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Complex Band Structure of 1-D Solid Phononic Crystals with Viscoelasticity

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Abstract: The wave propagation in a 1-D solid phononic structure (PnS) with viscoelasticity is investigated. This solid PnS, composed by an epoxy matrix and steel inclusions, is capable of filtering the propagation of elastic bulk waves over a specified range of frequency, called band gaps. The real and complex band structures are obtained by the improved plane wave expansion (IPWE) and extended plane wave expansion (EPWE) methods, respectively. The viscoelasticity is considered by using the standard linear solid model. The viscoelasticity influences significantly the band gap formation, and the propagating and evanescent modes. The results can be useful for the understanding of the elastic bulk wave attenuation in 1-D solid periodic structures with viscoelasticity.

Keywords: wave attenuation, bulk elastic waves, band gaps, standard linear solid model.

INTRODUCTION

The study of viscoelasticity is old (Holland, 1967) and well known in the context of wave propagation (Charlier and Crowet, 1986; Carcione, Kosloff and Kosloff, 1988). However, only after the emergence of the phononic structures (PnSs) (Sigalas and Economou, 1992) the researchers have started to investigate the influence of the viscoelastic effect on the band structure (Zhao and Wei, 2009; Wei and Zhao, 2010), even though some authors had already reported interesting aspects of the viscoelasticity in periodic structures (Wilm *et al.*, 2002, Orr *et al.*, 2008). Moreover, the effect of the viscoelasticity on the evanescent Bloch waves was firstly reported by Hussein (2009), and Moiseyenko and Laude (2011).

The study of Chen *et al.* (2019) presents a brief review of the wave propagation in viscoelastic PnSs (VPnSs). They obtained novel topological patterns of the optimized VPnSs by using the bi-directional evolutionary structure optimization (BESO) method considering the maximum attenuation and stiffness.

Li, Zhang and Wang (2021) showed that the viscosity of the matrix material affects the position and the width of the band gaps, which provides an alternative way to tune the band gaps of PnSs by adjusting the viscoelastic parameters combined with other influencing parameters.

Zhu, Zhong and Zhao (2016) found that a wider and lower initial forbidden frequency or lower and higher quality factor resonant frequency can be obtained by adjusting the two viscous parameters combined considering the effect of volume fraction and shapes of scatters. They also investigated the presence of a defect in VPnSs.

In this investigation, the real and complex band structures of the VPnSs are numerically obtained by the improved plane wave expansion (IPWE) and extended plane wave expansion (EPWE) methods, respectively. First, the EPWE is validated by comparing its real part with the IPWE. Next, the influence of the relaxation time on the evanescent Bloch waves is investigated.

1-D VISCOELASTIC PHONONIC CRYSTAL MODELLING

Figure 1 (a) sketches the front view of the 1-D VPnS unit cell composed by steel inclusions (blue shaded area) in an epoxy matrix (white). The standard linear solid (SLS) model (Fig. 1 (b)) is used to consider the viscoelasticity of the epoxy matrix, since it represents a more realistic behaviour (Lakes, 2009). It should be highlighted that the SLS model has two forms, *i.e.*, the Maxwell and Kelvin forms. In this study, the Maxwell form is used, and the term "SLS model" means the Maxwell form of the SLS model (Lin, 2020).

For the modelling, the IPWE (Zhao and Wei, 2009) is used to compute the real part of the band structure, *i.e.*, without considering the viscoelasticity. The proposed EPWE will be formulated in a future publication and it is not derived for brevity. The general idea of the EPWE can be found, for instance, in Laude *et al.* (2009) and Miranda Jr., Rodrigues and Dos Santos (2022) for 2-D PnSs. In addition, it should be underlined that the evanescent modes obtained by the EPWE are associated with the wave attenuation in the unit cell, since it is defined as $\Im\{\mathbf{k}\}a$ (Miranda Jr. *et al.*, 2020), where $\mathbf{k}$ is the Bloch wave vector, also known as wave number, and a is the unit cell length. Next, it is presented only some concepts associated with the viscoelastic materials.

