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# Representations of the frequency response of linear time-periodic systems: adapting to stochastic excitation

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*Abstract: We study a class of mechanical system expressed in terms of non-autonomous second-order systems of Ordinary Differential equations with periodic coefficients under stochastic excitation. From the idea underlying the Harmonic Balance Method, we explore alternative representation basis and transformations to facilitate the study of the response of this class of systems when stochastic forcing functions are considered. The strengths and limitations of different time-frequency techniques are reviewed and discussed, including the multi-scale properties revealed by the Wavelet Transform, and the Short-Time Fourier Transform.*

***Keywords: LTP systems, Stochastic Process, Uncertainty Propagation, Harmonic Balance Method, Time-Frequency analysis***

## INTRODUCTION

The motivation of this research work lies at the junction of two problems: one is the study of the response of deterministic mechanical systems with periodic parameters, the other one is the study of the response of deterministic systems subjected to stochastic excitation. Mechanical systems with periodic parameters include a wide variety of machinery consisting of a component undergoing large rotations to which elastic bodies, like blades, are radially attached: these include various types of turbines, wind and hydro-kinetic generators, helicopter propulsion systems and mechatronic devices (for some examples, see Christensen and Santos (2006); Junkins (1993)). Owning to the periodicity of the effective parameters, these systems display particular dynamical characteristics that limit the effectiveness of more traditional analysis techniques that are highly efficient on so-called natural (*i.e.* constant coefficient) systems. Modal analysis and frequency domain techniques need to be considerably extended to yield useful results. When the excitation acting on these systems are modeled as stochastic processes, additional limitations become evident since the standard techniques do not capture vital information of the response.

The systems of interest for this study can be generally modeled by systems of second-order Ordinary Differential Equations with periodic coefficients. This broader problem has been under study since at least the 1800s (see Floquet (1883) for example), where Floquet and Hill provided substantial contributions. These contributions led to the development of the Floquet-Lyapunov, Hill-Floquet, or Harmonic Balance Method approaches.

In practice, the study of linear time-invariant (LTI) systems to harmonic excitation can be greatly simplified by converting the problem to the frequency domain, where the frequency response of the system to a given excitation is reduced to an algebraic problem, as opposed to the solution of the given differential equation in the time domain. Notions like the frequency response function, or the transmissibility function, can be defined and constitute a practical tool for the engineering practice (see Preumont (1994) for a concrete reference). This approach can be expanded to treat linear time-periodic (LTP) systems in a similar manner. The general idea for the determination of the frequency response of LTP systems consists of ( Xu and Gasch (1995)):

1. Expanding the time periodic coefficients in a complex Fourier series:

2. Formulating and solving the corresponding hyper-eigenvalue problem to obtain the coefficients of dynamic or time-varying eigenvectors and the corresponding eigenvalues (or poles);

3. Expressing the impulse response, often called Duhamel's integral for the LTI case, of the LTP system with the help of the modal basis found in the previous step;

4. Transforming the impulse response function of the system into the frequency domain via Fourier transform. This is sometimes called Harmonic Transfer Function for a formal mathematical description of the generalization;

5. Transforming the excitation into the frequency domain via Fourier transform;

6. The frequency response can then be obtained, as in the LTI case, by algebraic operations: products and sums of the frequency domain excitation and the Harmonic Transfer Function, and shifts in frequency.

In principle, this approach allows for the description of the response to more general excitations, not exclusively harmonic ones. The key idea of this procedure is the transformation of the time-periodic system into the frequency domain, by projection onto a Fourier basis which in practice is truncated. The procedure previously described utilizes a Fourier basis, but a similar idea can be employed with different basis functions. As show in Wereley and Hall (1990), the steady state response of this type of system is a so-called modulated complex exponential. The frequency content of the response will include elements associated not only with the system's intrinsic frequency and excitation driving frequency, but will also display the effects of the parametric terms, these are associated with the periodic nature of the coefficients.

If a stochastic excitation is considered as the input of such systems, the most immediate result is that in general the response is non-stationary, owning to the time-varying characteristics of the system. Specialized techniques are required to effectively analyze this type of response: for instance, the time evolution of the frequency content of the response constitutes a useful description from a design and analysis perspective, but it requires the shift from a representation purely on the frequency domain to a time-frequency or mixed domain.

Inspired by the overview by Scholl (2021), the following representations of the response are considered:

- Wavelet basis with the associated Wavelet Transforms: wavelets provide time-frequency representation as well as multi-resolution characteristics;
- Short-Time Fourier Transform: it offers a localized distribution of the energy with time-frequency resolution;
- Wigner-Ville Distribution: it provides a time-frequency representation with many additional properties useful to stochastic analysis.

The motivation of this study is to determine convenient generalizations of the procedure previously outlined taking into account: 1) the stochasticity of the input and its possible spectral content; 2) the stochasticity and non-stationarity of the response. The generalizations consist of applying the same procedure while changing the basis representation/transformation selected: in other words, replacing the complex Fourier basis (changing the representation), and replacing the Fourier Transform by the appropriate corresponding transformation. Comparisons will be possible between the Wavelet scalogram, the STFT spectrogram, the Wigner-ville distribution and the frequency response resulting from the known results of the Harmonic Balance Method (HBM) with Fourier Series and Transform.

