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Invariant Solutions to the Mode Shape Equation of Uniform Rectangular Kirchhoff Plates

Afonso W. Nunes[1], **Samuel da Silva**[1], **and Stylianos Dimas**[2]

[1] Departamento de Engenharia Mecânica, Faculdade de Engenharia de Ilha Solteira, UNESP – Universidade Estadual Paulista, 15385-000 Ilha Solteira, SP, Brazil
[2] Departamento de Matemática, Divisão de Ciências Fundamentais, ITA - Instituto Tecnológico da Aeronáutica, 12228-900 São José dos Campos, SP, Brazil

Abstract: The present work derives exact — invariant — solutions to the mode shape equation of uniform rectangular Kirchhoff plates via enhanced group analysis techniques. The study of the optimal system of the finite-dimensional Lie algebra of the admitted classical Lie point symmetries yields three nonequivalent invariant solutions to the studied equation. An additional invariant solution arises from nonclassical Lie point symmetries.

Keywords: exact invariant solutions, mode shape equation, Kirchhoff plate model, enhanced group analysis, Lie point symmetries

INTRODUCTION

The growing interest in thin-walled structures for industrial purposes mainly concerns their shape versatility that allows a balance between design, weight, and structural rigidity, e.g., in civil, acoustic, and aeronautical applications (Ma et al., 2021; Kumar and Sathiya, 2021). Within this framework, the literature extensively carries out vibration analysis of thin plate-like elastic structures via analytical, numerical, and experimental approaches. For the first one, the difficulty in finding exact solutions for the mode shapes of such plates, i.e., the displacement patterns of the given vibrating structure, usually precludes an analytical study of its modal parameters. The case of thin, uniform rectangular plates modeled by the Kirchhoff plate theory follows accordingly: literature exact solutions for their mode shapes are applicable only for a few plate boundary configurations (Navier, 1823; Voigt, 1893; Zeissig, 1898; Wu, Liu, and Chen, 2005). The nonexistence of exact general solutions for studying thin plates brings about the use of approximate methods, e.g., the Rayleigh-Ritz and superposition methods, prone to elevated errors as the wavelength becomes small (Warburton, 1954; Leissa, 1973; Gorman, 1976; Gorman, 1977).

Recent computer algebra advances have assisted symmetry methods — or, more generally, modern group analysis — to gain a new perspective on practical applications (Dimas and Tsoubelis, 2005; Cheviakov, 2007). Symmetry methods from group analysis encompass changing coordinate procedures based on transformations for classifying, simplifying, and solving analytically differential equations (DEs). For partial differential equations (PDEs), each symmetry transformation that acts nontrivially provides a reduced-order equation with fewer independent variables, whose resolution yields the so-called invariant solutions (Olver, 1986; Hydon, 2000). This class of solutions is valuable for the analytical investigation of particular phenomena in physical problems modeled by PDEs mainly because only a handful of such equations have definite general solutions (Kawahara; Uhlmann, and van Veen, 2011; Kaur and Wazwaz, 2018).

This conference paper presents new exact solutions to the mode shape PDE of uniform rectangular plates modeled according to the Kirchhoff plate theory. The study begins using tools from the enhanced group analysis toolbox, in particular equivalence transformations, to classify the PDE in question. The obtained knowledge allows dealing with a version of the PDE free of parameters for which classical and nonclassical Lie point symmetry transformations are sought. The former spawns a Lie algebra whose corresponding one-dimensional optimal system leads to reduced-order equations that provide three exact invariant solutions to the mode shape equation. From the latter, a nonclassical transformation succeeds in finding one more invariant solution. For all the calculations involved the symbolic packages SYM and GEM were utilized, (Dimas and Tsoubelis, 2005; Cheviakov, 2007).

MODE SHAPE EQUATION OF UNIFORM RECTANGULAR PLATES

Mode shape equations derive from elastodynamic equations when assuming harmonic motion, characterizing displacement patterns by dissociating time-varying and spatial terms in free vibration. According to the Kirchhoff model for thin plates, the mode shape equation of a rectangular vibrating plate follows as (Richardson, 1997; Love, 2013):

$$\nabla^2\left(\nabla^2 u\right) - \lambda^4 u = 0, \tag{1}$$

where $u = u(x_1, x_2)$ is the deflection shape, $\lambda = (\rho\,\omega^2/D)^{\frac{1}{4}}$ is the wavenumber, $D = E\,h^3/[12\,(1-\nu^2)]$ is the flexural stiffness, E is the Young's modulus, ν is the Poisson's ration, h is the thickness, ρ is the density, and ω is the natural angular frequency of the plate for a given wavenumber. Here, x_1 and x_2 are the spatial coordinates and ∇^2 is the twodimensional Laplacian for (x_1, x_2).

