



# Dynamics of pendula base-excited system for wave energy harvesting application

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*Abstract: The vast wave energy potential has positioned Wave Energy Converters (WEC) as emerging technologies; therefore, this study presents a novel two-pendula device that will be able to harvest energy from pendula rotatory motions when harmonically excited by sea waves. The dynamics of a vertical harmonically excited pendula system, when coupled through an elastic base, is investigated to limit the system parameters which lead to pendula rotatory motions. System parameters, such as wave frequency and amplitude, and the difference in pendula lengths (included as parameter mismatch) are constrained to define ranges where pendula rotations exist. Bifurcation analysis of the dynamical system and one-parameter continuation studies are performed with the computational tool ABESPOL, developed at the Centre for Applied Dynamics Research (CADR) at the University of Aberdeen, to find stability ranges of parameters for pendula rotations. The dynamic studies shown the complexity of the system with coexistence between pendula antiphase, phase rotatory motions and pendula oscillations. To reach the desired attractors several basins of attraction are computed to find optimum system initial conditions.*

**Keywords:** *Pendula system, rotations, bifurcation analysis, energy harvesting.*

## INTRODUCTION

New wave energy harvesting technologies have been investigated and deployed in the past decade to harvest energy from the marine environment due to the vast energy potential stored in sea waves. At the Centre for Applied Dynamics Research at the University of Aberdeen, a novel Wave Energy Converter (WEC) concept is under investigation (Terrero *et al.*) which will utilize pendula motions to harvest energy from sea waves. Therefore, an understanding of the physics governing pendula dynamics is reviewed and investigated in this study.

The dynamics of a single parametric vertically excited pendulum was studied by Clifford and Bishop (1995) and the experimental studies related to their initial work achieving pendulum rotatory motions when harmonically excited in the vertical direction, were further investigated by Alevras (2015). Horton (2013) and Garira and Bishop (2003) studied the stability of pendula for period-1 rotations, while the birth identification for pendula rotatory responses was defined with the saddle node bifurcations by Lenci *et al.* (2008). Further studies with harmonically excited two-pendula systems identified the ranges where stable rotations exist, (Klimina *et al.*, 2017). Dynamics of the pendulum and the behaviours of a parametric pendulum under stochastic excitation were performed by Andreeva *et al.* (2016) with the possible applicability of energy harvesting from the random quasi-random sea wave excitation. Moreover, previous works at the University of Aberdeen (Najdecka *et al.*, 2015) proved the superiority of pendulum rotatory motions in comparison with oscillations in regards of energy harvesting, thereby this work will define the stable rotatory solutions for a two-pendula system.

This work is inspired by previous research presented by Najdecka *et al.* (2015), where the dynamics of a two-pendula system coupled with a common elastic base was experimentally investigated for stochastic excitation and periodic excitation and the work by Mazal *et al.* (2017), where the rotational orbits for both parametrically excited pendulums and the boundaries limiting pendula rotations were studied. The system parameters that are varied to target rotations are the excitation amplitude and frequency, and a new parameter mismatch which is the difference in pendulums lengths.

The next section of this work describes the physical system under investigations and the equations of motion governing the physical system, followed by a bifurcation analysis computed with the ABESPOL package (Chong, 2016). One-parameter continuation studies are performed for a specific range of wave amplitude and frequency of excitation and parameter mismatch. The continuation is initialized with a range of system parameters obtained with direct integration in Matlab, where rotatory motions for both pendulums are identified. Conclusions and research outcomes are summarized in the last section showing specific ranges for which stable rotatory motions for the pendula system exist and presenting the saddle nodes identified for the range of parameters studied in this work.

## PHYSICAL AND MATHEMATICAL MODELLING

The physical system is a two-pendula system coupled with a common elastic base and harmonically vertically excited from the base, approximating the pseudo-harmonic motions exerted from sea waves. Figure 1(b) shows the experimental rig manufactured at the University of Aberdeen used for the experimental data acquisition. The system considers that each pendulum is able to rotate independently, the pendula masses are equal and the coupled base allows synchronization between pendula. The system scheme is presented in Fig. 1(a) and the equations of motion are derived with Lagrange method;  $X$  and  $Y$  denote horizontal and vertical displacement of the base, while  $\theta_1$  and  $\theta_2$  denotes the angular displacement of pendulum 1 and pendulum 2, respectively. The following Eq. 1 represents the motion of the base in the horizontal direction and Eq. 2 in the vertical direction, Eq. 3 and Eq. 4 represents the pendulum 1 and pendulum 2 angular displacement respectively and they are expressed as follows,

$$(M + 2m)\ddot{X} + c_x\dot{X} + k_xX + ml(\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2\sin\theta_1) + m(l + \delta)(\ddot{\theta}_2\cos\theta_2 - \dot{\theta}_2^2\sin\theta_2) = 0 \quad (1)$$

