_{max} at the transition instant. So,

$$y_j(t = t_{0,l}) + (t - t_{0,l})\,v = \begin{cases} y_{max}, & \text{if } v > 0 \\ y_{min}, & \text{if } v < 0 \end{cases} \tag{12}$$

and depending on the sign of the belt velocity, the transition instants from stick-to-slip are given by

$$t_{\text{end},l} = \begin{cases} \dfrac{y_{max} - y_j(t = t_{0,l})}{v} + t_{0,l}, & \text{if } v > 0 \\[2ex] \dfrac{y_{min} - y_j(t = t_{0,l})}{v} + t_{0,l}. & \text{if } v < 0 \end{cases} \tag{13}$$

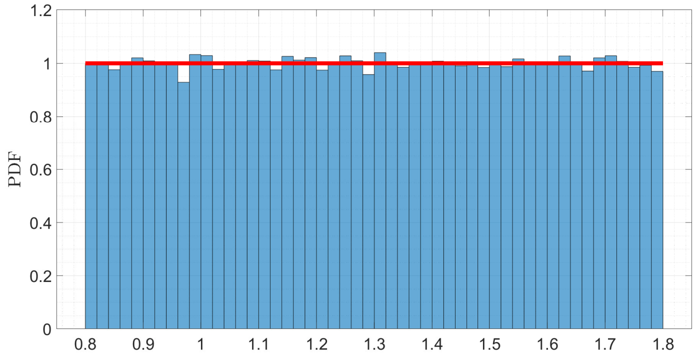

Figure 4 – PDF of F_d and a normalized histogram with $5 \cdot 10^5$ realizations.

To calculate the transition instants from slip-to-stick, the time derivative of Eq. (11) is stated as

$$\dot{y}_p(t) \approx -C_{3,p} \sin\left(\sqrt{k}(t - t_{0,l}) - C_{4,p}\right)\sqrt{k}\,, \tag{14}$$

and the mass velocity should be equal to velocity of the belt, so

$$-C_{3,p} \sin(\sqrt{k}(t - t_{0,l}) - C_{4,p})\sqrt{k} = v. \tag{15}$$

Thereby, the parameter t is isolated, obtaining

$$t_{\text{end},l} = \frac{\left[asin\left(\frac{-v}{C_{3,p}\sqrt{k}}\right) + C_{4,p}\right]}{\sqrt{k}} + t_{0,l}. \tag{16}$$

It should be noted that the condition that leads to Eqs. (15) and (16), that the mass velocity equals the belt velocity, is a necessary but not sufficient condition to determine the transition instants. To complete the condition, the position of the mass must not be in the interval $(y_{min}\,,\ \ y_{max})$.

STOCHASTIC APPROACH - THE MONTE CARLO METHOD

The Monte Carlo method permits to construct statistical models for random object transformations. The principal idea of the MC is to transform a stochastic problem in several deterministic problems. The first step is to define a sample of the input random object with known distribution. Then, each realization is transformed according to the deterministic problem. Lastly, the outputs of interest are saved and used to construct their statistical models. The resolution procedure is stopped after some convergence criterion is met, otherwise more realizations are simulated, as detailed in Sampaio and Lima (2012). Due to the elevated amount of data required to construct an accurate statistical model, among other factors, stochastic problems can be classified as big data problems.

In this paper, the dynamic friction force is modeled as a random variable. The uncertainties in this random variable influence the output variables of interest, such as the transition instants, the phase duration, the number of sticks, the position of the system, and the computational cost. In this study, the dynamic friction is modeled as a random variable with a uniform distribution, and its probability density function (PDF) is

$$p_{F_d}(x) = \begin{cases} \frac{1}{L_s - L_i}, & L_i \leq x \leq L_s \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

where L_i is the inferior limit of the support and L_s its upper limit. These limits are chosen to respect two features: the range must be positive and the upper limit must not exceed the static friction force, f_e. So, the support is defined as $[0.8,\ 1.8]$. Figure 4 shows the normalized histogram with $5 \cdot 10^5$ realizations and the PDF of the uniform distribution, used to define the dynamic friction as a random variable.

