



# A comparative Study of Dispersion Curves in Cylindrical Waveguide Using the Semi-Analytical Finite Element Method

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*Abstract: Ultrasonic inspections are a non-destructive technique widely used in the analysis of civil and mechanical structures, without affecting the operation of the equipment or harming its integrity. In the procedure, a transmitter emits acoustic pulses that propagate through the structural elements. According to the response recorded by the receivers, it is possible to detect pathologies, such as damage, because it changes the shape of the dispersion curves and wave modes, compared to the acoustic response of a structure without anomalies. Obtaining the dispersion curves means finding the roots of a system of equations, for which iterative computational algorithms are used. One way to solve the system is to use numerical methods such as finite elements. However, in more complex or very long waveguides, the computational costs are high, or unfeasible. This difficulty can be overcome with a combination of analytical solutions and the finite element method (SAFE or semi-analytical finite element). In this work, a comparison is made considering the responses obtained in the propagation of waves in cylindrical guides using the FEM of the COMSOL software and the SAFE implemented in Matlab.*

**Keywords:** Ultrasonic Guided Waves, Cylindrical Waveguide, Semi-Analytical Finite Element, Structural Integrity

## INTRODUCTION

Non-destructive testing (NDT) allows structural evaluation for maintenance or repair purposes, without the need to affect the performance of systems in operation. The application of NDT occurs in the most varied types of structures and one of the existing techniques is the acoustic inspection. In the oil industry, when a well reaches its limit and the production rate is no longer profitable, the activities of that well must be terminated (Mian (1992)). In plugging and abandonment operations (P&A), it is necessary to verify the structural integrity of the components of the well. In the acoustic inspection method, pulses that propagate through the layers of the well are used. With the information of the traveling mechanical waves and the characteristics of the medium, from the signal registered in the receivers it is possible to identify damages.

Guided wave modeling does not always have an analytical solution (Wilcox et al. (2002)). There are a variety of waveguides such as T-beams, rails, drill pipes in which waves cannot be modeled analytically (Elishakoff (2007); Hayashi et al. (2003); Rose (2000)). To find the properties of the system that are part of the construction of the dispersion curves, numerical algorithms are used that locate the roots of the equation. Normally these algorithms are iterative and use a step, which must have an adjusted size in order to find all the waves. If the distance between the steps is large, some roots may be lost. However, if the roots are widely spaced, or repeated, when determining a step too small, there is an unnecessary computational cost.

Some numerical methods are used to solve the problem, such as the finite element method (FEM), the finite difference method (FDM), or the boundary element method (BEM). These conventional techniques generally require four to six nodes per wavelength to accurately express their waveform (Elishakoff (2007)). Consequently, to represent the propagation of waves in guides that are large structures, a high number of nodes is required in the direction in which the wave propagates, which leads to long processing times and the need for a lot of computational memory.

For more complex geometries, as in the case studied in this work, a technique is used that can reduce computational costs. It consists of a combination of an analytical solution with the finite element method, or SAFE method (semi-analytical finite element). The discretization of the structure in finite elements is done in the cross section, allowing to find an approximate displacement field. Along the structure, in the direction of wave propagation, the analytical solution with the harmonic representation of the wave in the time domain is used.

In this work, the SAFE method was implemented in matlab to solve the problem of wave propagation in a hollow cylinder. To validate the implementation, the contours traced were compared with the response of the analytical solution and with the numerical results of a computational simulation in COMSOL. The results are presented in the next sections.

## 1 PROBLEM FORMULATION

The structure analyzed is a stress-free hollow cylinder (Fig. 1) with linear elastic material behavior modeled in cylindrical coordinates. Considering the already mentioned semi-analytical method, the exact analytical harmonic solutions

are used in both the  $\theta$  and  $z$  directions and the finite element approximation will be applied at the radial dimension.

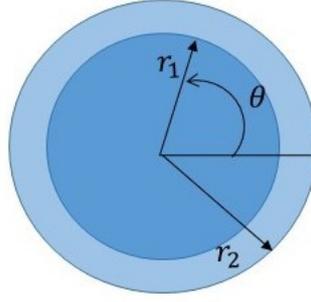


Figure 1 – Hollow cylinder with an inner radius of  $r_1 = 0,30\text{m}$  and an outer radius of  $r_2 = 0,33\text{m}$ .