For alternative approaches to the problem, relevant results are those of Bachelet et al. (2007); Wan (1973).

## PROBLEM FORMULATION

### Dynamical system

Our focus is a class of deterministic mechanical system whose dynamics is expressed in terms of a system of non-autonomous second-order Ordinary Differential Equations (ODE) with periodic coefficients:

$$m(t)\,\ddot{u}(t) + c(t)\,\dot{u}(t) + k(t)\,u(t) = p(t), \tag{1}$$

where $m(t) = m(t+T)$ is periodic with period $T$ and similarly $c(t)$ and $k(t)$. This is a linear time-periodic (LTP) single degree of freedom system. It is convenient to express the system in the state-space form, with $\mathbf{y} = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}$ and $\dot{\mathbf{y}} = \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix}$:

$$\dot{\mathbf{y}} = \mathbf{A}(t)\,\mathbf{y} + \mathbf{B}(t)\,p, \tag{2}$$

where:

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \qquad \mathbf{B}(t) = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}. \tag{3}$$

This transformation converts a system of one second order equation into a system of 2 first order equations.

In our study, the excitation $p(t)$ in Eq. (1) is the stochastic input of the system: a stationary zero mean gaussian process defined by its auto-covariance function $\Sigma_{pp}(\tau)$, or its auto-Power Spectral Density (PSD) $S_{pp}(f)$. Hence, introducing the expectation operator $\mathrm{E}[\bigstar]$, $\mathrm{E}[p(t)] = 0$, $\Sigma_{pp}(t_1,t_2) = \mathrm{E}[(p(t_1) - \mu_p(t_1))(p(t_2) - \mu_p(t_2))] = \Sigma_{pp}(\tau)$ with $\tau = t_2 - t_1$ and $S_{pp}(f) = \mathrm{TF}[\Sigma_{pp}(\tau)]$ where $\mathrm{TF}[\bigstar]$ stands for the Fourier transform operator.

### Synthesis of analytical tools

#### *Hill-Floquet-Lyapunov theory and Harmonic Balance Method*

Let us first synthesize the classical approach (see Xu and Gasch (1995); Christensen and Santos (2006) for a detailed exposition of the method), beginning by the approximation of the periodic coefficients in Equation (2) in a truncated Fourier Series as:

$$\mathbf{A}(t) = \mathbf{A}(t+T) = \sum_{a=-N}^{+N} \mathbf{A}_a \exp\left[\mathrm{i}a(2\pi f_\Omega)t\right], \tag{4}$$

where i is the imaginary unit and $f_\Omega = \frac{1}{T}$ is called the pumping frequency of the system. The subscript $\Omega$ in the pumping frequency $f_\Omega$ denotes the association of this frequency with the circular frequency $\Omega = 2\pi f_\Omega$. Often in practice $\Omega$ stands for the rotational speed that is imposed on the system and is the source of parametric excitation. When $p(t) = 0$, assuming a solution of the form $\mathbf{y}_k(t) = \mathbf{r}_k(t)\exp[\lambda_k t]$ with $k$ contributions, where $\mathbf{r}_k$ is a dynamic (time-dependent) right mode, the following relationship can be established:

$$\lambda_k \sum_{j=-N}^{+N} \mathbf{r}_{k,j} \exp\left[\mathrm{i}j(2\pi f_\Omega)t\right] + \sum_{j=-N}^{+N} \mathrm{i}j2\pi f_\Omega \mathbf{r}_{k,j} \exp\left[\mathrm{i}j(2\pi f_\Omega)t\right] - \sum_{a=-N}^{+N} \mathbf{A}_a \exp\left[\mathrm{i}a(2\pi f_\Omega)t\right] \sum_{j=-N}^{+N} \mathbf{r}_{k,j} \exp\left[\mathrm{i}j(2\pi f_\Omega)t\right] = \mathbf{0}, \tag{5}$$

which can be casted into a truncated hyper-eigenvalue problem:

$$\left(\lambda_k \hat{\mathbf{I}} - \hat{\mathbf{A}}\right)\hat{\mathbf{r}}_k = \hat{\mathbf{0}}, \tag{6}$$

where the circumflex symbol ˆ denotes the truncated version of what is in principle an infinitely large matrix or vector, accordingly. The solution of this eigenvalue problem provides the Fourier Series coefficients of the dynamic modes, and the steady state response to a generalized excitation can be expressed as:

$$\mathbf{y}(t) = \sum_{k=1}^{2K} \mathbf{r}_k(t) \int_{-\infty}^{t} \exp\left[\lambda_k t\right] \boldsymbol{l}_k^{\mathrm{T}}(\tau)\mathbf{B}(\tau)p(\tau)\,\mathrm{d}\tau, \tag{7}$$

where $l_k$ denotes the dynamic (time-dependent) left mode, such that $\boldsymbol{l}_k^{\mathrm{T}}\mathbf{r}_k = 1$.

