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Nonlinear normal modes of flexible risers in catenary configuration

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Abstract: Risers are offshore pipelines that connect a floating vessel to the bottom of the ocean, allowing oil to be extracted. Because the oil exploration is already being held at extreme deep levels, some of the risers are being design as flexible structures. Flexible beams are usually modeled under the assumption of large displacements and large rotations of the cross sections, but with small strains. Such hypotheses lead to a geometric nonlinearity that can be efficiently incorporated to the model using co-rotational finite element. Instead of performing numerical integration of the equation of motion to analyze the dynamics of flexible beams, this paper proposes a procedure to compute the nonlinear normal modes of this structure. The Rosenberg's definition is used, which allows the nonlinear normal modes to be computed using the Harmonic Balance Method and a numerical continuation scheme. The procedure is validated for a flexible riser in catenary configuration.

Keywords: Nonlinear normal modes, Harmonic Balance, Flexible beams, Co-rotational finite elements, Continuation methods

INTRODUCTION

For several industries, the dimensions of important structures are being extended to new limits in order to satisfy their new challenging needs. Those new dimensions led many structures to exhibit significant nonlinearities in their motion. A typical example is the offshore pipelines that are being used to explore oil at deep see levels (Yazdchi and Crisfield, 2002)(Albino et al., 2018). The extended length of the pipes turns them into extremely slim beams with significant flexibility. The large displacements and finite rotations of the beam's cross sections are then responsible for the generation of a geometric nonlinearity in the structure.

Flexible beams are usually modeled under the assumption of large displacement, finite rotation, but with small strains. Such hypotheses allow the equation of motion to be built using co-rotational finite elements. The main idea of the co-rotational formulation is to decompose the motion of each element as a small elastic deformation added to a rigid body motion. Several researches have already analyzed the dynamic responses of flexible beams using co-rotational finite elements (Hsiao and Jang, 1991)(Iura and Iwakuma, 1992)(Iura and Atluri, 1995)(Behdinam, Stylianou and Tabarrok, 1998), but all of them restricted to numerical integration of the equation of motion. The main differences between them lies in the choice of the interpolation functions used in the local frame when computing the kinetic and potential energies of the system.

An alternative tool to analyze the dynamics of nonlinear structures consists in the computation of nonlinear normal modes (NNM). The knowledge of NNMs allows a thoroughly understanding of a system's vibratory response in the nonlinear regime. It considers phenomena that are exclusive to nonlinear system, such as bifurcations, instability, super-harmonic, subharmonic and internal resonances. This paper presents a procedure to compute NNMs of flexible beams, with emphasis on flexible riser (offshore flexible oil pipes) in catenary configuration. The procedure combines several numerical methods, including: the Harmonic Balance Method (HBM), the Alternating frequency-time (AFT), a numerical continuation method and the Newton-Raphson solver. An example of a NNM of a flexible riser is presented and analyzed.

EQUATION OF MOTION

The co-rotational finite element is an efficient and simple way to incorporate the geometric nonlinearity emanated from the high flexibility of a beam. In the co-rotational framework, the motion of a finite element is decomposed into two parts: a rigid body displacement and a linear elastic deformation. A local coordinate system is then incorporated to each element and forced to move with it according to its rigid body part of the motion. Then, the small deformation is written with respect to this local coordinate system using typical linear beam elements, and later transformed to an inertial frame. This transformation between frames generates the geometric nonlinearities in the model that are associated with the large displacements and rotations of the cross sections.

In this paper, for the elastic potential energy, a cubic interpolation function for the transverse displacement of the beam and a quadratic interpolation of the cross-section rotation was adopted along the local coordinate system. This interpolation functions corresponds to the Interdependent Interpolation Element (IIE), and it was chosen since it corresponds to a lock-free element (Reddy,1997). For the kinetic and gravitational potential energy of the element, a linear interpolation for the transverse displacement of the beam and for the cross-section rotation was adopted along the local coordinate system, which represents the Timoshenko beam element. This lower order interpolation was chosen to simplify the model.