First of all, to solve problems involving elastic mechanical waves that propagate in solid materials The Navier-Cauchy equation (1) is used. For its resolution, the Helmholtz theorem (2) is applied

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2 \mathbf{u} = \rho \ddot{\mathbf{u}} \quad (1)$$

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi}. \quad (2)$$

For cylindrical coordinates, the potential vector of equation 2 can be decomposed into two scalar potentials Miklowitz (1978); Morse and Feshbach (1954), which results is equation (3) below:

$$\boldsymbol{\psi} = \eta \mathbf{e}_3 + \nabla \times (\boldsymbol{\chi} \mathbf{e}_3), \quad (3)$$

thus:

$$\mathbf{u} = \nabla\phi + \nabla \times (\eta \mathbf{e}_3) + \nabla \times \nabla \times (\boldsymbol{\chi} \mathbf{e}_3) \quad (4)$$

where

$$\nabla^2 \phi - \frac{1}{c_L^2} \ddot{\phi} = 0, \quad \nabla^2 \eta - \frac{1}{c_T^2} \ddot{\eta} = 0 \quad \text{e} \quad \nabla^2 \boldsymbol{\chi} - \frac{1}{c_T^2} \ddot{\boldsymbol{\chi}} = 0, \quad (5)$$

In polar coordinates, the displacement field is given by:

$$\begin{aligned} u_r &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \eta}{\partial \theta} + \frac{\partial^2 \boldsymbol{\chi}}{\partial z \partial \theta} \\ u_\theta &= \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} - \frac{\partial \eta}{\partial r} + \frac{1}{r} \frac{\partial^2 \boldsymbol{\chi}}{\partial z \partial \theta} \\ u_z &= \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \boldsymbol{\chi}}{\partial z^2} - \frac{1}{c_T^2} \frac{\partial^2 \boldsymbol{\chi}}{\partial t^2} \end{aligned} \quad (6)$$

It is necessary to solve the system of differential equations and find the field of displacements over time. According to the nature of the problem in question and in view of recognizing the constant physical properties of the waveguide along the propagation direction, using the semi-analytical approach, the displacement field that represents the wave motion is considered as follows:

$$\mathbf{u}^{(e)}(r, \theta, z, t) = \mathbf{U}(x, y) e^{i(kz + n\theta - \omega t)}, \quad (7)$$

For this paper purpose, numerical methods can be used, such as the finite element method. The finite element discretization leads to a weak formulation of the energy equation. The problem of wave propagation is reduced to a system

of algebraic equations, from which the dispersion can be obtained. In this work, the SAFE method is used. In another words, this is the analytic solution together with the finite element method. The solution, which depends on the time  $t$  and the propagation coordinate  $z$ , and the problem takes the following form (8):

$$\mathbf{u}^{(e)}(r, \theta, z, t) = \sum_{j=1}^n \mathbf{N}(r) \mathbf{q}^{(e)} e^{i(kz+n\theta-\omega t)}, \quad (8)$$

where  $\mathbf{u}^{(e)}$  is the displacement field,  $x, y$  e  $z$  are the space coordinates,  $\mathbf{q}^{(e)}$  is the nodal displacement vector,  $k$  is the wave number in axial direction,  $n$  an interger is the circumferential harmonic,  $\omega$  is the frequency, and  $t$  corresponds to time. The shape functions  $\mathbf{N}$  that corresponds to finite element in Fig, is given by:

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix}, \quad (9)$$

with linear shape function

$$N_1 = \frac{1}{2}(1 - \xi) \quad \text{and} \quad N_2 = \frac{1}{2}(1 + \xi). \quad (10)$$