METHODOLOGY

This section introduces the symmetry methods of classical and nonclassical Lie point symmetries and continuous equivalence transformations via group analysis for the classification of PDEs. For a comprehensive description of the symmetry theory, please refer to the classic works of Ovsiannikov (1982), Olver (1986), Bluman and Kumei (1989), Hydon (2000), Ibragimov (2009), and Bluman, Cheviakov and Anco (2010). In what follows, we adapt the theory to the class of fourth-order PDEs with two independent variables of the form:

$$\mathscr{F}(z_1, z_2, \phi, \partial\phi, \ldots, \partial^4\phi) = 0, \tag{2}$$

where $\phi = \phi(z_1, z_2)$ and $\partial^k\phi$ comprises all kth-order partial derivatives of ϕ with respect to its independent variables $(k = 1, 2, 3, 4)$. For brevity, we define the range of each index introduced only once and we adopt the usual notation for the partial derivatives, e.g., $\phi_{z_i} = \partial\phi/\partial z_i$.

Classical Lie Point Symmetries

Consider the *infinitesimal transformations* for the (z_1, z_2, ϕ)-space of variables (Olver, 1986; Bluman and Kumei, 1989):

$$z_i^* = z_i + \varepsilon\,\xi_{z_i} + \mathscr{O}(\varepsilon^2), \quad \phi^* = \phi + \varepsilon\,\eta + \mathscr{O}(\varepsilon^2), \quad (i = 1, 2) \tag{3}$$

for $\xi_{z_i} = \xi_{z_i}(z_1, z_2, \phi)$ and $\eta = \eta(z_1, z_2, \phi)$ being the *infinitesimals*, $\varepsilon \in \mathbb{R}$ being the continuous transformation parameter, and $\mathscr{O}(\varepsilon^2)$ comprising neglected terms of order higher than one. That is, the infinitesimal generators come from the Taylor series expansion of a *(local) Lie group of point transformations*,

$$z_i^* = z_i^*(z_1, z_2, \phi, \varepsilon), \quad \phi^* = \phi^*(z_1, z_2, \phi, \varepsilon), \tag{4}$$

about its *neutral element* $\varepsilon = 0$. Acting on the entire space of variables of Eq. (2) with such infinitesimal transformations requires understanding its effects upon the derivatives of ϕ, in other words, to obtain the suitable *prolongation* of Eq. (3). For instance, applying the Taylor series expansion to the first-order partial derivatives of ϕ:

$$\phi_{z_i}^* = \phi_{z_i} + \varepsilon\,\eta_{z_i}^{(1)} + \mathscr{O}(\varepsilon^2), \tag{5}$$

yields the first prolongation of the infinitesimals, where

$$\eta_{z_i}^{(1)} = \mathscr{D}_{z_i}\eta - (\mathscr{D}_{z_i}\xi_{z_j})\,\phi_{z_j}, \quad (j = 1, 2), \tag{6}$$

with

$$\mathscr{D}_{z_i} = \frac{\partial}{\partial z_i} + \phi_{z_i}\frac{\partial}{\partial\phi} + \phi_{z_i z_j}\frac{\partial}{\partial\phi_{z_j}} + \cdots + \phi_{z_i z_{i_1}\cdots z_{i_k}}\frac{\partial}{\partial\phi_{z_{i_1}\cdots z_{i_k}}} + \cdots \quad (i_k = 1, 2), \tag{7}$$

being the total derivative operator. Hereinafter, we adopt the Einstein summation convention (Einstein, 1916) for summations over repeated indices. Similarly, one can obtain the kth-order prolongations from the Taylor expansions:

$$\phi_{z_{i_1}\cdots z_{i_k}}^* = \phi_{z_{i_1}\cdots z_{i_k}} + \varepsilon\,\eta_{z_{i_1}\cdots z_{i_k}}^{(k)} + \mathscr{O}(\varepsilon^2), \tag{8}$$

where

$$\eta_{z_{i_1}\cdots z_{i_k}}^{(k)} = \mathscr{D}_{z_{i_k}}\eta_{z_{i_1}\cdots z_{i_{k-1}}}^{(k-1)} - (\mathscr{D}_{z_{i_k}}\xi_{z_j})\,\phi_{z_{i_1}\cdots z_{i_{k-1}} z_j}. \tag{9}$$