To derive the equation of motion, the Lagrange equation is used. Considering the co-rotational beam element adopted here, the Lagrange equation becomes

$$\frac{d}{dt}\left(\frac{\partial \mathscr{T}}{\partial \dot{\mathbf{q}}}\right)+\underbrace{\frac{\partial \mathscr{V}}{\partial \bar{\mathbf{q}}}\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{q}}}_{\frac{\partial \mathscr{V}}{\partial \mathbf{q}}}+\frac{\partial \mathscr{U}}{\partial \mathbf{q}}=0, \tag{1}$$

where $\mathscr{T}$ corresponds to the total kinetic energy of the flexible beam, $\mathscr{V}$ is the total elastic potential energy and $\mathscr{U}$ is the total gravitational potential energy. $\mathbf{q} \in \mathbb{R}^n$ corresponds to the global generalized coordinates of the beam, where n represents the number of degrees of freedom in the structure, and $\bar{\mathbf{q}} \in \mathbb{R}^{n_l}$ corresponds to the local generalized coordinates, where n_l represents the number of local coordinates in the structure. Because of the adopted interpolation function, $\mathscr{T}$ and $\mathscr{U}$ could be written directly in terms of $\mathbf{q}$, while $\mathscr{V}$ had to be written in terms of $\bar{\mathbf{q}}$. Therefore, the chain rule had to be used to evaluate $\frac{\partial \mathscr{V}}{\partial \mathbf{q}}$. The term $\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{q}}$ establishes the relationship between the local and global coordinates and takes the rigid body motion into consideration. This term is also responsible for the appearance of nonlinearity in the equation of motion.

By performing the derivatives in of Eq. 1, the following equation of motion is obtained:

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{f}(\mathbf{q})=\mathbf{0}, \tag{2}$$

where $\mathbf{M} \in \mathbb{R}^{n\times n}$ is the mass matrix and $\mathbf{f} \in \mathbb{R}^n$ is the vector with nonlinear forces, which includes the elastic and gravitational forces. A complete description of how to derive the equation of motion is found in (Wagner 2022). Instead of performing a numerical integration of this equation of motion, as already performed in previous publications, this paper incorporates this co-rotational model in a procedure to calculate nonlinear normal modes (NNM).

1 NONLINEAR NORMAL MODES

For the computation of the NNMs of nonuniform flexible beams, the Rosenberg's definition of NNM will be used here. It consists in families of periodic solutions of autonomous systems. One important characteristic of NNMs is its dependence on the energy level. Therefore, the periodic solutions must be computed for a pre-defined range of energies, which can be done combining the Harmonic Balance Method (HBM) (Krack and Gross, 2019) to find periodic solutions with a prediction-correction scheme to perform the numerical continuation of the periodic solutions (Seydel, 2009).

The HBM is a popular method used to solve periodic boundary value problems. For the flexible beam considered in this paper, the periodic boundary value problem used to compute periodic solution can be constructed adding a periodic boundary restriction to the equation of motion, which leads to:

$$\begin{aligned} \text{Solve:}\quad & \mathbf{M}\ddot{\mathbf{q}}+\mathbf{f}(\mathbf{q})=\mathbf{0},\quad t\in[0,T]\\ \text{With:}\quad & \mathbf{q}(0)=\mathbf{q}(T)\\ & \dot{\mathbf{q}}(0)=\dot{\mathbf{q}}(T). \end{aligned} \tag{3}$$

Here, $T=\frac{2\pi}{\Omega}$ is the unknown fundamental period of the solution. Instead of seeking the periodic solution of Eq. (3) directly, the HBM starts with the definition of an *Ansatz* function $\mathbf{q}_H(t)$, written as a truncated Fourier series, that converges to $\mathbf{q}(t)$ as the truncation order increases. The *Ansatz* is defined as

$$\mathbf{q}_H(t)=\sum_{k=-H}^{H}\mathbf{Q}_k e^{\mathbf{i}k\Omega t}, \tag{4}$$

where H is the truncation order of the series, $\{\mathbf{Q}_k\}_{k=-H}^{H} \in \mathbb{C}^n$ are the respective Fourier coefficients of the *Ansatz* and Ω is the fundamental frequency. The vector basis used to span the *Ansatz* corresponds to a set of Fourier functions, which are intrinsically periodic, and therefore automatically satisfy the periodic boundary conditions required by Eq. (3). Since the *Ansatz* is only an approximation of $\mathbf{q}(t)$, a residual $\mathbf{r}(t)$ is expected when introducing Eq. (4) into the equation of motion:

$$\mathbf{r}(t):=\sum_{k=-H}^{H}-\omega^2k^2\mathbf{M}\mathbf{Q}_k e^{\mathbf{i}k\Omega t}+\mathbf{f}(\mathbf{q}_H)\neq\mathbf{0}. \tag{5}$$

Since the nonlinear elastic force vector, $\mathbf{f}(\mathbf{q}_H)$, is a function of the *Ansatz* only, it is also periodic. Therefore, the residual can be expressed as

$$\mathbf{r}(t)=\sum_{k=-\infty}^{\infty}\mathbf{R}_k\left(\Omega,\{\mathbf{Q}_l\}_{l=0}^{H}\right)e^{\mathbf{i}k\Omega t}, \tag{6}$$

where $\mathbf{R}_k$ corresponds to the k-th Fourier coefficient of the residual.