RESULTS AND DISCUSSIONS

The stochastic problem is tackled with two different strategies: 1) a 'numerical' case that combines the Monte Carlo method with numerical integration based on the Runge-Kutta of 4^{th} and 5^{th} order and 2) an 'analytical' case that combines

Monte Carlo with an analytical approximation based on the Multiple Scale method. As already mentioned, the numerical integration used MATLAB's built-in Runge-Kutta method of 4^{th} and 5^{th} order, which is a variable step integrator. As such, one can define a maximum time step, although the integrator is yet free to choose smaller ones if required. In this case, the maximum time step used was 10^{-3} s. The numerical and analytical approximations were simulated using the parameters shown in Tab. 1 and the simulated time is $[0, 2000]$s. A total number of $5 \cdot 10^5$ realizations was used.

Parameter	Value	Unit	Parameter	Value	Unit
m	1	Kg	k	0.1	N/m
v	-2	m/s	$y(0)$	1	m
γ	1	(N s)/m	$\dot{y}(0)$	4	m/s
a	0.1	(Kg s)/(m^2)	f_e	2	N
ε	0.0001	-	g	9.81	m/s^2

Table 1 – Parameters used in the analytical and numerical simulations.

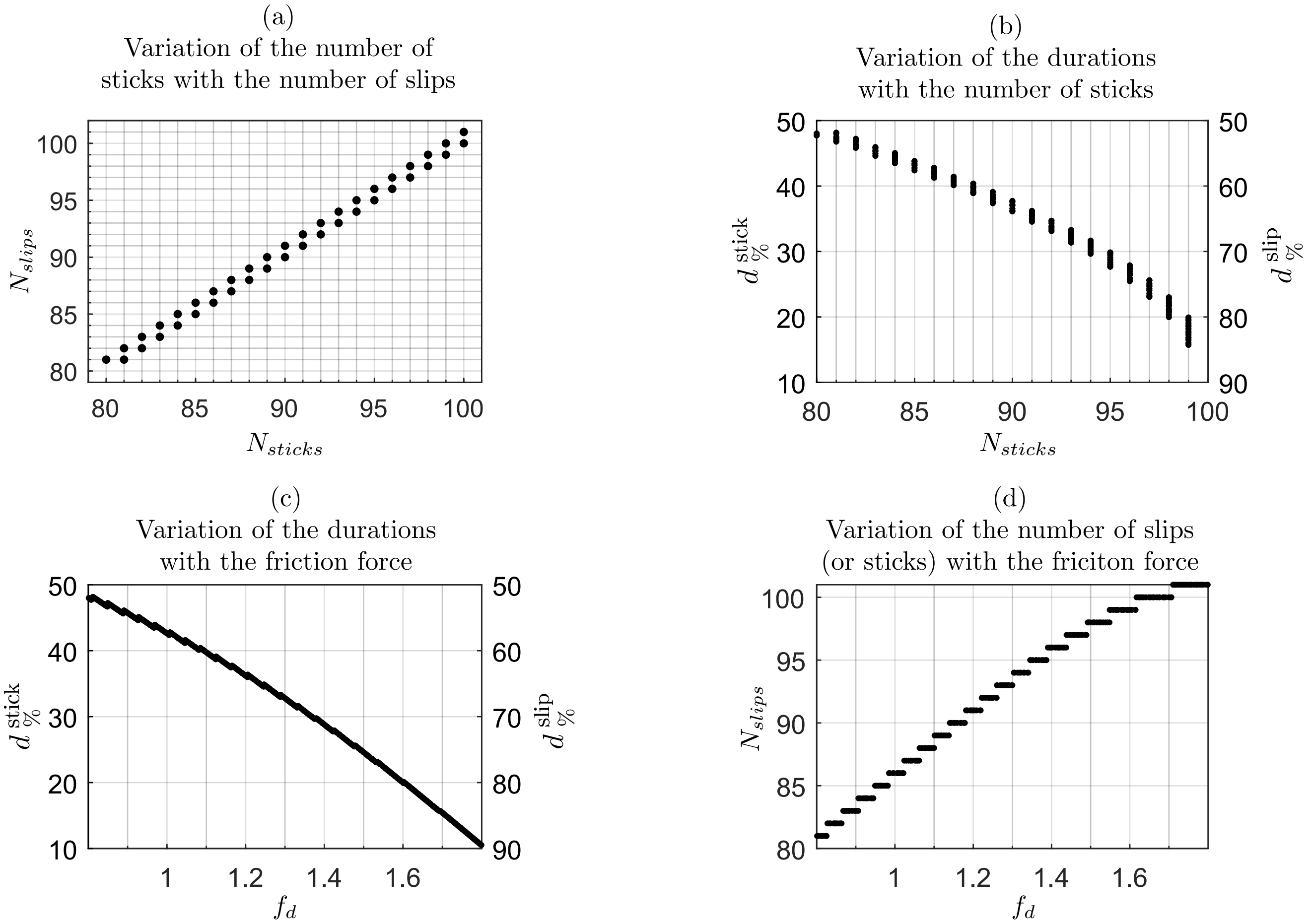

Figure 5 – Scatter plot showing: (a) the relation between the number of sticks (N_{sticks}) and slips (N_{slips}); (b) the variation of the duration of all the sticks in each realization ($d_{\%}^{stick}$) and the duration of all the slips in each realization ($d_{\%}^{slip}$), with the number of sticks. The durations are expressed as a percentage of the total simulation time; (c) the variation of the durations with the friction force f_d; (d) the variation of number of slips with the friction force f_d.

In Fig. 6 one of the simulated realizations is shown. The simulation show that both methods, analytical and numerical, provide very similar results, a behavior that is observed for any of the realizations. As shown in Fig. 3, the stick-slip oscillator's response can be divided into j stick and p slip phases. Each phase has an associated duration d_j^{stick} or d_p^{slip}. Therefore, the total stick duration and total slip duration (d^{stick} and d^{slip}, in seconds, or $d_{\%}^{\text{stick}}$ and $d_{\%}^{\text{slip}}$, as a percentage of the simulation's duration), are given by