Using this modal basis, the frequency response to a general deterministic excitation can be expressed as follows:

$$\mathbf{Y}(f) = \sum_{j=-N}^{+N}\sum_{l=-N}^{+N}\sum_{w=-N}^{+N} \mathbf{R}_j \left[\frac{1}{\mathrm{i}(2\pi f + (l+w)2\pi f_\Omega) - \lambda_k}\right] \mathbf{L}_l \begin{bmatrix} \mathbf{0} \\ \mathbf{W}_w \end{bmatrix} P(2\pi f + (j+l+w)2\pi f_\Omega), \tag{8}$$

where $\mathbf{R}$ and $\mathbf{L}$ are matrices containing the Fourier Series coefficients of the right and left dynamic eigenvectors as columns respectively, $\mathbf{W}$ contains the Fourier Series coefficients of the inverse of $m(t)$, and $P(f) = \mathrm{FT}[p(t)] = \int_{-\infty}^{\infty} p(t)\exp[-\mathrm{i}2\pi f t]\,\mathrm{d}t$ is the Fourier Transform of the input. From this formulation, the dynamic compliance matrices $\mathbf{H}_{j,l,w}(f)$ can be defined as an extension of the analogous quantity common in linear time-invariant systems. They have the form:

$$\mathbf{H}_{j,l,w}(f) = \mathbf{R}_j \left[\frac{1}{\mathrm{i}(2\pi f + (l+w)2\pi f_\Omega) - \lambda_k}\right] \mathbf{L}_l \begin{bmatrix} \mathbf{0} \\ \mathbf{W}_w \end{bmatrix}. \tag{9}$$

For stochastic excitations, we propose to evaluate:

$$\mathbf{S}_{yy}^{HBM}(f) = \sum_{j=-N}^{+N}\sum_{l=-N}^{+N}\sum_{w=-N}^{+N} \mathbf{H}_{j,l,w}(f) S_{pp}(2\pi f + (j+l+w)2\pi f_\Omega)\mathbf{H}_{j,l,w}^{\mathrm{H}}(f) \tag{10}$$

where the superscript H denotes the conjugate transpose of the dynamic compliance submatrix $\mathbf{H}_{j,l,w}(f)$. Note that, from the matrix $\mathbf{S}_{yy}^{HBM}(f)$, one can extract the term $S_{uu}^{HBM}(f)$ in the first rank and column, related to the stochastic process $u(t)$. As we will see later with the application, this quantity gives useful informations on the frequency contents of $u(t)$, but it should be stressed that $\mathbf{S}_{yy}^{HBM}(f)$ is not a PSD of the non-stationary stochastic process $\mathbf{y}(t)$, even if the function $S_{pp}(f)$ in Eq. (10) is effectively the Power Spectral Density (PSD) of the input stationary stochastic process $p(t)$: for non-stationary stochastic processes, different generalizations of the PSD must be made. Some proposals include a PSD resulting from a two-dimensional Fourier transform, into a domain of two frequencies (see for instance the methodology in Bachelet et al. (2007)). But here, the results are expressed only on the frequency domain (or the time domain only from an inverse FT), and so no detail of the time evolution of the frequency content is provided, which is key information given that the system under study has known time-varying characteristics.

### *Periodic Generalized Harmonic Wavelet*

The Periodic Generalized Harmonic Wavelets (PGHW) constitute the basis of a time-frequency representation of a given signal. The larger class of Wavelet is described in Cattani and Kudreyko (2010), while the particular family used in this paper is detailed in Spanos et al. (2016). The time-domain representation of the PGHW is:

$$\Psi_{m,n,k}(t) = \frac{1}{n-m}\sum_{q=m}^{n} \exp\left[\mathrm{i}q\left(2\pi\Delta f_{\Omega}\right)\left(t-\frac{kT}{n-m}\right)\right], \tag{11}$$

where $n, m$ are parameters defining the bandwidth or also the wavelet scale, and the parameter $k$ is associated to the time shift of the wavelet. For a real-valued, zero-mean signal $\mathbf{y}(t)$, the following representation is possible:

$$u(t) = \mathbf{C}_{\psi}\psi = 2\mathrm{Re}\left[\sum_{i}\sum_{k}\mathbf{C}_{n_i,m_i,k}\Psi_{m,n,k}(t)\right], \tag{12}$$

and the wavelet coefficients are obtained by means of the wavelet transform:

$$\mathbf{C}_{n,m,k} = \frac{n-m}{T}\int_0^T u(t)\,\hat{\Psi}^*_{m,n,k}(t)\,\mathrm{d}t, \tag{13}$$

here $\hat{\Psi}^*_{m,n,k}(t)$ is the complex conjugate of $\Psi_{m,n,k}(t)$. The wavelet scalogram is defined by the time-frequency (or time-scale) representation of $\left|\mathbf{C}_{n,m,k}\right|^2$, that leads to $S_{uu}^{PGHW}(f_{mn}, t_k) = \mathrm{E}\left[\left|\mathbf{C}_{n,m,k}\right|^2\right]$. For a wavelet representation with the PGHW family, $S_{uu}^{PGHW}(f_{mn}, t_k)$ is associated to a corresponding frequency band $f_{mn} \in (\Delta fm, \Delta fn)$ and time shift interval $t_k \in \left(\frac{kT}{n-m}, \frac{(k+1)T}{n-m}\right)$. It should be stressed that the definition utilized here represents the Wavelet scalogram, but in practice it is closely related to some of the definitions for the estimation of evolutive PSD pioneered in the PGHW by publications like Spanos et al. (2016).