The coordinate fluxes characterized by these infinitesimal transformations generate a smooth vector field known as the *infinitesimal generator*:

$$\mathscr{X} = \xi_{z_i}\frac{\partial}{\partial z_i} + \eta\frac{\partial}{\partial u}, \tag{10}$$

whose kth-order prolongation is:

$$\mathscr{X}^{(k)} = \mathscr{X} + \eta_{z_i}^{(1)}\frac{\partial}{\partial\phi_{z_i}} + \cdots + \eta_{z_{i_1}\cdots z_{i_k}}^{(k)}\frac{\partial}{\partial\phi_{z_{i_1}\cdots z_{i_k}}}. \tag{11}$$

Finally, Eq. (2) is invariant under the said (local) Lie group if the corresponding infinitesimal generator satisfies the following *linearized symmetry condition*:

$$\mathscr{X}^{(4)}[\mathscr{F}(z_1, z_2, \phi, \partial\phi, \ldots, \partial^4\phi)]\Big|_{\mathscr{F}=0} = 0. \tag{12}$$

The condition above leads to an overdetermined set of homogeneous linear PDEs called the *determining equations* by considering Eq. (12) as a polynomial concerning the set $\{\partial\phi, \ldots, \partial^4\phi\}$. The integration of this set of differential equations yields the set of infinitesimal generators — which form a *Lie algebra* — that keep invariant Eq. (2) and, hence, by the method of exponentiation, the corresponding admitted (local) Lie group of point transformations (Ovsiannikov, 1982; Bluman and Kumei, 1989; Hydon, 2000).

Invariant Solutions

Local *invariants* of the given group action satisfies the condition (Olver, 1986):

$$\mathscr{X}\,\Theta = 0, \tag{13}$$

for all the functionally independent solutions $\Theta = \Theta(z_1, z_2, \phi) = \{r, \psi\}$, i.e., the complete set of invariants, that arise as the general solution of Eq. (13) by integrating the *characteristic system*:

$$\frac{\mathrm{d}z_1}{\xi_{z_1}} = \frac{\mathrm{d}z_2}{\xi_{z_2}} = \frac{\mathrm{d}\phi}{\eta}, \tag{14}$$

from where $r(z_1, z_2, \phi) = c_1$ and $\psi(z_1, z_2, \phi) = c_2$, for c_1 and c_2 being the constants of integration from the characteristic system's resolution. In the practical regard, each admitted said set of invariants works as new coordinates to the studied Eq. (2) that reduce its number of independent variables by one. This systematic approach aids to the construction of group-invariant solutions, which arise from the resolution of the reduced equations, thus obtaining explicit, *invariant solutions* of the problem.

Optimal System of Subalgebras

Seeking invariant solutions directly from the finite-dimensional Lie algebra $\mathscr{A}$ may lead to redundancy in the solution set due to existing maps between them. Classification of the group elements from the algebra according to their simplest equivalent form produces an *optimal system* from which nonequivalent invariant solutions arise. First, let the commutative operation between infinitesimal generators be defined as (Ovsiannikov, 1982; Hydon, 2000):

$$[\mathscr{X}_p, \mathscr{X}_q] = \mathscr{X}_p \mathscr{X}_q - \mathscr{X}_q \mathscr{X}_p, \tag{15}$$

and the *adjoint representation* as the sum of the Lie series of the argument exponentiation:

$$\mathrm{Ad}(\mathrm{e}^{\varepsilon\,\mathscr{X}_p})\,\mathscr{X}_q = \mathscr{X}_q - \varepsilon\,[\mathscr{X}_p, \mathscr{X}_q] + \frac{\varepsilon^2}{2!}\,[\mathscr{X}_p, [\mathscr{X}_p, \mathscr{X}_q]] - \cdots \tag{16}$$