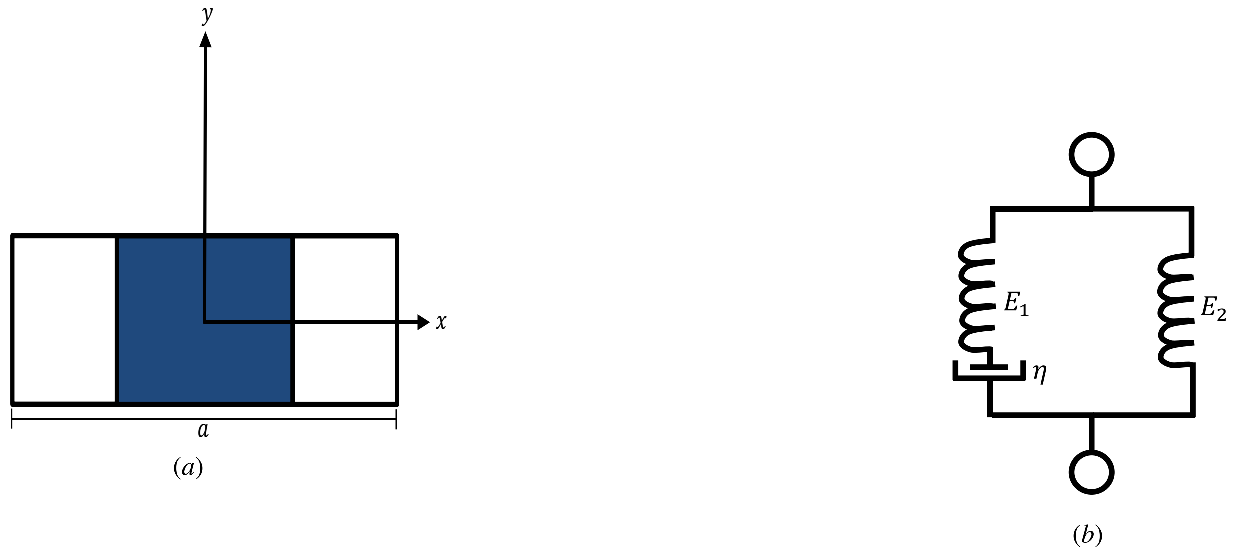


Figure 1 – Front view of the 1-D VPnS unit cell (a) **and the standard linear solid model** (b)**.**

The constitutive equations, in the time domain, for a linearly viscoelastic material are (Gurtin and Sternberg, 1962, Li, Zhang and Wang, 2021):

$$\hat{\sigma}_{ij} = \int_{-\infty}^{t} \hat{c}_{ijkl}(t-\tau)\frac{d\hat{\varepsilon}_{kl}}{d\tau}d\tau, \tag{1}$$

where $i, j, k, l = 1, 2, 3$, $\hat{\sigma}_{ij}$ is the elastic stress tensor, $\hat{c}_{ijkl}$ is the elastic stiffness tensor, $\hat{\varepsilon}_{ij}$ is the elastic strain tensor, t is the time and τ is a constant. The time domain dependence of $\hat{\sigma}_{ij}$, $\hat{c}_{ijkl}$ and $\hat{\varepsilon}_{ij}$ is omitted for brevity. The standard tensor notation is used with Latin indices running from 1 to 3. They obey Einstein's summation convention when repeated. Moreover, the integration in Eq.1 is known as Boltzmann (Lakes, 2009) or hereditary integral (Kelly, 2022), which expresses a convolution.

After a mathematical manipulation, the Eq. 1 can be rewritten as (Hwu and Cheng, 2011, Kelly, 2022):

$$\hat{\sigma}_{ij} = \hat{c}_{ijkl}\hat{\varepsilon}_{kl}(0) + \int_{0}^{t} \hat{c}_{ijkl}(t-\tau)\frac{d\hat{\varepsilon}_{kl}}{d\tau}d\tau, \tag{2}$$

where the "0" in the lower limit of the integral means 0^+ just after any possible non-zero initial strain. In that sense, the components of the elastic strain tensor $\hat{\varepsilon}_{kl}(t)$ in the Eq. 2 is to be regarded as a continuous function, *i.e.*, with no jumps over $[0,t]$. However, Lakes (2009) does not consider the term $\hat{c}_{ijkl}\hat{\varepsilon}_{kl}(0)$ in Eq. 2.