When projecting the residual into the subspace of the *Ansatz* (performing a Fourier-Galerking projection), the time dependency of Eq. (6) is removed. Also, using the orthogonality of the Fourier functions and imposing that the residual must be perpendicular to the subspace of the *Ansatz* (i.e., balanced up to the H-th harmonic), a system of nonlinear algebraic equation is constructed

$$\mathbf{R}_m\left(\Omega, \{\mathbf{Q}_l\}_{l=0}^{H}\right) = \mathbf{0} \quad \text{for} \quad m = -H, \dots, H. \tag{7}$$

Since the Fourier coefficients $\mathbf{R}_m$ shares the conjugate mirror property, it is sufficient to solve Eq. (7) only for $m = 0, \dots, H$.

This system of nonlinear algebraic equations is under-determined and requires two additional equations, one related to a phase restriction and other to an amplitude restriction, which are defined here as η_p and η_a, respectively. It is important to highlight that the amplitude restriction is the equation that defines the energy level of the periodic solution. Combining the Eq. (7) with the phase and amplitude, the resulting system of nonlinear algebraic equation can be solved uniquely. This system of equations can be written in a compact form as

$$\mathscr{R}(\mathbf{x}, \varepsilon) = \begin{bmatrix} \mathbf{R}_0(\mathbf{x}) \\ \Re\{\mathbf{R}_1(\mathbf{x})\} \\ \Im\{\mathbf{R}_1(\mathbf{x})\} \\ \vdots \\ \Re\{\mathbf{R}_H(\mathbf{x})\} \\ \Im\{\mathbf{R}_H(\mathbf{x})\} \\ \eta_p(\mathbf{x}) \\ \eta_a(\mathbf{x}, \varepsilon) \end{bmatrix} = \mathbf{0}. \tag{8}$$

where $\mathbf{x} = [\Omega, \mathbf{Q}_0, \Re\{\mathbf{Q}_1\}, \Im\{\mathbf{Q}_1\}, \dots, \Re\{\mathbf{Q}_H\}, \Im\{\mathbf{Q}_H\}]$ are the vector of unknowns and ε is the energy level of the periodic solution. The solution of Eq. (8) is usually found numerically using the Newton-Raphson method.

The system of nonlinear algebraic equations defined in Eq. (8) gives a good approximation of the periodic solution for a specific energy level. To completely characterize the NNMs, the periodic solution must be evaluate for a wide range of energy, i.e., considering $\varepsilon \in \left[\varepsilon_{min}, \quad \varepsilon_{max}\right]$. This can be efficiently done using a numerical path continuation and using ε as free parameter. The set of solutions at different energy levels creates a periodic solution branch. In this paper, the predictor-corrector scheme was chosen as continuation method to evaluate the periodic solution in a predefined energy interval. The tangent method was used for the prediction step and the arc-length for the correction step. A complete description of this continuation method is given in (Seydel, 2009).

NUMERICAL EXAMPLE

The numerical example discussed in this paper consists of a flexible riser in catenary configuration. It corresponds to a 350m pipe that connects a ship to a sub-sea tower at a depth of 150m. The ship is dislocated 150m horizontally from the sub-sea tower, as showed in Fig. 1a. This example highlight the potential of the proposed procedure to handle non-academic problems. Since the main purpose of this example is to illustrate the versatility of the co-rotational finite element formulation, the forces generated by the interaction of the structure with the water are not considered at this moment (including hydrostatic and hydrodynamic forces). The material and geometric properties of the structure are presented in Fig. 1a. The riser was modeled using 40 equally spaced co-rotational finite elements, leading to a model with 119 degrees of freedom. The displacements of some of the nodes (6th, 21th and 32th) were chosen for a deeper analysis later on and they are marked as red dots in the figure.

Before computing the NNMs of this riser, the catenary configuration must be found first, which corresponds to the static equilibrium of the structure. To find it, we first set the riser in horizontal position and compute the static deformation caused by the gravity. Then, the position of the right hand-side support is gradually changed from its initial position (350,0) to its final position (150,150) in 100 discrete steps. For each step, the static equilibrium is computed using the Newton-Raphson solver, where the previous solution is used as initial guess of the new solution (sequential continuation). Some of the computed equilibrium positions are presented in Fig. 1b.

To compute the NNMs, the first step consists in calculate the underlying LNMs of the riser in its final catenary configuration. Those linear modes are important because they are used as initial guess for the first periodic solution at the lowest energy levels. The nonlinear elastic force is then linearized around the final catenary position to obtain the tangent stiffness matrix. This stiffness matrix, together with the mass matrix, is then used to form the eigenvalue problem able to extract the LNMs.