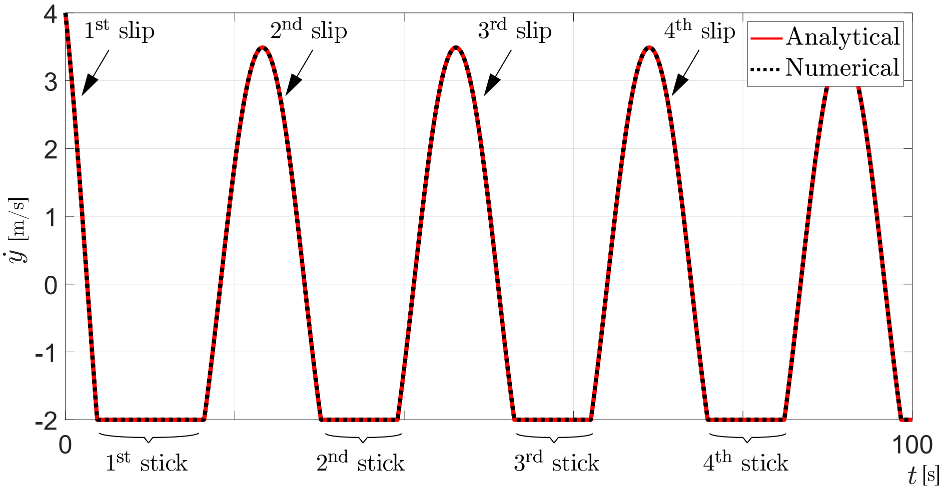

Figure 6 – The speed $\dot{y}(t)$ for one of the realizations until the 4^{th} stick and slip phases.

$$
\begin{aligned}
d^{\text{stick}} &= \sum_{j=1}^{N_{\text{stick}}} d_j^{\text{stick}}, \quad d^{\text{slip}} = \sum_{p=1}^{N_{\text{slip}}} d_p^{\text{slip}} \\
d_{\%}^{\text{stick}} &= \sum_{j=1}^{N_{\text{stick}}} \frac{d_j^{\text{stick}}}{t_n} 100, \quad d_{\%}^{\text{slip}} = \sum_{p=1}^{N_{\text{slip}}} \frac{d_p^{\text{slip}}}{t_n} 100
\end{aligned}
\tag{18}
$$

Figure 5(a) shows that the number of stick and slip phase, in this problem, present a relation given by

$$
|N_{sticks} - N_{slips}| = \begin{cases} 0 \\ 1 \end{cases} \tag{19}
$$

Fig. 5 (b) depicts a scatter plot with the stick durations ($d_{\%}^{stick}$) and the slip durations ($d_{\%}^{slip}$) against the number of sticks (N_{sticks}). All the durations are expressed as a percentage of the total simulated time. The graphic shows a tendency for the stick duration to reduce when the number of sticks (or slips) increases. Also, this is equivalent to stating that the slip duration increases when the number of sticks (or slips) increases. Figure 5(c) illustrates a scatter plot of $d_{\%}^{stick}$ and $d_{\%}^{slip}$ against the dynamic friction force f_d. The duration of each phase is related to the difference between the static and dynamic friction forces. In this problem, the static friction is fixed and the only the dynamics one varies. Fig. 5(c) shows that the higher the dynamic friction force, the smaller the difference between the friction coefficients and the higher the percentage of the slip phases during a simulation. Also, as a consequence, the smaller the percentage of stick phases is. Figure 5(d) depicts a scatter plot for the number of slips (N_{slips}) against the friction force f_d.