The algorithm for finding the optimal system of generators consists of taking the most general infinitesimal generator:

$$\hat{\mathscr{X}} = \kappa_\ell\,\mathscr{X}_\ell, \quad (\ell = 1, 2, \ldots, n) \tag{17}$$

for a nth-dimensional algebra, applying the adjoint representation a finite number of times to it,

$$\mathrm{Ad}(\mathrm{e}^{\varepsilon_1\,\mathscr{X}_{l_1}}) \cdots \mathrm{Ad}(\mathrm{e}^{\varepsilon_m\,\mathscr{X}_{l_m}})\,\hat{\mathscr{X}}, \tag{18}$$

for obtaining the simplest form of $\hat{\mathscr{X}}$ that is to zero as many coefficients κ_ℓ as possible. *Videlicet*, we find the quotient space of the algebra of Lie under the equivalent relation for $x, y \in \mathscr{A}$

$$x \sim y \iff \exists\, z \in \mathscr{A}, \varepsilon \in \mathbb{R} : x = \mathrm{e}^{\varepsilon z} y \mathrm{e}^{-\varepsilon z},$$

and we choose the set of the best representatives of this quotient space as the optimal system.

Nonclassical Lie Point Symmetries

The admitted classical Lie point symmetry transformations are often insufficient or nonexistent for obtaining invariant solutions of the equation under analysis. Fortunately, the toolkit of group analysis contains even more advanced kinds of symmetries: one of them is the *nonclassical symmetries*. In a few words, nonclassical symmetries are (point) symmetries admitted by particular families of solutions of a DE but not by the proper DE. To find the said kind of symmetries, we need to add the condition $Q = \eta - \xi_{z_i}\,\phi_{z_i} = 0$, where Q is the *characteristic* of a symmetry $\mathscr{X}$, see Eq. (10), to the linearized symmetry condition of Eq. (12), that is (Bluman and Cole, 1969; Olver and Rosenau, 1987; Hydon, 2000):

$$\mathscr{X}^{(4)}[\mathscr{F}(z_1, z_2, \phi, \partial\phi, \ldots, \partial^4\phi)]\Big|_{\substack{\mathscr{F}=0\\ Q=0}} = 0. \tag{19}$$

The algorithm follows as for the classical case with the only difference that now the determining equations consist of nonlinear PDEs. Consequently, it is common not to be able to find the complete solution set for these kinds of determining equations.

Equivalence Transformations from Enhanced Group Analysis

Suppose symmetries are transformations of differential equations that leave invariant their solution set. In that case, equivalence transformations are transformations that leave invariant collections of differential equations, i.e., symmetries on whole families of differential equations. The Lie algorithm for finding symmetries extends naturally to the equivalence transformations case by expanding the variable space of the infinitesimal transformations for Eq. (2) to include also its arbitrary constants and functions. Consider, for instance, that Eq. (2) includes an arbitrary constant, say γ. We extend the variable space by first assuming that $\gamma = \gamma(z_1, z_2, \phi)$ and then by including the coordinate flux (Ibragimov, 2009; Bluman, Cheviakov, and Anco, 2010):

$$\gamma^* = \gamma + \varepsilon\mu + \mathscr{O}(\varepsilon^2), \tag{20}$$

with the infinitesimal $\mu = \mu(z_1, z_2, \phi, \gamma)$. From this, Eqs. (3) and (20) form a one-parameter Lie group of point equivalence transformations of Eq. (2).

The infinitesimal generator associated with such equivalence transformation now has the form:

$$\mathscr{Y} = \xi_{z_i}\frac{\partial}{\partial z_i} + \eta\frac{\partial}{\partial \phi} + \mu\frac{\partial}{\partial \gamma}, \tag{21}$$

prolonged up to the k-th order as follows:

$$\mathscr{Y}^{(k)} = \mathscr{Y} + \eta^{(1)}_{z_i}\frac{\partial}{\partial \phi_{z_i}} + \mu^{(1)}_{w_j}\frac{\partial}{\partial \gamma_{w_j}} + \cdots + \eta^{(k)}_{z_{i_1}\dots z_{i_k}}\frac{\partial}{\partial \phi_{z_{i_1}\dots z_{i_k}}} + \mu^{(k)}_{w_{j_1}\dots w_{j_k}}\frac{\partial}{\partial \gamma_{w_{j_1}\dots w_{j_k}}} \quad (j = 1,2,3,\ j_k = 1,2,3), \tag{22}$$

where $(w_1, w_2, w_3) = (z_1, z_2, \phi)$. The prolongation formulae for μ follow Eq. (9) with the only difference that one must now consider ϕ as an independent variable.