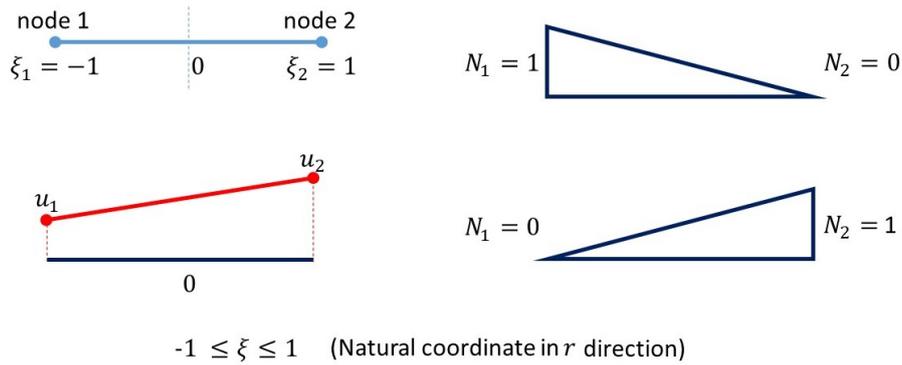


Figure 2 – Shape functions adopted.

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} \\ \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \\ \frac{\partial u_z}{\partial z} \\ \frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \end{bmatrix} \Rightarrow \boldsymbol{\varepsilon} = (\mathbf{L}_1 + \mathbf{L}_2) \mathbf{u}, \quad (11)$$

with

$$\mathbf{L}_1 = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ \frac{1}{r} & \frac{1}{r} \frac{\partial}{\partial \theta} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} \frac{\partial}{\partial \theta} \\ 0 & 0 & \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial r} - \frac{1}{r} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{L}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & 0 \\ \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (12)$$

Substituting the linear form functions in the strain-displacement relationships, in another words, replacing (8) into (11), the strain on the element can be expressed by:

$$\boldsymbol{\varepsilon} = (\mathbf{L}_1 + \mathbf{L}_2) \mathbf{N}(r) \mathbf{q}^{(e)} e^{i(kz+n\theta-\omega t)} = (\mathbf{B}_1 + ik\mathbf{B}_2) \mathbf{q}^{(e)} e^{i(kz+n\theta-\omega t)}, \quad (13)$$

where matrix  $\mathbf{B}$  represents the geometric compatibility matrix, composed of the shape functions and theirs derivatives:

$$\mathbf{B}_1 = \begin{bmatrix} \frac{dN_1}{dr} & 0 & 0 & \frac{dN_2}{dr} & 0 & 0 \\ \frac{N_1}{r} & i\frac{n}{r}N_1 & 0 & \frac{N_2}{r} & i\frac{n}{r}N_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\frac{n}{r}N_1 & 0 & 0 & i\frac{n}{r}N_2 \\ 0 & 0 & \frac{dN_1}{dr} & 0 & 0 & \frac{dN_2}{dr} \\ i\frac{n}{r}N_1 & \frac{dN_1}{dr} - \frac{N_1}{r} & 0 & i\frac{n}{r}N_2 & \frac{dN_2}{dr} - \frac{N_2}{r} & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

The equation 13 can be substituted into the constitutive relation to obtain the stress tensor components as follows:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} = \mathbf{C}(\mathbf{B}_1 + ik\mathbf{B}_2) \mathbf{q}^{(e)} e^{i(kz+n\theta-\omega t)}, \quad (15)$$

where  $\boldsymbol{\sigma}$  is the stress matrix,  $\mathbf{C}$  is the constitutive tensor for elastic materials.

Substituting the displacements, strains, and stresses values into the governing equation, we have the elementary equation:

$$\left( \mathbf{K}_1^{(e)} + ik\mathbf{K}_2^{(e)} + k^2\mathbf{K}_3^{(e)} \right) \mathbf{U}^{(e)} - \omega^2 \mathbf{M}^{(e)} \mathbf{U}^{(e)} = 0, \quad (16)$$

with

$$\begin{aligned} \mathbf{K}_1^{(e)} &= \int_r \int_\theta \mathbf{B}_1^T \mathbf{C} \mathbf{B}_1 r dr d\theta \\ \mathbf{K}_2^{(e)} &= \int_r \int_\theta (\mathbf{B}_1^T \mathbf{C} \mathbf{B}_2 - \mathbf{B}_2^T \mathbf{C} \mathbf{B}_1) r dr d\theta \\ \mathbf{K}_3^{(e)} &= \int_r \int_\theta \mathbf{B}_2^T \mathbf{C} \mathbf{B}_2 r dr d\theta \\ \mathbf{M}^{(e)} &= \rho \int_r \int_\theta \mathbf{N}^T \mathbf{N} r dr d\theta \end{aligned} \quad (17)$$