The results for the run-time taken by the numerical integration and the analytical approximation, are shown in Fig. 7. The support of the normalized histogram of the numerical integration run-time is about $[23.5 \;\; 50.3]$ while the support of the normalized histogram of the analytical approximation run-time is about $[7.8, \;\; 15.3]$. The mean values are 39.25 s and 10.66 s, and the standard deviations 3.28 s and 1.13 s, for the numerical and the analytical methods, respectively. These results show that the run-time with the analytical approximation are, in mean, 3.68 times faster than the numerical ones, and also that they are less sensitive to variations, considering that the standard deviation is 2.9 times smaller than that of the numerical integration.

The standard deviation of the run-time with the analytical approximation being smaller than the numerical one indicates that the numerical integration is more affected by the maximum value of the dynamic friction and stick phase duration than the analytical one. This observation agrees with the results depicted in Fig. 8, where the influence of the dynamic force F_d on the run-time is analyzed. Figure 8(a) shows a top of view for the joint normalized histogram of the random variables run-time and dynamic force F_d for the analytical case, and Fig. 8(b) for the joint normalized histogram of the random variables run-time and dynamic force F_d for the numerical case. In each plot, the line in black represents the moving mean, as we can see in the Fig. 8(b) , while in Fig. 8(a) the analytical one remains closer to being a horizontal line, thus, they are less prone to variations with this parameter. This validates the second part of the hypothesis that the friction not only affects the duration of the sticks and slips phases, but also that it has an effect over the run-time.

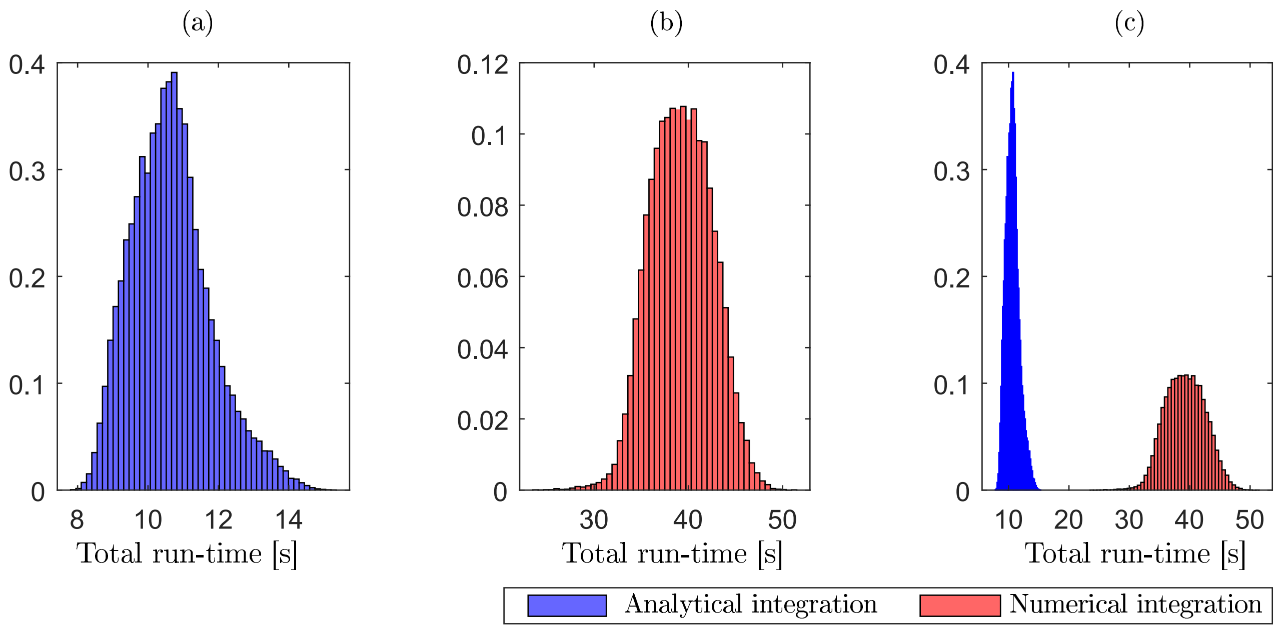

Figure 7 – Normalized histograms of the run-time to simulate the Monte Carlo method using the analytical approximation (blue) and numerical integration (red).