Similarly to Eq. (12), the existence of point equivalence transformations relies on the validity of the condition:

$$\mathscr{Y}^{(4)}[\mathscr{F}(z_1, z_2, \phi, \gamma, \partial\phi, \partial\gamma, \dots, \partial^4\phi, \partial^4\gamma)]\Big|_{\mathscr{F}=0} = 0, \tag{23}$$

whose evaluation yields a set of determining equations built and solved exactly like the classical case.

EQUIVALENT DIFFERENTIAL EQUATION VIA EQUIVALENCE TRANSFORMATIONS

In this section, we apply the method of equivalence transformations to Eq. (1). First, we extend the PDE

$$\nabla^2(\nabla^2 u) - \lambda^4(x_1, x_2, u(x_1, x_2))\, u = 0, \tag{24}$$

and since we work with a family of PDEs where λ is a constant, we need to add this information to the extended equation, namely:

$$\begin{cases} \nabla^2(\nabla^2 u) - \lambda^4(x_1, x_2, u(x_1, x_2))\, u = 0, \\ \lambda_{x_1} = 0, \\ \lambda_{x_2} = 0, \\ \lambda_u = 0. \end{cases} \tag{25}$$

The invariance condition from Eq. (23) yields the following determining equations:

$$\begin{gathered} \frac{\partial \xi_{x_1}}{\partial x_1} - \frac{\mu}{\lambda} = 0, \quad \frac{\partial \xi_{x_2}}{\partial x_2} - \frac{\mu}{\lambda} = 0, \quad \frac{\partial \xi_{x_2}}{\partial u} = 0, \quad \frac{\partial \xi_{x_1}}{\partial x_2} + \frac{\partial \xi_{x_2}}{\partial x_1} = 0, \quad \frac{\partial^2 \xi_{x_1}}{\partial x_2^2} = 0, \\ \frac{\partial^2 \eta}{\partial x_1 \partial u} = 0, \quad \frac{\partial^2 \eta}{\partial x_2 \partial u} = 0, \quad \frac{\partial^2 \eta}{\partial u^2} = 0, \quad \frac{\partial^4 \eta}{\partial x_1^4} + \frac{\partial^4 \eta}{\partial x_2^4} + 2\frac{\partial^4 \eta}{\partial x_1^2 \partial x_2^2} + \lambda^4\left(u\frac{\partial \eta}{\partial u} - \eta\right) = 0. \end{gathered} \tag{26}$$

Its resolution gives the equivalence infinitesimal transformations admitted by Eq. (1) as follows:

$$\begin{gathered} \mathscr{Y}_1 = uF_1(\lambda)\frac{\partial}{\partial u}, \quad \mathscr{Y}_2 = F_2(\lambda)\frac{\partial}{\partial x_1}, \quad \mathscr{Y}_3 = F_3(\lambda)\frac{\partial}{\partial x_2}, \quad \mathscr{Y}_4 = F_4(\lambda)\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right), \\ \mathscr{Y}_5 = -F_5(\lambda)\left(\frac{x_1}{\lambda}\frac{\partial}{\partial x_1} + \frac{x_2}{\lambda}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial \lambda}\right), \quad \mathscr{Y}_6 = F_6(x_1, x_2, \lambda)\frac{\partial}{\partial u}, \end{gathered} \tag{27}$$

where $F_1, \dots, F_5$ are arbitrary functions of λ and F_6 is a solution to Eq. (1).

Observe that the generator $\mathscr{Y}_5$ corresponds to the continuous transformation $(x_1, x_2, u, \lambda) \to (x_1 \mathrm{e}^{\varepsilon}, x_2 \mathrm{e}^{\varepsilon}, u, \lambda \mathrm{e}^{-\varepsilon})$, so by taking $\mathrm{e}^{\varepsilon} = \lambda \neq 0$ we obtain a transformation that maps Eq. (1) into:

$$\nabla^2(\nabla^2 w) - w = 0, \tag{28}$$

for $u(x_1, x_2) = w(y_1, y_2)$, $x_1 = y_1/\lambda$, $x_2 = y_2/\lambda$. The preference for dealing with Eq. (28) arises from the fact that only the case $\lambda = 1$ of Eq. (1) needs to be considered, hence reducing the analysis of the family Eq. (1) to the one of its best representative, Eq. (28).