### *Short-Time Fourier Transform with Gaussian window*

The Short-Time Fourier Transform (STFT) consists of introducing a function of time as a multiplicative factor into the integrand, this function being concentrated around its temporal argument and decaying over points in time that are far from its center. This function, or window, is applied to the function under analysis and then the Fourier Transform is applied. The process is repeated with the window moving through the entire temporal interval. The result is a time-localized representation of the frequency content of the signal $\mathbf{y}_l(t)$ (see Scholl (2021) or also Sandsten (2016) and Boashash (2003)):

$$U^{STFT}(t,f) = \int_{-\infty}^{\infty} h(r-t)\,u(r)\exp\left[-\mathrm{i}2\pi fr\right]\mathrm{d}r, \tag{14}$$

where $h(t)$ is the window function. If a Gaussian window is selected, then $h(t) = \exp\left[-\alpha t^2\right]$, where the parameter $\alpha$ shortens the window length as it increases. The spectrogram is defined as $S_{uu}^{STFT}(t,f) = \left|U^{STFT}(t,f)\right|^2$, which is a time-frequency distribution of the power of the original function. For a stochastic process we introduce: $S_{uu}^{STFT}(t,f) = \mathrm{E}\left[\left|U^{STFT}(t,f)\right|^2\right]$.

### *Smoothed Wigner-Ville distribution*

The Wigner-Ville Distribution (WVD) (a detailed exposition of this representation can be found in Boashash (2003)) is based on the Instantaneous Auto-Correlation (IAC) function of the process being represented. For the deterministic case of $u(t)$, it is:

$$R_{uu}(t,\tau) = u\left(t+\frac{\tau}{2}\right)u^*\left(t-\frac{\tau}{2}\right), \tag{15}$$

which is a function of proper time $t$ and lag $\tau$. Here $u^*$ is the complex conjugate of $u$. A very natural extension to the case of random processes is:

$$R_{uu}(t,\tau) = \mathrm{E}\left[u\left(t+\frac{\tau}{2}\right)u^*\left(t-\frac{\tau}{2}\right)\right], \tag{16}$$

and then the Wigner-Ville distribution is obtained by taking the Fourier Transform in the lag dimension:

$$S_{uu}^{WVD}(t,f) = \int_{-\infty}^{\infty} R_{uu}(t,\tau)\exp\left[-\mathrm{i}2\pi f\tau\right]\mathrm{d}\tau. \tag{17}$$

In practice, it is convenient to introduce two variations of this proposal. First, the analytical associate of $u(t)$ is used, that is $z(t) = u(t) + \mathrm{iHT}(u(t))$, where $\mathrm{HT}[\star]$ is the Hilbert Transform operator. Second, a similar idea to that of the STFT can be applied to the transformation, with the objective of reducing one of the main drawbacks of this representation: the existence of artifacts, or components in the solution that convey little physical information about the system. With these two variations, the Smoothed Pseudo Wigner-Ville Distribution (SP-WVD) reads:

$$S_{uu}^{SP-WVD}(t,f) = \int_{t_i}\int_{f_i}\int_{-\infty}^{\infty} h_t(t-t_i)\, h_f(f-f_i)\, R_{uu}(t,\tau)\exp[-\mathrm{i}2\pi f\tau]\,\mathrm{d}\tau \mathrm{d}f_i \mathrm{d}t_i. \tag{18}$$

## Comments

We conclude the section with some observations regarding the methods introduced. Assuming $u(t)$ is to be represented by a suitable normalized basis:

$$u(t) = \sum_{k=1}^{\infty} \mathbf{C}_k \Phi_k(t,f), \tag{19}$$

with the coefficients of this sum given by $\mathbf{C}_k = \int_{-\infty}^{\infty} u(t)\,\Phi_k(t,f)\,\mathrm{d}t$, we can establish an analogy between the Fourier, Wavelet and STFT representations based on their basis functions, coefficients and related integral transform:

- For Fourier: the basis are of the form $\Phi_k(t,f) = \exp[-\mathrm{i}2\pi f t]$; the Fourier coefficients provide information exclusively on the frequency domain, and the integral transform associated is the Fourier transform: $\mathbf{C}_k = \int_{-\infty}^{\infty} u(t)\exp[-\mathrm{i}2\pi f t]\,\mathrm{d}t$;

- For the PGHW type of wavelet: the basis are of the form $\Phi_{m,n,k}(t) = \frac{1}{n-m}\sum_{q=m}^{n} \exp\left[\mathrm{i}q2\pi\Delta f\left(t - \frac{kT}{n-m}\right)t\right]$; the wavelet coefficients provide information related to time and frequency, and the integral transform associated is the Wavelet transform $\mathbf{C}_{n,m,k} = \frac{n-m}{T}\int_0^T u(t)\,\Psi_{m,n,k}^{*}(t)\,\mathrm{d}t$. Clearly, in this particular case the synthesis of the represented function is more conveniently expressed in terms of a double sum, and does not precisely follow the single summation formula;