Finally, the eigenvalue problem in the global coordinate system can be obtained by adopting to a conventional finite element assembly methodology:

$$(\mathbf{K}_1 + ik\mathbf{K}_2 + k^2\mathbf{K}_3 - \omega^2\mathbf{M}) \mathbf{Q} = 0, \quad (18)$$

## METHODOLOGY AND RESULTS

The SAFE method was implemented in Matlab, using one-dimensional elements in the radial direction of the cylinder's cross section and the analytical solution in the longitudinal and circumferential directions. To validate the implementation, a comparison was made with the dispersion curves obtained analytically by de Magalhães Correia et al. (2020), as shown in Fig 3.

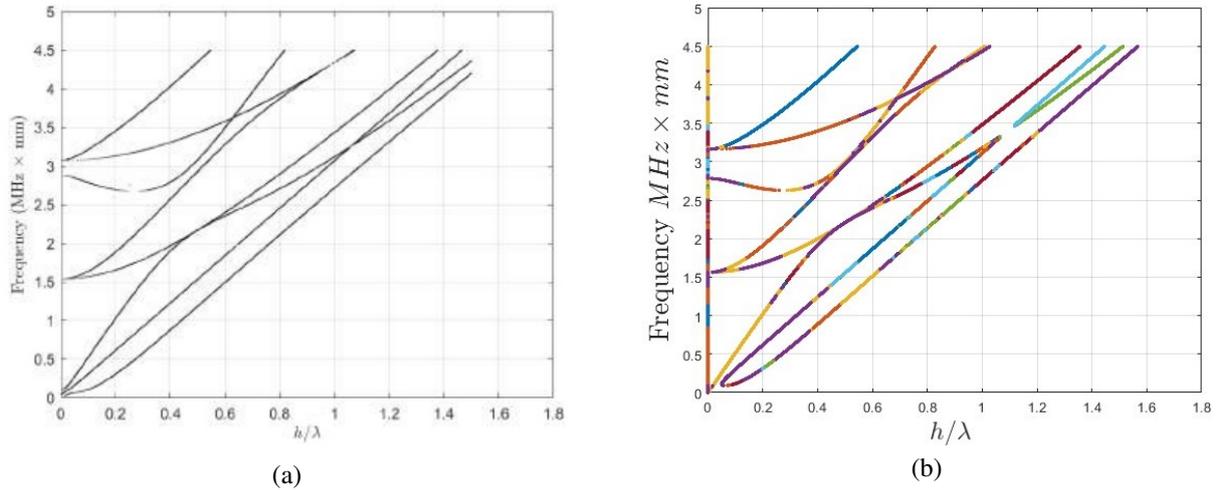


Figure 3 – The image (a) shows the analytical solution and (b) shows the dispersion curves calculated using the SAFE method.

The dispersion curves calculated with the SAFE method have a good agreement with the curves obtained analytically. Once the implementation was validated, a different model was built in COMSOL so that the results could be evaluated using finite element method. However, due to computational memory limitations and the need for fine discretization, the numerical model needs to have different characteristics compared to the initial model. The first modification was carried out in the frequency range used. It was reduced from 5 MHz to 50 kHz. The geometry has also been modified. According to the Braga et al. (1990), the relationship between radius and thickness affects the dispersion curve, making it possible to obtain more points in a lower frequency range with a higher  $R/d$  ratio. In this way, the dimensions of the model were changed.