ADMITTED LIE POINT SYMMETRIES OF THE EQUIVALENT EQUATION

This section investigates the admitted classical and nonclassical Lie point symmetries of Eq. (28) and its associated invariant solutions. The said invariant solutions are then written for the (x_1,x_2,u)-space of variables in order to get explicit solutions of Eq. (1).

Classical Symmetries, Associated Optimal System, and Invariant Solutions

The formulation for the classical Lie point symmetries of Eq. (28) via Eq. (12) yields:

$$\begin{gathered}\frac{\partial \xi_{y_1}}{\partial y_1}=0,\quad \frac{\partial \xi_{y_1}}{\partial w}=0,\quad \frac{\partial \xi_{y_2}}{\partial y_2}=0,\quad \frac{\partial \xi_{y_2}}{\partial w}=0,\quad \frac{\partial \xi_{y_1}}{\partial y_2}+\frac{\partial \xi_{y_2}}{\partial y_1}=0,\quad \frac{\partial^2 \xi_{y_2}}{\partial y_1^2}=0,\\ \frac{\partial^2 \eta}{\partial y_1\,\partial w}=0,\quad \frac{\partial^2 \eta}{\partial y_2\,\partial w}=0,\quad \frac{\partial^2 \eta}{\partial w^2}=0,\quad \frac{\partial^4 \eta}{\partial y_1^4}+\frac{\partial^4 \eta}{\partial y_2^4}+2\frac{\partial^4 \eta}{\partial y_1^2\partial y_2^2}+w\frac{\partial \eta}{\partial w}-\eta=0.\end{gathered} \tag{29}$$

The solution of Eq. (29) provides the following basis for the Lie algebra of symmetries of Eq. (28):

$$\mathscr{X}_1=\frac{\partial}{\partial y_2},\quad \mathscr{X}_2=\frac{\partial}{\partial y_1},\quad \mathscr{X}_3=w\frac{\partial}{\partial w},\quad \mathscr{X}_4=y_2\frac{\partial}{\partial y_1}-y_1\frac{\partial}{\partial y_2},\quad \mathscr{X}_5=F(y_1,y_2)\frac{\partial}{\partial w}, \tag{30}$$

where $F(y_1,y_2)$ is a solution to Eq. (28). Let the set $\{\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{X}_4\}$ spans the finite-dimensional Lie algebra represented in Table 1.

Table 1 – Commutative table.

$[\mathscr{X}_p,\mathscr{X}_q]$	$\mathscr{X}_1$	$\mathscr{X}_2$	$\mathscr{X}_3$	$\mathscr{X}_4$
$\mathscr{X}_1$	0	0	0	$\mathscr{X}_2$
$\mathscr{X}_2$	0	0	0	$-\mathscr{X}_1$
$\mathscr{X}_3$	0	0	0	0
$\mathscr{X}_4$	$-\mathscr{X}_2$	$\mathscr{X}_1$	0	0

For the said algebra with $a_1,a_2\in\mathbb{R}$ and the described algorithm, the optimal system is given by:

$$\{\mathscr{X}_1,\quad a_1\mathscr{X}_1+\mathscr{X}_3,\quad a_2\mathscr{X}_3+\mathscr{X}_4\}, \tag{31}$$

from where the corresponding one-parameter groups of local transformations arise:

$$\mathrm{e}^{\gamma\mathscr{X}_1}:(y_1,y_2,w)\to(y_1,y_2+\gamma,w), \tag{32}$$

$$\mathrm{e}^{\varepsilon(a_1\mathscr{X}_1+\mathscr{X}_3)}:(y_1,y_2,w)\to(y_1,y_2+a_1\varepsilon,\mathrm{e}^{\varepsilon}w), \tag{33}$$

$$\mathrm{e}^{\varepsilon(a_2\mathscr{X}_3+\mathscr{X}_4)}:(y_1,y_2,w)\to(y_1\cos(\varepsilon)+y_2\sin(\varepsilon),y_2\cos(\varepsilon)-y_1\sin(\varepsilon),\mathrm{e}^{a_2\varepsilon}w). \tag{34}$$