- For the STFT representation: one can think of the basis functions as being the complex Fourier basis but replacing the traditional inner product by a weighted inner product with the window functions as weights; a less intuitive approach would be considering the entire kernel of the transformation as the basis, having the form $\Phi_k(t,f) = h(t_i - t)\,u(t_i)\exp[-\mathrm{i}2\pi f t_i]$. The coefficients $\mathbf{C}_{k,j}$ have localization in both time and frequency, and the integral transform associated to the Short Time Fourier Transform is $\mathbf{Y}^{STFT}(f,t) = \int_{-\infty}^{\infty} h(t_1 - t)\,u(t_i)\exp[-\mathrm{i}2\pi f t_1]\,\mathrm{d}t_1\mathrm{d}t$.

In each case, Parseval's theorem is valid and so we can establish a representation of the power of the system: $\int_{-\infty}^{\infty}|u(t)|^2\,\mathrm{d}t = \sum_{k=1}^{\infty}|\mathbf{C}_k|^2$. The Wigner-Ville Distribution, on the other hand, starts with the IAC function, which is associated to the power of the system in the sense that, traditionally, the conventional PSD is its dual as reflected in the Wiener-Khinchin theorem.

## System and input description

For the purposes of our study, we use a simplified particular case where only one of the coefficients are periodic. The values are shown in Table 1, while the excitation $p(t)$ is a Gaussian stationary stochastic process that will be described in more detail in the following subsection.

**Table 1 – Parameters of the system**

| Parameter | Value |
|---|---|
| $m$ | 1 |
| $c$ | 0.2 |
| $k$ | $(0.5\times 2\pi)^2 + \sin(0.2\times 2\pi t)$ |
| $f_s$ | 0.5 [Hz] |
| $f_\Omega$ | 0.2 [Hz] |

The excitation selected as input for the application is a strictly stationary, centered Gaussian process of unit variance $p(t) \to \mathcal{N}(\mu_p = 0, \sigma_p = 1)$. The covariance function and Power Spectral Density (PSD) of the process can be seen in Fig. 1. One realization of the input process is show in Fig. 2. It can be seen from the two-sided PSD that the process has a constant frequency content on intervals that contain the frequencies associated with the model under study.