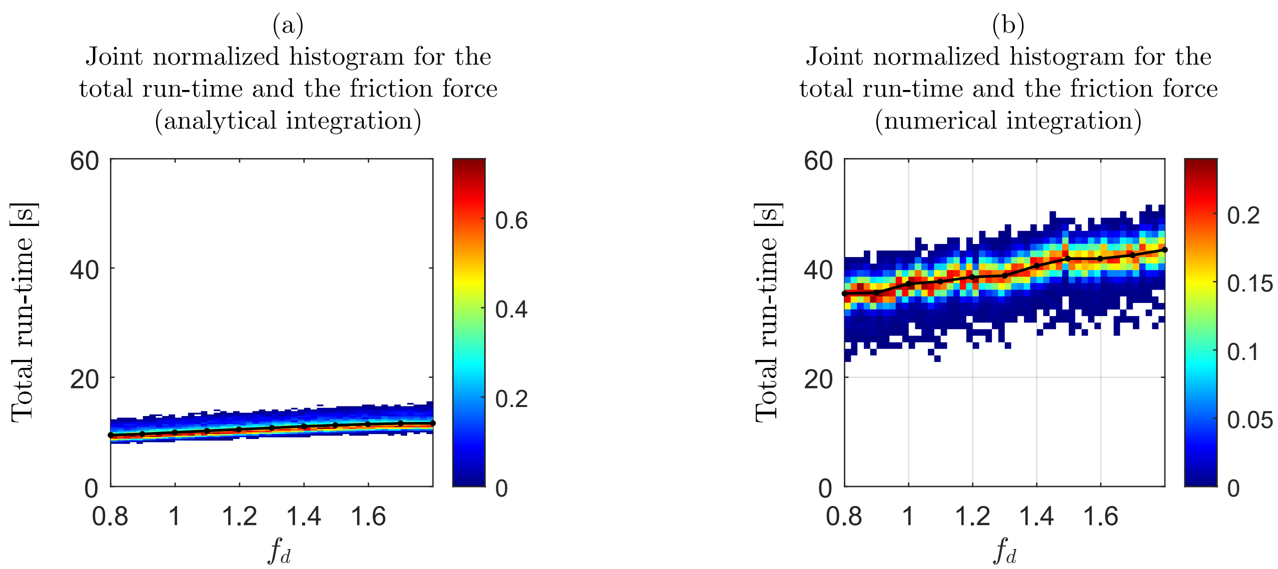

Figure 8 – Top view of the joint normalized histograms of the run-time to simulate the Monte Carlo method using the analytical approximation (a) and numerical integration (b) and random variable F_d.

Finally, Fig. 9(a) shows a top of view of joint normalized histogram of the random variables analytical run-time and percentage of the total stick duration ($d_{\%}^{sticks}$) ; and Fig. 9(b) shows a top of view of joint normalized histogram of the numerical run-time and percentage of the total stick duration ($d_{\%}^{sticks}$) . In Figs. 9(a) and (b), the black lines are the moving means. In this figure, it is observed that the longer the percentage of total stick duration, the faster the run-time is. Again, the moving mean in the analytical approximation tends to be more horizontal than the numerical one, which means that it is less sensitive to variations.