Only the first two NNMs are analyzed hereafter. Each of them was characterized by the continuation branch of the periodic solution with 50 discrete points. The HBM was used with a truncation order $H = 10$ to approximate the periodic solutions. Given the high number of degrees of freedom in the model, the number of unknowns to be found with the HBM becomes $n(2H+1)+1 = 2500$. The computational time to compute each NNMs of the riser in catenary was approximately one hour.

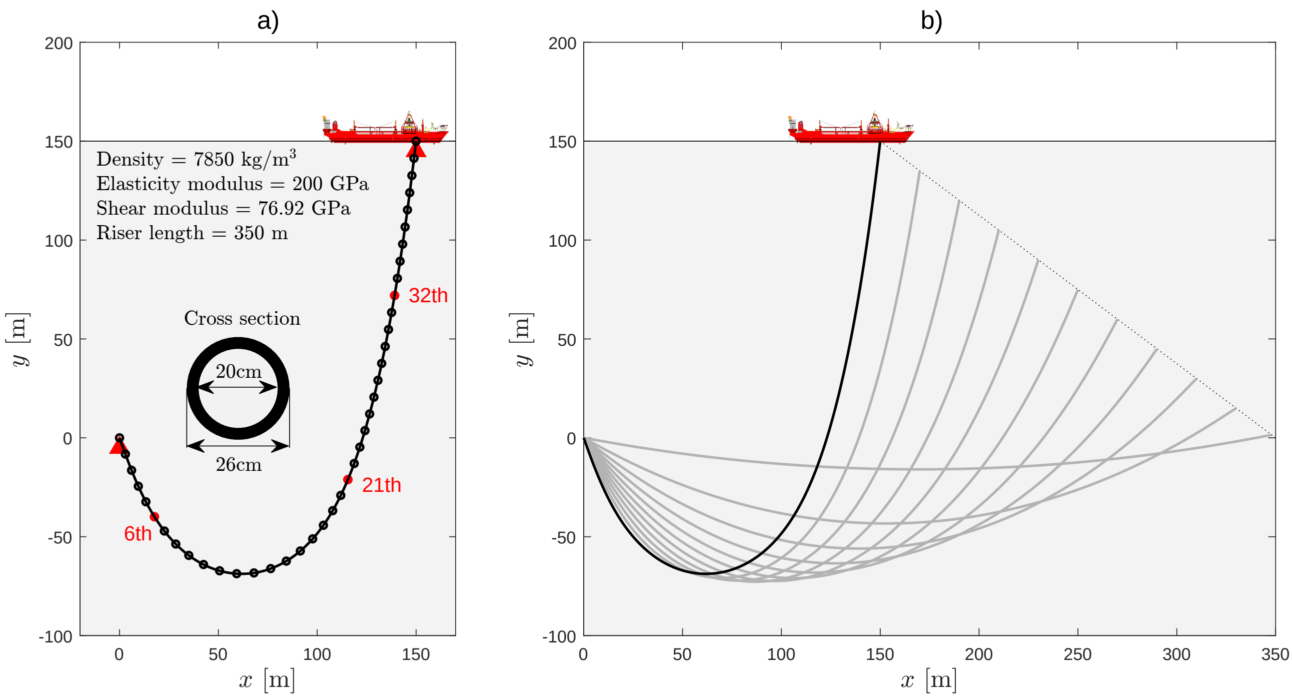

Figure 1 – Flexible riser in catenary configuration. a) Geometric and material properties of the model. Representation of the adopted mesh, highlighting three nodes in which the motion is analyzed in detail. b) Static equilibrium of the flexible riser from horizontal to its final installation position.

Figure 2a shows the FEP of the first NNM. The fundamental frequency is approximate constant in the analyzed energy range. Each dot represents a computed periodic solution and the respective initial riser configuration of the periodic solutions are presented in Fig. 2b. The black curve represents the solution with the lowest energy and red the highest. Figure 2c shows different positions of the riser during one oscillation of the first NNM at the highest energy level evaluated.

Figure 3 shows how much each harmonic contributes in the periodic motion of the NNM in the x and y direction of three nodes (6th, 21th and 32th). For low energy levels, the motion is composed by only the first harmonic, so that the NNM is approximately equal to the underlying LNM. But, as the energy increases, the participation of this harmonic drops up to values between 60% and 70%. The rest of the motion is composed by the DC term and the 2nd to 5th harmonics.

Figure 4a show the FEP of the second NNM. This time, the fundamental frequency of the periodic solutions drops with the energy level, characterizing a softening behavior. Once again the initial riser configuration of the periodic solutions are presented in Fig. 4b for different energy levels, where the increasing in the energy is represented by the changing in colors from black to red. Figure 4c also shows the positions of the riser during one oscillation of the NNM at the highest energy level. One interesting characteristic of the mode at this high energy level is the fact that there is no nodal point, i.e., there is no point in the structure that has no motion.