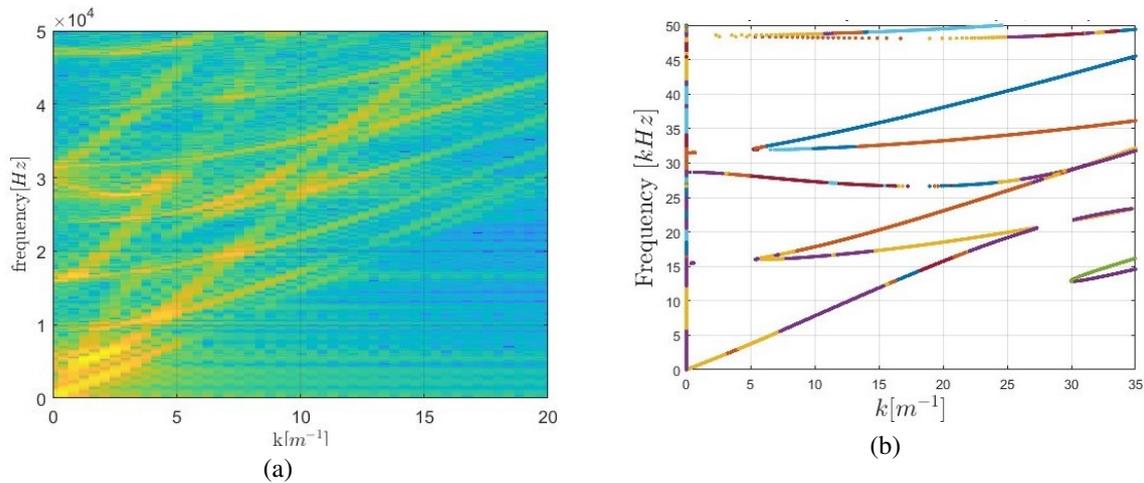
The cylindrical structure similar as showed in the Fig. 1 with the characteristics presented in Tab. 1 was modeled in COMSOL Multiphysics, which is a platform for solving differential equations using numerical methods. The responses obtained were used in a matlab algorithm to obtain the dispersion curves. These dispersion curves were compared with the result obtained by the SAFE method, also implemented in matlab.

Table 1 – COMSOL model characteristics

	parameter	value	units
<b>Young's modulus</b>	$E_0$	$193 \times 10^9$	Pa
<b>Poisson's ratio</b>	$\nu$	0.27	-
<b>Density</b>	$\rho$	7860	kg/m <sup>3</sup>
<b>Frequency</b>	$f_0$	100,000	Hz
<b>Cylinder length</b>	$L$	1.0	m
<b>Inner radius</b>	$r_1$	0.10	m
<b>Outer radius</b>	$r_2$	0.20	m
<b>Time</b>	$t_f$	$9.23 \times 10^{-3}$	s
<b>Time step size</b>	$\Delta t$	$7.69 \times 10^{-7}$	s
<b>Shear wave velocities</b>	$c_s$	4,955	m/s

From the two images shown in Fig. 4 it is possible to see an acceptable correlation. To obtain the curves shown in

(a), a processing time of 104575 s was necessary, that is, 1 day, 5 hours, 2 minutes and 55 seconds. The curves shown in (b) were obtained after a processing of 474 seconds. Both models were calculated on a computer with 3.60 GHz (4 processors) and 2.0 TB of RAM. To improve the comparison, it would be necessary to increase the time intervals and further reduce the discretization of the structure modeled in COMSOL, which would incur even higher computational costs.



**Figure 4 – In the graph of (a) the dispersion curves obtained with COMSOL are shown and in (b) the dispersion curves obtained with the SAFE method in matlab.**

## FINAL REMARKS

The study of elastic wave propagation in very long waveguides can result in high computational costs due to the discretization necessary to represent the wavelengths in the longitudinal direction. Therefore, the implementation of the SAFE method has a good acceptance for the analysis of this type of structure. The semi-analytical method was implemented in Matlab and validated through the conference with the analytical solution. Its efficiency was proven through comparison with the approximation of the numerical solution processed in COMSOL Multiphysics resulting in a saving of processing time. As future works we propose to use the results generated by numerical simulation using finite element method to perform inspection analysis, applying machine learning techniques (Abyani et al. (2022); Saikia et al. (2022)).

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