Applying the Laplace transform to the Eq. 2, one can obtain (Hwu and Chen, 2011):

$$\tilde{\sigma}_{ij}(s) = s\tilde{c}_{ijkl}(s)\tilde{\varepsilon}_{kl}(s), \tag{3}$$

where $s = \sigma + i\omega$ is a complex number frequency, σ is a real number, ω is the angular frequency and $i = \sqrt{-1}$. It should be highlighted that the temporal Fourier transform applied to the Eq. 2 does not "eliminate" the term $\hat{c}_{ijkl}\hat{\varepsilon}_{kl}(0)$. Thus, this is the reason of using Laplace transform instead of Fourier transform to the Eq. 2, *e.g.*, $\mathscr{L}\{\dot{f}(t)\} = s\mathscr{L}\{f(t)\} - f(0^+)$ and $\mathscr{F}\{\dot{f}(t)\} = i\omega\mathscr{F}\{f(t)\}$.

The elastic strain tensor, $\hat{\varepsilon}_{kl}$, for linear media is calculated by:

$$\hat{\varepsilon}_{kl} = \frac{1}{2}(\hat{u}_{k,l} + \hat{u}_{l,k}), \tag{4}$$

where $\hat{u}_i$ is the elastic displacement vector.

Applying the simplification of $\sigma = 0$ (*i.e.*, $s = i\omega$) in Eq. 3, applying the temporal Fourier transform to the Eq. 4, combining both equations and omitting the frequency dependence for brevity, yields:

$$\sigma_{ij} = i\omega c_{ijkl}u_{kl}. \tag{5}$$

The differential equations of motion (in the frequency domain) in the absence of body forces are expressed by:

$$\sigma_{ij,i} = -\omega^2\rho u_j, \tag{6}$$

where ρ is the mass density.

Substituting the Eq. 5 in the Eq. 6, considering an isotropic elastic solid (bulk waves, *i.e.*, a 3-D model) and for a 1-D periodicity (Fig. 1 (a)), $\partial/\partial x_i = 0$, $i = 2,3$, results in:

$$-\omega^2 \rho u_1 = (i\omega c_{11} u_{1,1})_{,1}, \tag{7}$$

$$-\omega^2 \rho u_2 = (i\omega c_{66} u_{2,1})_{,1}, \tag{8}$$

$$-\omega^2 \rho u_3 = (i\omega c_{44} u_{3,1})_{,1}, \tag{9}$$

where $c_{66} = \frac{1}{2}(c_{11} - c_{12}) = c_{44} = \mu = G$, $G = \frac{E}{2(1+\nu)}$, $c_{11} = \lambda + 2\mu$, $c_{12} = \lambda$ and $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ (Miranda Jr. and Dos Santos, 2017). The λ and μ are the Lamé constants, E is the Young's modulus, G is the shear modulus and ν is the Poisson's ratio. The Eqs. 7-9 are written using Voigt notation. One can observe that the Eqs. 7-9 are the same of those obtained by Zhao and Wei (2009). It should be pointed out that the other components of the elastic tensor in Eqs. 7-9 do not need to be included because of the 1-D periodicity hypothesis (see Zhao and Wei, 2009, for instance).

For the SLS model, the elements ($\hat{c}_{ij}$) of the elastic stiffness tensor can be written as (Zhao and Wei, 2009, Hwu and Chen, 2011, Li, Zhang and Wang, 2021):

$$\hat{c}_{ij} = c_{ij_\infty} + (c_{ij_0} + c_{ij_\infty})^{-\frac{t}{\tau_{\hat{c}_{ij}}}}, \tag{10}$$

where c_{ij_0} and c_{ij_∞} are the initial and final states of the elastic constants and $\tau_{\hat{c}_{ij}}$ is the relaxation time.

As example, and considering the SLS model of the Fig. 1 (b), one can rewrite Eq. 10 in terms of the Young's modulus (Lakes, 2009):

$$\hat{E} = E_2 + E_1^{-\frac{t}{\tau_{\hat{E}}}}, \tag{11}$$

where the relation between the Eqs. 10 and 11, in terms of $\hat{E}$, is $\hat{E}_2 = \hat{E}_\infty$ and $\hat{E}_1 = \hat{E}_0 - \hat{E}_\infty$.