The participation of each harmonic in the motion at the same nodes (6th, 21th and 32th) in the x and y direction is presented in Fig. 5. Once again, at low energies, the motion starts having the participation of only the first harmonic, with the exception of the motion of the 21th node in the x direction that already have some participation of the DC term and the second harmonic. As the energy increases, the participation of the first harmonic in the motion of this node decreases quickly, going up to 10% at the highest energy level. This node is more sensitive to the higher harmonics because it is closer to the nodal point of the underlying LNM, where the fundamental harmonic should be close to zero.

CONCLUSIONS

Flexible beams were modeled here using the co-rotation finite element formulation. This model was only possible because large displacement, finite rotation and small strain was assumed. As a result of this modeling strategy, the motion of flexible beams with geometric nonlinearities could be efficiently computed.

The main contribution of this paper consists in the incorporation of the co-rotation finite element in a standard procedure to compute NNMs. The Rosenberg's definition of NNM was used. A periodic restriction was incorporated to the equation of motion to generate a periodic boundary value problem, in which the solution was approximated with the Harmonic Balance Method. The computation of the Fourier coefficients of the nonlinear elastic term was done numerically using the Alternating Frequency-Time method. The tangent method and the arc-length parametrization were adopted in

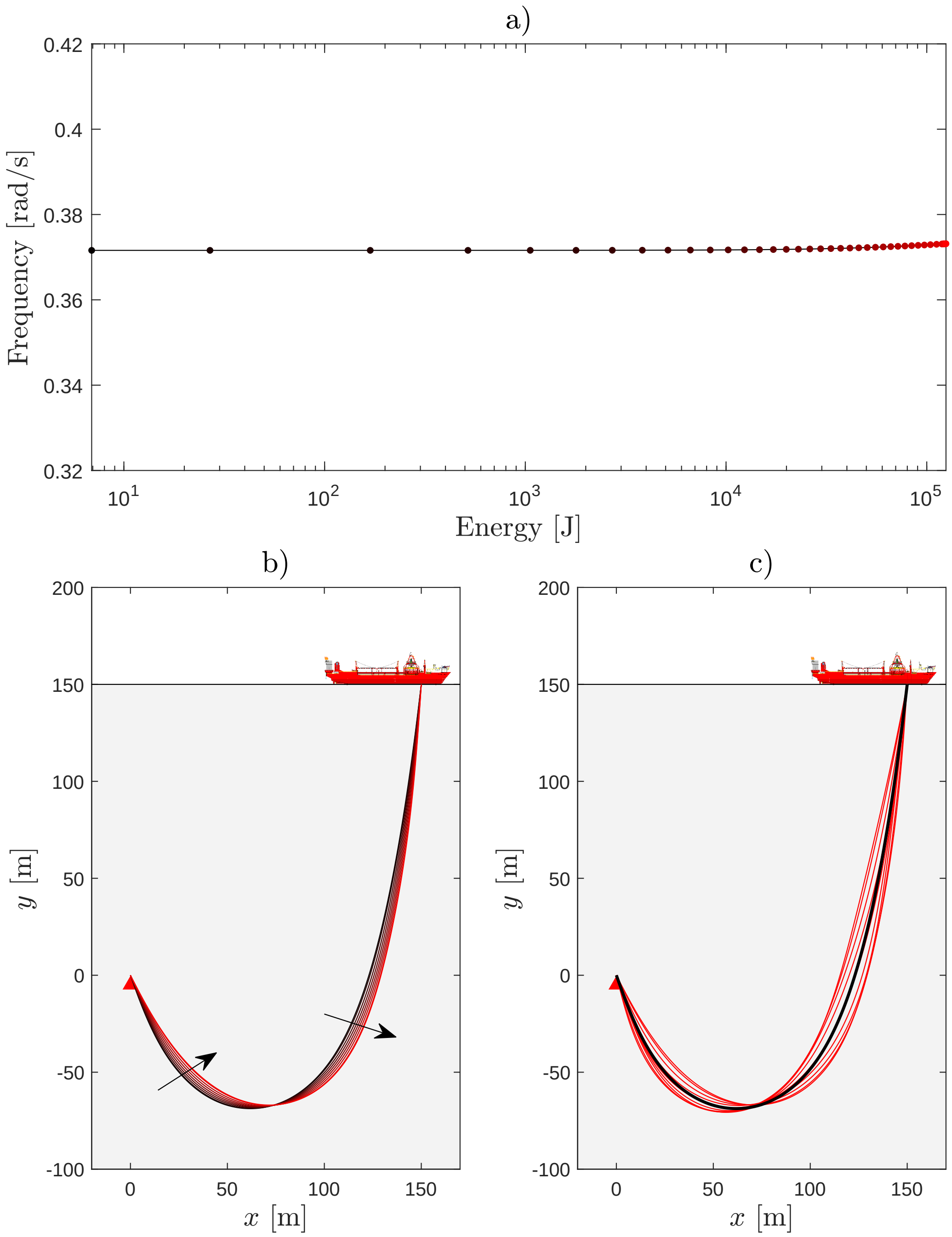

Figure 2 – First mode of the flexible riser in catenary configuration. a) Frequency-energy of the first NNM. b) Initial position of the first NNM for different energy levels. c) Different positions of the riser during one oscillation of the first NNM at the highest energy level.

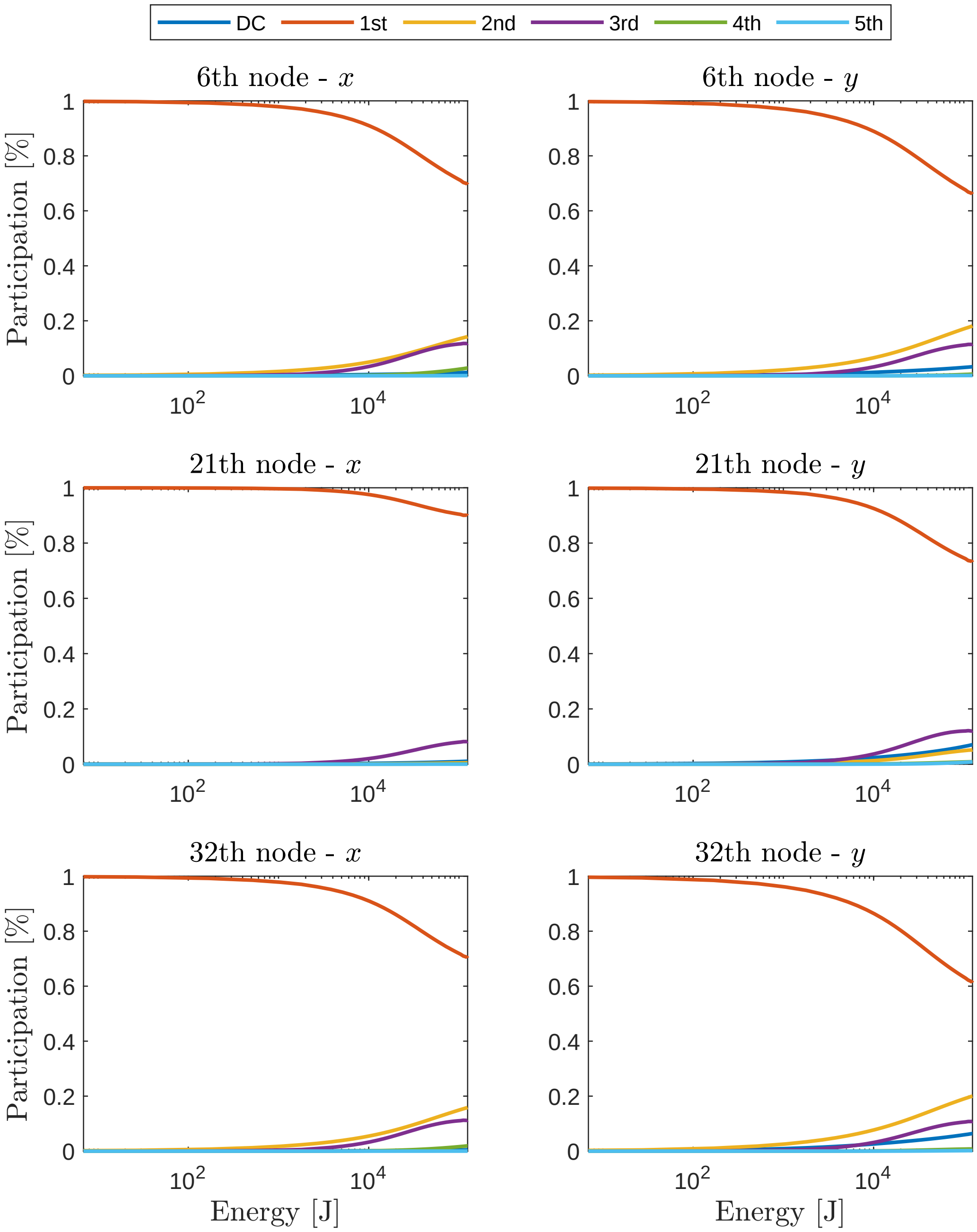

Figure 3 – Participation of each harmonic in the periodic motion of the first NNM in terms of the energy in the system. Analysis of the motion in the x and y direction of the 6th, 21th and 32th nodes.