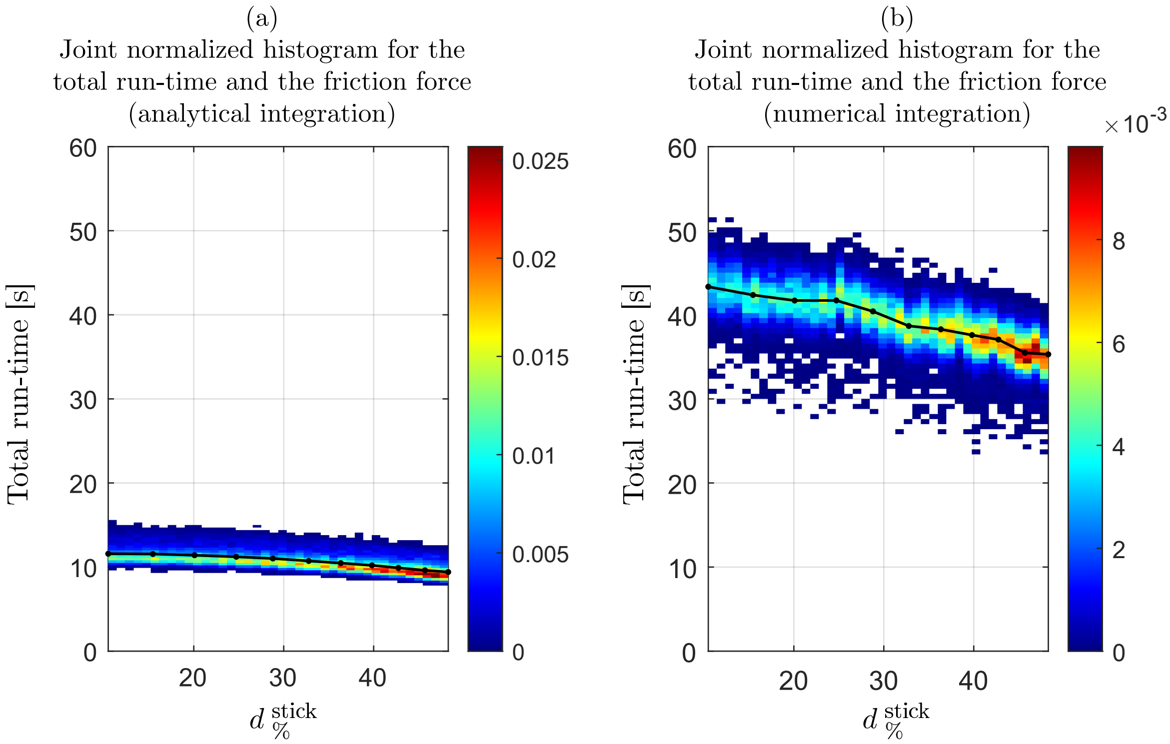

Figure 9 – Top view of the normalized histograms of the run-time to simulate the Monte Carlo method using the analytical approximation (a) and numerical integration (b) and Total stick duration.

CONCLUSIONS

In this study, the problem of an oscillator that exhibits stick-slip was addressed. A stochastic dry friction was considered, with the dynamic friction modelled as a uniform probability distribution. The focus of this study was set on the computational aspects of the stochastic simulations.

Considering the deterministic problem, which is non-linear due to the dry friction, an exact closed-form solution does not exist, and approximation techniques must be used. As a consequence, the stochastic problem was solved by means of two strategies: a combination of the Monte Carlo method with a numerical integration, and the combination of the Monte Carlo method with an analytical integration, where an approximation is obtained by using the Multiple Scale method.

With the numerical integration, an approximation was obtained by discretizing the equation in the time domain following the Runge-Kutta integration scheme. The method relies in taking small time-steps to guarantee the accuracy of the solution. Whereas with the analytical integration, an approximation given by a closed-form expression can be obtained for the stick and the slip phases.

The analytical approximation is constructed as a piece-wise function, due to the sudden change in the behavior between stick and slip phases. But once the general form of the closed-form expression is known, only the information at the transitions instants and the initial conditions at the beginning of each phase are required to completely define the behavior of the system at any time. This has a direct impact in the computational costs involved with the analytical approximation, and it is advantageous if compared to the numerical one where many intermediate time steps in-between the transitions instants are required to assure accuracy in the approximation. Moreover, it can be a decisive factor to define the feasibility of a stochastic study, given the large number of realizations that need to be performed, and the run-times involved in such calculations.

All these previous aspects are reflected in the results obtained for the stochastic oscillator. The mean and standard deviation of the run-times show that the Monte Carlo simulation using the analytical approximation is 3.68 times faster than that with the numerical integration, and that the analytical case is less sensitive to variation, given that the standard deviation obtained with the analytical approximation is 2.9 times smaller than the standard deviation obtained with numerical integration. These results can also be observed in the joint histograms of the run-times and the magnitude of the dynamic forces, as well as in the joint histograms of the run-times and the stick duration, where the supports for the analytical and the numerical cases are visibly different.

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