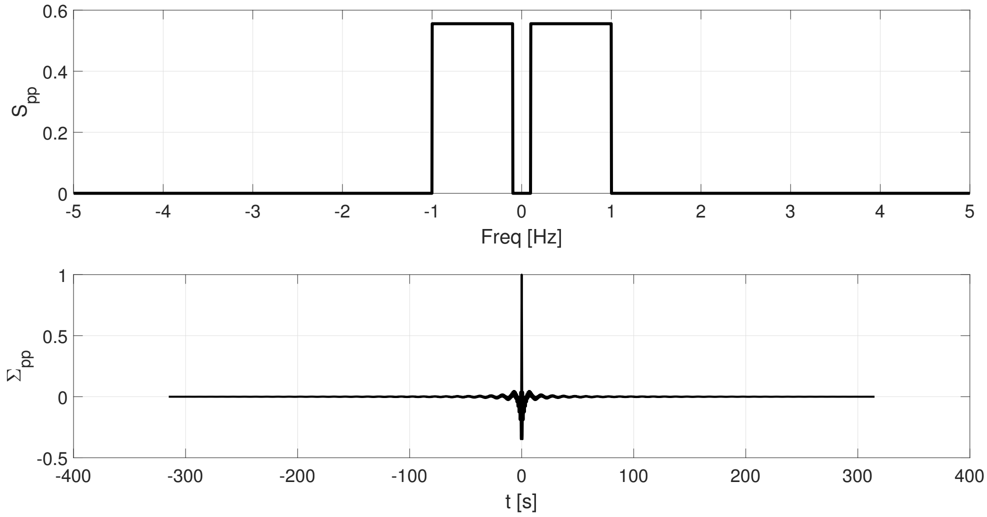


**Figure 1 – Input process descriptors: PSD (top) and Covariance function (bottom)**

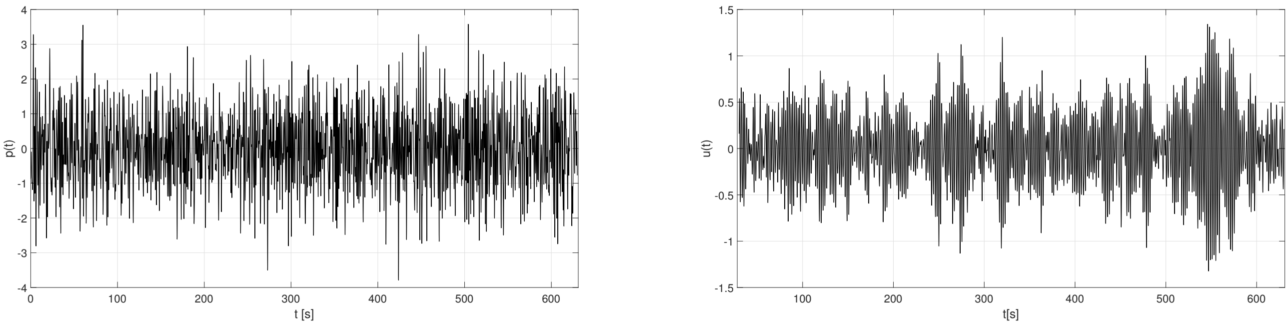


**Figure 2 – Temporal realization of the input process (left) and response (right)**

# RESULTS

## Simulation characteristics

The system described in the previous section is excited with the corresponding Gaussian process. First, the ODE system is solved numerically for each sample of the excitation, resulting in a time-ensemble array of data describing the response. The parameters of this resolution scheme are listed in Table 2. The temporal realizations of the input process have been generated utilizing the algorithm outlined in Suptille et al. (2012).

**Table 2 – Simulation parameters**

| Parameter | Value | Detail |
|---|---|---|
| Samples | 2000 | |
| Time step | $0.0600[s]$ | Interpolation step |
| Interval | $[0, 599.9400]$ | |
| Relative tolerance | $10^{-8}$ | |
| Algorithm | Adams-Bashforth-Moulton | Matlab's implementation |

We determine the mean and variance function of the response ensemble to obtain a basic statistical characterization of the response. The time-varying mean and variance function are shown in Fig. 3. We remark that the variance function has an average value of 0.1397.

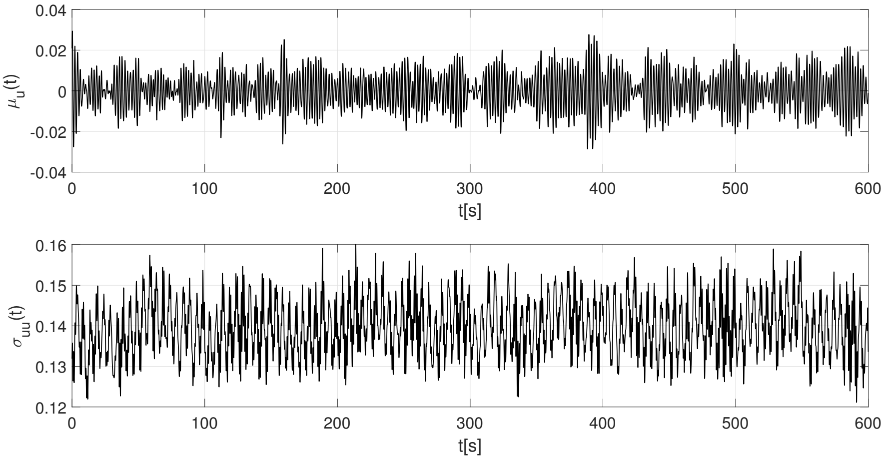


**Figure 3 – Empirical mean and variance of the sampled response**

Following the initial statistical description, we proceed with the application of the HBM method. Using a harmonic number equal to 3 : $[N_{-1}, N_0, N_1]$, we use the established relationships in terms of the dynamical compliance matrix to obtain the $\mathbf{S}_{yy}^{HBM}(f)$ to obtain the amplitude $\left|S_{uu}^{HBM}(f)\right|$ shown in Fig. 4. We again remark that this representation lacks any detail of time, as can be readily noticed.