Consider now $\mathscr{X}_1$ from the set of Eq. (31). The differential invariant method provides $r=y_1$ and $\psi=w$ as new independent and dependent variables for the reduction of Eq. (28), respectively, which turn the PDE into the ODE:

$$\frac{d^4\psi}{dr^4}-\psi=0, \tag{35}$$

having the solution:

$$\psi=C_1\mathrm{e}^{-r}+C_2\mathrm{e}^{r}+C_3\sin(r)+C_4\cos(r), \tag{36}$$

finally by using the transformations (34) and reverting to the original coordinates we get:

$$u=C_1\mathrm{e}^{-\lambda(\alpha x_1-\beta x_2)}+C_2\mathrm{e}^{\lambda(\alpha x_1-\beta x_2)}+C_3\sin(\lambda(\alpha x_1-\beta x_2))+C_4\cos(\lambda(\alpha x_1-\beta x_2)), \tag{37}$$

where $\alpha^2+\beta^2=1$, $\alpha,\beta\in\mathbb{R}$.

Similarly, the remaining symmetries from the optimal system give the nonequivalent invariant solutions:

$$u=\mathrm{e}^{\frac{\lambda(\alpha x_2+\beta x_1)-\gamma}{a_1}}\left(C_5\,\mathrm{e}^{\frac{\lambda(\alpha x_1-\beta x_2)}{a_1}\sqrt{-1-a_1^2}}+C_6\,\mathrm{e}^{-\frac{\lambda(\alpha x_1-\beta x_2)}{a_1}\sqrt{-1-a_1^2}}+C_7\,\mathrm{e}^{\frac{\lambda(\alpha x_1-\beta x_2)}{a_1}\sqrt{-1+a_1^2}}+C_8\,\mathrm{e}^{-\frac{\lambda(\alpha x_1-\beta x_2)}{a_1}\sqrt{-1+a_1^2}}\right), \tag{38}$$

and letting $a_2=0$ to avoid a complex solution from the last element of the optimal system:

$$u=C_9\left[\mathrm{I}\left(\lambda\sqrt{x_1^2+(x_2+\gamma)^2}\right)+\mathrm{J}\left(\lambda\sqrt{x_1^2+(x_2+\gamma)^2}\right)\right], \tag{39}$$

where J and I are the first-kind ordinary and modified Legendre functions, respectively, $\alpha^2+\beta^2=1,\ \alpha,\beta,\gamma\in\mathbb{R}$, and C_m for $m=1,2,\ldots,9$ the constants of integration (Abramowitz and Stegun, 1970).

Nonclassical Symmetries and Invariant Solutions

Assuming $\xi_{y_1} = 1$, the invariance condition from Eq. (19) results in the determining equations for the (y_1, y_2, w)-space:

$$\frac{\partial^2\eta}{\partial y_1\,\partial w} - \frac{\partial^2\eta}{\partial y_2\,\partial w} = 0, \quad \frac{\partial^2\eta}{\partial^2 w} = 0, \quad \frac{\partial^3\eta}{\partial y_2^2\,\partial w} + \frac{\partial\eta}{\partial w}\frac{\partial^2\eta}{\partial y_2\,\partial w} = 0,$$
$$\frac{\partial^4\eta}{\partial y_1^4} + \frac{\partial^4\eta}{\partial y_2^4} + 2\frac{\partial^2\eta}{\partial y_1\,\partial w} + 4\frac{\partial^2\eta}{\partial y_2\,\partial w}\left(\frac{\partial^2\eta}{\partial y_1^2} + \frac{\partial^2\eta}{\partial y_2^2} - \frac{\partial\eta}{\partial y_2}\frac{\partial\eta}{\partial w} - \frac{\partial\eta}{\partial y_1}\frac{\partial\eta}{\partial w} + \eta\frac{\partial^2\eta}{\partial w^2}\right) + w\frac{\partial\eta}{\partial w} - \eta = 0, \tag{40}$$

which admit the solution:

$$\mathscr{X} = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \sqrt{2b_1}\tanh\left(\sqrt{\frac{b_1}{2}}(y_1 + y_2 + b_2)\right)\frac{\partial}{\partial w}, \tag{41}$$

for the constants $b_1 \geq 0$ and b_2.