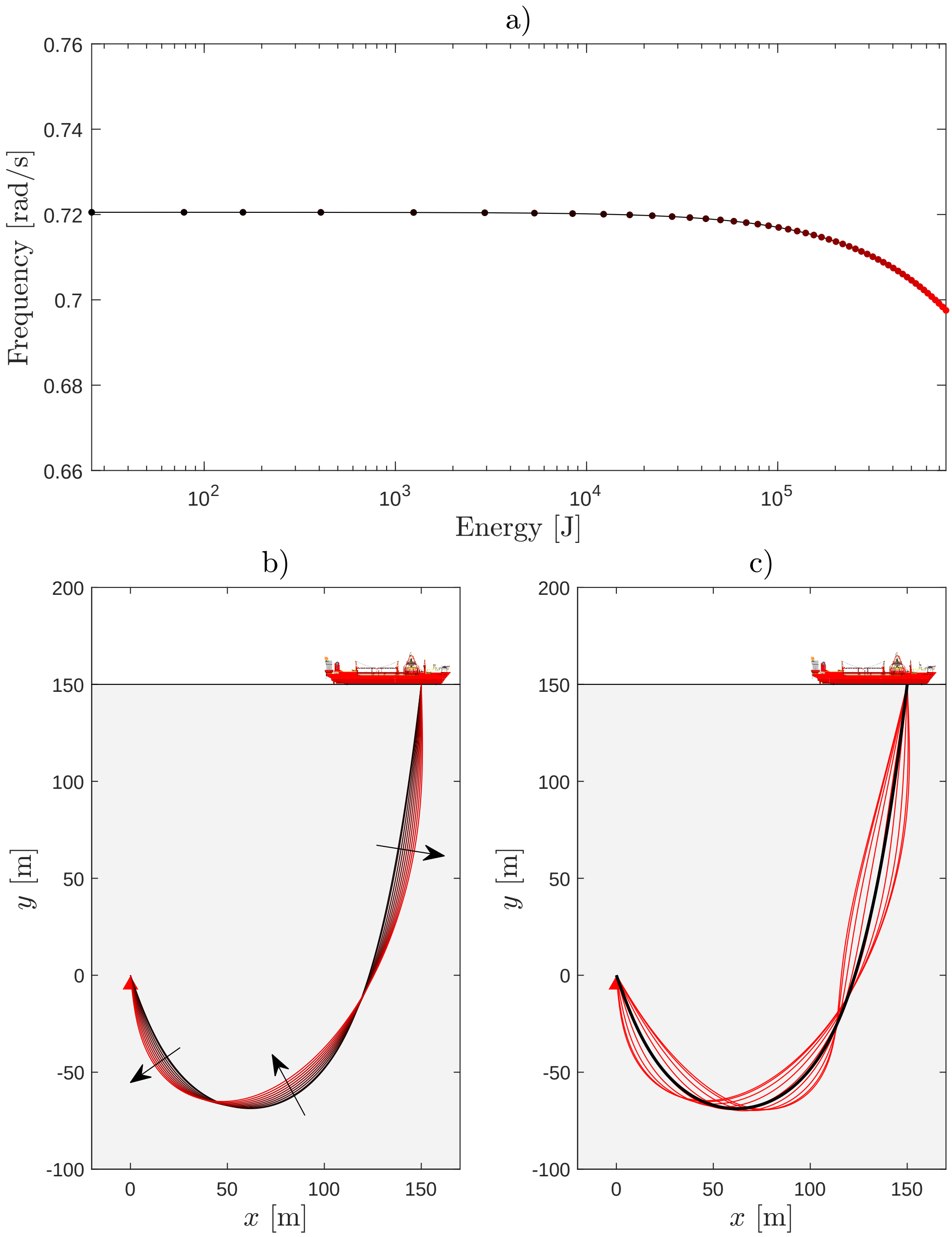

Figure 4 – Second mode of the flexible riser in catenary configuration. a) Frequency-energy of the second NNM. b) Initial position of the second NNM for different energy levels. c) Different positions of the riser during one oscillation of the second NNM at the highest energy level.