Applying the temporal Fourier transform to the Eq. 10, results in:

$$c_{ij} = \frac{(c_{ij_0} - c_{ij_\infty})\tau_{\hat{c}_{ij}}}{1 + \omega^2 \tau^2_{\hat{c}_{ij}}} + i\frac{(c_{ij_\infty} - c_{ij_0})\omega\tau^2_{\hat{c}_{ij}}}{1 + \omega^2 \tau^2_{\hat{c}_{ij}}}. \tag{12}$$

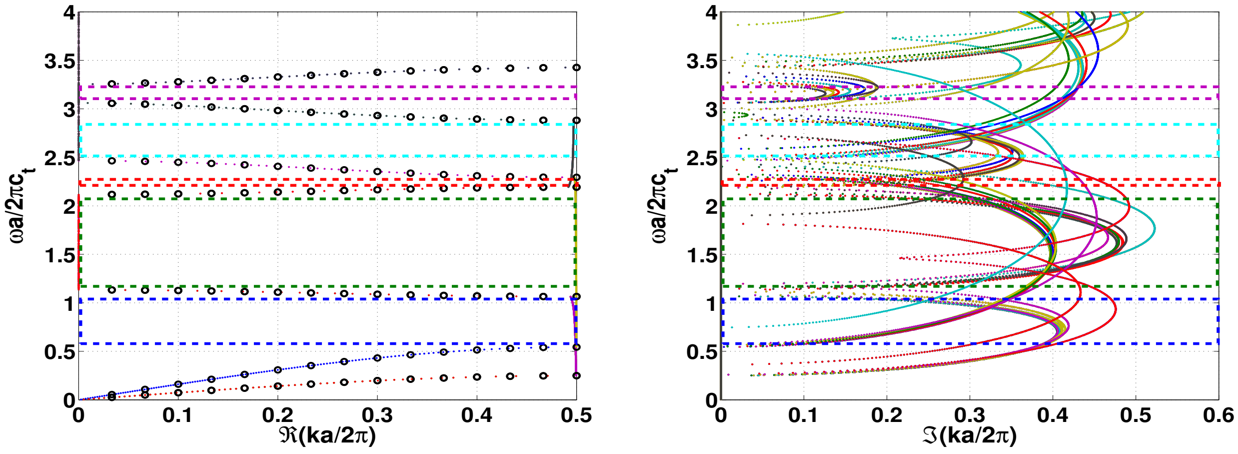
SIMULATED EXAMPLES

The properties of the steel inclusions (A) and epoxy matrix (B) chosen for the simulation are $\rho_A = 7890$ kg/m^3, $\rho_B = 1180$ kg/m^3, $c_{44_A} = 81.8 \times 10^9$ N/m^2, $c_{44_B} = 1.3 \times 10^9$ N/m^2, $c_{11_A} = 263.6 \times 10^9$ N/m^2, $E_B = 3.56 \times 10^9$ N/m^2, $f_f = 0.5$, where f_f is the filling fraction (*i.e.*, the inclusion length divided by the unit cell length).

In Fig. 2, it is shown the complex band structure of the VPnS without viscoelasticity computed by (a) IPWE (black circles) and $(a-b)$ EPWE (coloured points) approaches, considering 21 plane waves for the Fourier series expansion. The Bragg scattering band gaps are identified by the coloured dashed lines.

The normalized frequency $\omega a/2\pi c_t$ (where $c_t = c_{44_B}/\rho_B$ is the transversal wave velocity in the epoxy) is used in order to obtain a general result independent of a, since there are only Bragg-type band gaps. The first five band gaps are opened up between the following frequencies – 0.5435-1.065 Hz, 1.133-2.117 Hz, 2.193-2.294 Hz, 2.464-2.881 Hz and 3.057-3.257 Hz, respectively. The real part of the modes obtained by the proposed EPWE presents a good matching with those computed by the IPWE in Fig. 2 (a). The first five full band gaps (*i.e.*, between longitudinal, u_1, and transversal modes, u_2 and/or u_3) can be observed in Fig. 2 (a). In Fig. 2 (b), it is illustrated the imaginary part of all modes, but it is not straightforward to identify the attenuation associated with the band gaps.

It should be emphasized that more modes are observed in Fig. 2 (b) than in Fig. 2 (a), since there are some evanescent waves which present $\Re\{k\} = 0$. Moreover, in Fig. 2, it is shown only the positive values of k, thus some evanescent waves that have $\Im\{k\} > 0$ and $\Re\{k\} < 0$ cannot be directly observed (their real parts) in Fig. 2 (a). Furthermore, for a deeper analysis of complex band structures of phononic structures, see Laude *et al.* (2009) and Miranda Jr. *et al.* (2022), for instance. The other results with viscoelasticity will be presented in a journal article (submitted).



(*a*) (*b*)

Figure 2 – Complex band structure of the VPnSs with steel inclusions in a an epoxy matrix, computed by (*a*) **IPWE (black circles) and** ($a - b$) **EPWE (coloured points) approaches. The band gaps are identified by the coloured dashed lines.**

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