The characteristic's invariance:

$$\sqrt{2b_1}\tanh\left[\sqrt{\frac{b_1}{2}}(y_1 + y_2 + b_2)\right]w - w_{y_1} - w_{y_2} = 0, \tag{42}$$

yields:

$$w = \cosh\left(\sqrt{\frac{b_1}{2}}(y_1 + y_2 + b_2)\right)\Psi(y_2 - y_1), \tag{43}$$

for the arbitrary function $\psi = \psi(y_2 - y_1)$. Substituting Eq. (43) into Eq. (28) with $\zeta = y_2 - y_1$ and $\psi = \psi(\zeta)$ reduces the equivalent PDE to:

$$\frac{\mathrm{d}^4\psi}{\mathrm{d}\zeta^4} + b_1\frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2} + \frac{b_1^2 - 1}{4}\psi = 0. \tag{44}$$

The reduced-order Eq. (44) via nonclassical Lie point symmetries admits the solution:

$$\psi = B_1\,\mathrm{e}^{\left(-\frac{1+b_1}{2}\right)^{1/2}\zeta} + B_2\,\mathrm{e}^{-\left(-\frac{1+b_1}{2}\right)^{1/2}\zeta} + B_3\,\mathrm{e}^{\left(\frac{1-b_1}{2}\right)^{1/2}\zeta} + B_4\,\mathrm{e}^{-\left(\frac{1-b_1}{2}\right)^{1/2}\zeta}, \tag{45}$$

where B_m are constants of integration for $m = 1, 2, 3, 4$.

Finally, the map $(\zeta, \psi) \to \left(\lambda x_2 - \lambda x_1, u/\cosh\left[\sqrt{b_1/2}(y_1 + y_2 + b_2)\right]\right)$ provides the last invariant solution to Eq. (1) with the chosen methodology:

$$u = \left(B_1\,\mathrm{e}^{\left(-\frac{1+b_1}{2}\right)^{1/2}(\lambda x_2 - \lambda x_1)} + B_2\,\mathrm{e}^{-\left(-\frac{1+b_1}{2}\right)^{1/2}(\lambda x_2 - \lambda x_1)} + B_3\,\mathrm{e}^{\left(\frac{1-b_1}{2}\right)^{1/2}(\lambda x_2 - \lambda x_1)} + B_4\,\mathrm{e}^{-\left(\frac{1-b_1}{2}\right)^{1/2}(\lambda x_2 - \lambda x_1)}\right)\cosh\left(\sqrt{\tfrac{b_1}{2}}\left[(x_1 + x_2)\lambda + b_2\right]\right). \tag{46}$$

FINAL REMARKS

A simpler, equivalent DE results from the enhanced group analysis of the mode shape equation of uniform Kirchhoff rectangular plates via equivalence transformations. The admitted classical Lie point symmetries from the finite-dimensional Lie algebra of the equivalent equation are structured according to their optimal system of subalgebras, providing the said equation with three invariant solutions and, hence, three exact solutions to the mode shape equation. Similarly, an additional invariant solution derives from nonclassical Lie point symmetries, written for the original space of variables and thus completing the set of four invariant solutions of the analyzed equation. From the literature, the works of Navier (1823), Voigt (1893), Zeissig (1898), and Wu, Liu, and Chen (2005) consider exact solutions that work for specific boundary configurations. Besides that, all found invariant solutions in this work are completely new or at least, in the case of Eq. (37), more general then the literature ones.

The special class of invariant solutions commonly comprises the only known exact solutions for PDEs. Also, it is worth emphasizing that the study of several mechanical problems relies on the resolution of their modeling PDEs. Within this framework, symmetry methods have been furnishing powerful tools for computing such solutions analytically, broadening the range of exact solutions for the given and similar problems. Extensions of the proposed plate approach may lead to original exact solutions to more complicated mechanical problems, e.g., finding exact mode shape solutions for thick plates or expanding the formulation for investigating the invariance of plate boundary conditions under the computed symmetry transformations. If feasible, the latter may symbolize a considerable advance in terms of defining exact mode shapes of plates for boundary configuration that does not admit the well-known plate solutions from the literature.

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