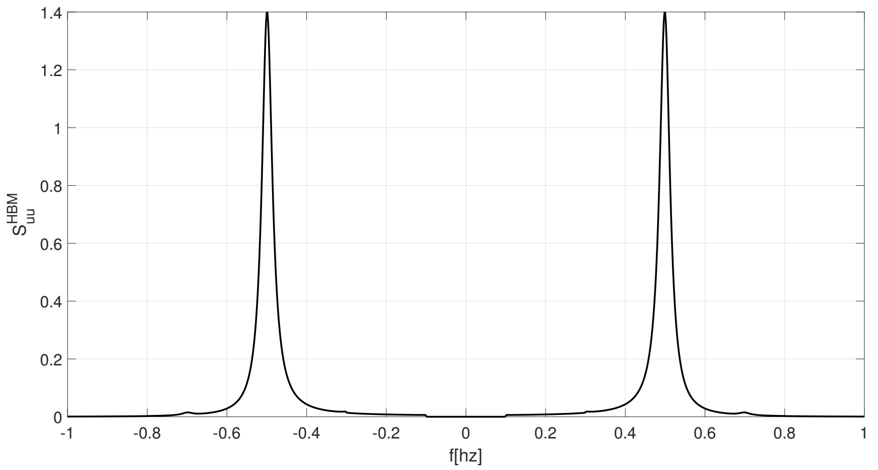


**Figure 4 – Plot of the $S_{uu}^{HBM}$ spectrum**

In terms of the frequency description of the system, as expected, the main frequency is in the vicinity of $f_s = 0.5$Hz, since this is by construction the fundamental frequency of the system, and the input process has a prescribed PSD containing this frequency. Further, the pumping frequency, or the frequency related to the periodicity of the coefficient, is $f_\Omega = 0.2$Hz, which gives origin to the harmonic terms 0.7 Hz and 0.3 Hz. The appearance of these harmonics is best conceptualized analyzing the denominator of the dynamic compliance matrix : $(2\pi f + (l+w)2\pi f_\Omega) - \lambda_k$, making $|\lambda_k| \approx 2\pi f_s$ (the equality is broken because the system is damped, so $\lambda_k$ contains a real part) and so, when $f = (f_s - (l+w) f_\Omega)$, the denominator approaches zero and the amplitude of the response at $(f_s - (l+w) f_\Omega)$ has a peak. Here $l, w$ are the harmonic numbers in the Fourier Series representation, for a 3 terms expansion being $[-1, 0, 1]$. With this initial analysis of the empirical solution, we proceed with the application of the alternative representations described in the previous sections.

## Time-frequency representations of the solution

The different time-frequency representations discussed in this section have been utilized for the analysis of the ensemble of solutions described in the previous section. Each realization is transformed with the corresponding integral transformation, and then the average of the coefficients are computed across realizations. The STFT and SP-WVD have been obtained using the corresponding Matlab analysis tools with suitable parameters, while the PGHW given its novelty, has been calculated using our own code. The selection of the parameters consists of a compromise between good resolution and acceptable computational time.

The main limitation of the procedure followed in this comparison stems from two main sources: the relatively long observation period selected to give a meaningful estimation of the usefulness of each tool, and the necessity of generating a sufficiently large ensemble as to provide meaningful probabilistic and spectral characteristics of the processes under analysis. Concerning the former point: while the simplified model and input process might not display additional complexity on this extended time window, it is the case for the more detailed models that this study will inform in the future.

Our main results are show in Fig. 5, where it should be noted that the scale of each transform is different, owning to the normalization of each basis and window employed. In both the PGHW and the STFT, it is noticeable that the frequency bands containing the expected frequencies $[0.3, 0.5, 0.7]$ vary in magnitude with time. The localization in frequency of the PGHW captures this detail more clearly, while the STFT provides a very localized description in time. The SP-WVD provides a more detailed visualization of both the activity of the central frequency band, and of the time-varying activation of the harmonics: these can be seen as the intermittent branches stemming from the central frequency line.

An important aspect concerns the computation time required for these representations. The PGHW and STFT are substantially faster to compute with respect to the SP-WVD, which suggests the possibility of adapting the algorithm to optimize for random inputs with prescribed or known correlation structures.

## CONCLUSION AND PROSPECTS

The modeling of the vibration of a broad category of mechanical systems characterized by rotodynamical behavior with elastic elements has lead us to the study of the response of systems of second order ordinary differential equations with periodic coefficients when these are excited by stochastic inputs. The nature of the system implies a response that is not stationary, requiring adapted techniques to extract useful information from the response. With a simplified typical model, we have synthesized the basic results obtained from traditional modal analysis and frequency domain representation, as well as those from the Harmonic Balance Method that can be interpreted as a generalization of the two areas. Using a stationary Gaussian stochastic process as input to the model, we have verified the non-stationarity of the response, and the limitations presented by HBM: it provides information contained exclusively in the frequency domain, when the time evolution of the frequency content is lost.

We have selected three time-frequency representations to describe the response of the system, contrasting their strengths and drawbacks with respect to three fundamental criteria: the quality of the time-frequency description provided, the practicalities of their implementation to the stochastic domain, and the computational exigences. We have also compared the characteristics of the descriptors offered by each representation with the expected results verified empirically (that is by numerical resolution of the system) and by the HBM, thus ensuring the consistency of each representation. The three representations provided a good description of the expected time-varying frequency content, in the particular frequency band that was anticipated. This suggests that the three methods provide some visualization of the harmonics that characterize the type of system under study.

The PGHW provides a good time-frequency resolution for the type of problem under study, as well as an efficient representation of the system. Its numerical cost is moderate, since it requires the numerical evaluation of an important number of integrals to approximate the inner product that provides the Wavelet coefficients. A reduction of the order of the expansion, to reduce the computational cost, would result in a poor resolution of the response representation. This limitation might be attenuated by implementing more efficient projection schemes. The adaptation of the method to stochastic quantities is not straightforward, but it is conceptually clear: when representing a stochastic process, the wavelet coefficients will be random variables. This opens the possibility of establishing analytical relationships to characterize the coefficients given the stochastic parameters of the inputs, thus greatly diminishing the computational requirements.

With respect to the STFT, the time-frequency resolution obtained was acceptable. The algorithm employed was of high computational efficiency. The main drawback is related to the interpretation of the method in the conceptual functional projection context established, in particular with respect to the interpretation of the characteristic window used in the method. Similarly, there is no straightforward extension of the method to stochastic problems. Some ideas to address these limitations would be to: explore the feasibility of interpreting the STFT as consisting of a Fourier basis with a weighted inner product for the computation of the coefficients, where the window functions play the role of the weight functions; with respect to the stochasticity, it is worth exploring the possibility of applying STFT over the traditional time-domain descriptors of non-stationary random processes like the correlation function.