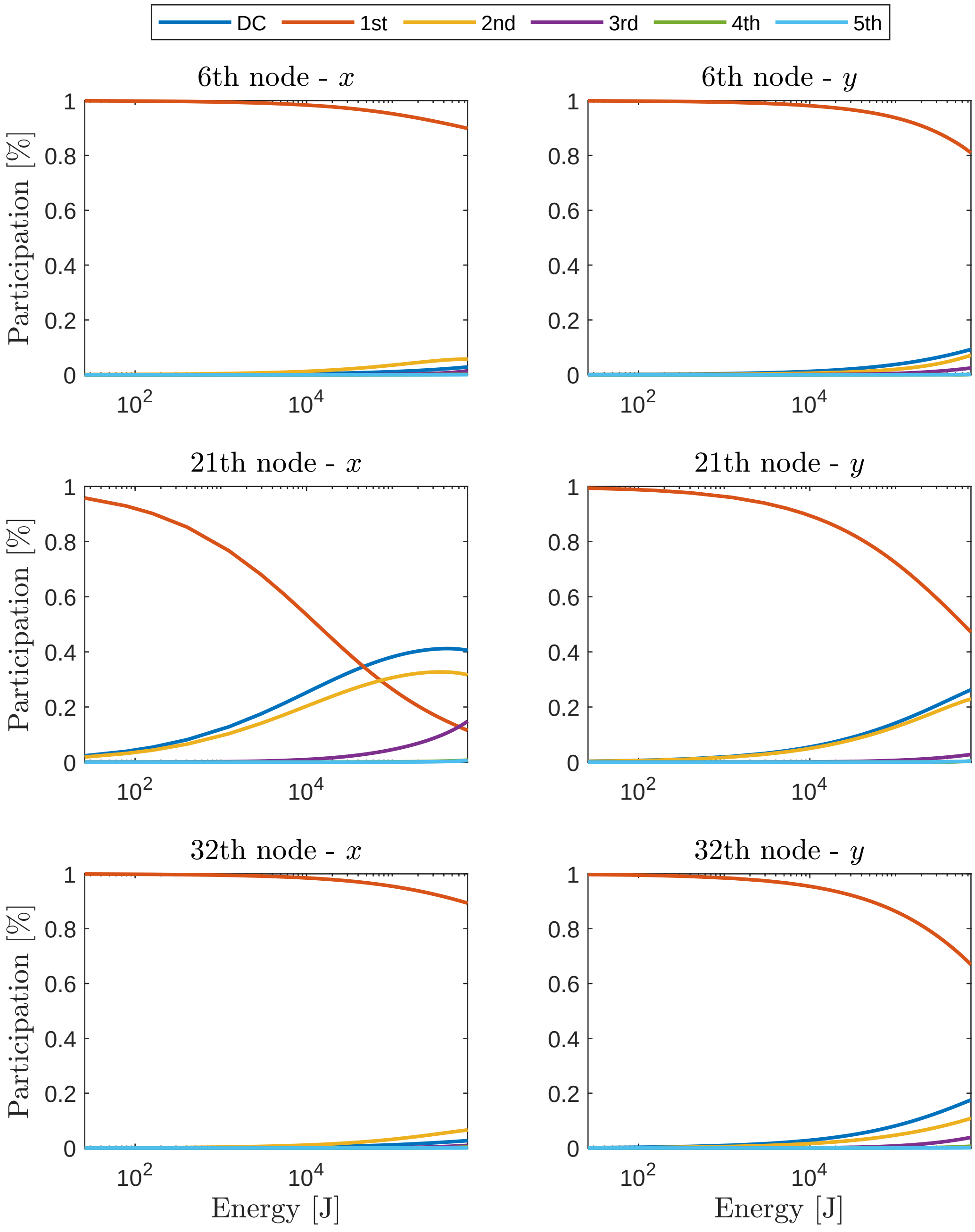

Figure 5 – Participation of each harmonic in the periodic motion of the second NNM in terms of the energy in the system. Analysis of the motion in the x and y direction of the 6th, 21th and 32th nodes.

the predictor-corrector scheme to perform the numerical continuation of the periodic solution at different energy levels. The overall system of nonlinear algebraic equation built during the corrector phase was solved numerically using the Newton-Raphson solver.

With the proposed method incorporating the co-rotational finite element modeling in the computation of NNMs, the overall method became able to handle more complex flexible beams. This important result was highlighted with a numerical example. The example illustrated the capability of the proposed method to deal with and industrial problem of flexible beams. It considered the case of a flexible riser in catenary configuration, a typical structure found in the oil and gas industry. In this example, the dimension of the problem was considerably large, increasing the computational time but without presenting convergence problems. The first two NNMs of the structure were computed. The participation of the harmonics in the NNM motion at particular nodes were analyzed, evidencing the contribution of the geometric nonlinearities in the modes. For the analyzed structure, the fundamental frequency of the first NNM was approximately constant in the analyzed frequency range, while the second NNM presented a softening behavior.

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