Regarding the SP-WVD, a high quality time-frequency resolution was obtained. The smoothing functions effectively attenuated any possible artifact arising from the classic version of the method. A very clear advantage is that this representation utilizes as starting point a version of the correlation function, a fundamental element in the study of stochastic processes. This makes the interpretation and extension of the method to the stochastic domain much more intuitive, and suggests the possibility of establishing optimized versions of the method aimed exclusively to stochastic phenomena. The main drawback of the representation was the very high computational cost involved.

The lines of inquiry following this study are the following:

- Establish a time-frequency representation of the dynamic modes of the system under study;
- Develop analytical results of the input-output type, to characterize the parameters of each representation by propagation of the probabilistic descriptors of the inputs, thus reducing overall computational requirements;
- Implement instances of the algorithms utilized to better suit stochastic quantities of analysis, in both the time/realization sense, and the spectral sense;
- Study the scaling of each method with increasing number of degrees of freedom of the system studied;
- Extend these developments to non-stationary inputs.

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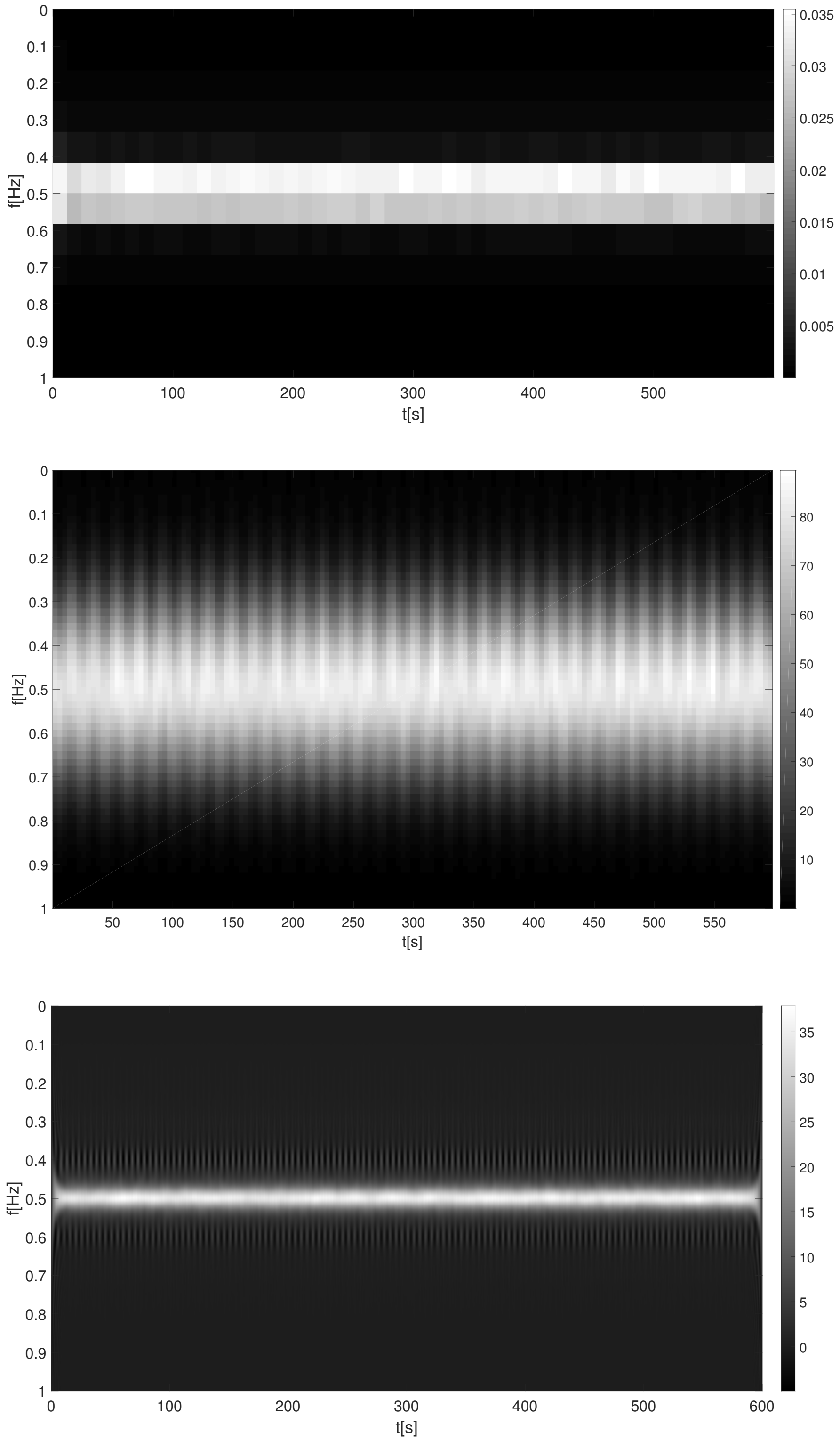


**Figure 5 – Spectral frequency representation of the stochastic response: a) PGHW representation ( top); b) STFT with Gaussian window (middle); c) Smoothed Pseudo Wigner-Ville Distribution (bottom)**