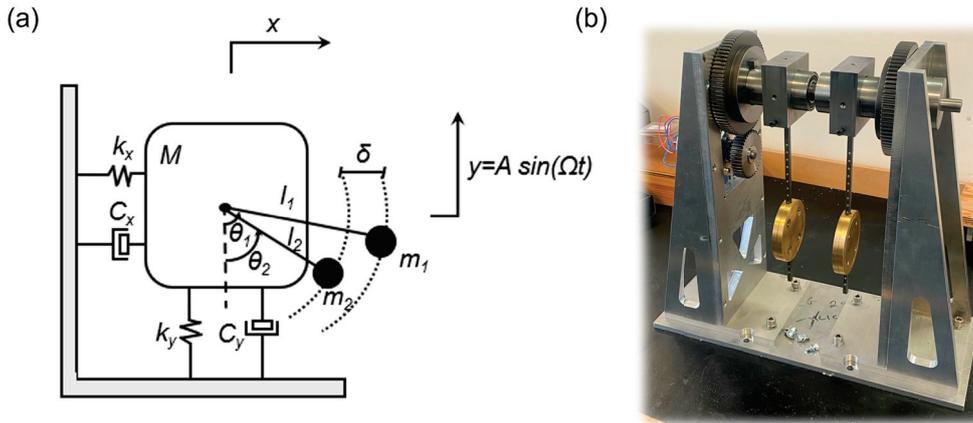
$$(M + 2m)\ddot{Y} + c_y(\dot{Y} - \dot{R}_y) + k_y(Y - R_y) + ml(\ddot{\theta}_1\sin\theta_1 + \dot{\theta}_1^2\cos\theta_1) + m(l + \delta)(\ddot{\theta}_2\sin\theta_2 - \dot{\theta}_2^2\cos\theta_2) = 0 \quad (2)$$

$$ml^2\ddot{\theta}_1 + ml(\ddot{X}\cos\theta_1 + \dot{Y}\sin\theta_1 + g\sin\theta_1) + c_\theta l\dot{\theta}_1 = 0 \quad (3)$$

$$m(l + \delta)^2\ddot{\theta}_2 + ml(l + \delta)(\ddot{X}\cos\theta_2 + \dot{Y}\sin\theta_2 + g\sin\theta_2) + c_\theta(l + \delta)\dot{\theta}_2 = 0 \quad (4)$$

where  $M$  and  $m$  denote the mass of the base and the pendulum's mass, respectively,  $l$  is the reference length of the pendulum and  $\delta$  the difference in pendulums lengths. The damping and stiffness coefficients in the horizontal and vertical direction for the base are denoted as  $c_x$ ,  $c_y$ ,  $k_x$  and  $k_y$  respectively and the damping coefficient for the pendulums is denoted as  $c_\theta$ .  $R_y$  represents the harmonic force exerted on the pendula system, where  $R_y = A\sin(\Omega t)$ ,  $A$  is the amplitude and  $\Omega$  frequency of excitation respectively and  $t$  representing time. The equations of motion are non-dimensionalised respect to a linear approximation of the pendula natural frequency; hence, the new non-dimensionalised variables and parameters that are considered for the study are the non-dimensionalised frequency of excitation ( $\omega$ ), amplitude of excitation ( $p$ ) and the parameter mismatch ( $\delta$ ), where the relation between the dimensionalised variables and non-dimensionalised variables are  $\omega = \frac{\Omega}{\omega_n}$ , where  $\omega_n$  is the natural frequency and  $p = \frac{A}{l}$ .

To study the dynamics of the pendula, direct integration with Runge Kutta is performed to bound a range of parameter values where pendula rotations exist. This range is then used to perform one-parameter continuation studies to identify the limit between oscillations and rotations. The following section presents the one-parameter continuation studies and the ranges of parameters utilized to start the direct integrations in ABESPOL to perform the continuation are  $\omega \in [2.00 - 2.20]$ ,  $p \in [0.05 - 0.50]$  and  $\delta \in [0.00 - 0.05]$ .



**Figure 1 – (a) Physical model of two pendula system harmonically excited from the base. The system comprises three masses, the base mass  $M$  and each pendulum mass denoted as  $m_1$  and  $m_2$ . The pendula lengths are denoted as  $l_1$  and  $l_2$  for pendulum 1 and 2, respectively. The parameter mismatch is represented by  $\delta$ - and the damping and stiffness coefficients in the horizontal and vertical directions are denoted as  $c_x$ ,  $k_x$ ,  $c_y$  and  $k_y$ , respectively. The angular displacement respect to the vertical is expressed as  $\theta_1$  and  $\theta_2$  for pendulum 1 and 2. (b) two pendula system experimental rig. Two identical pendulums supported by the same pivot and able to rotate independently are coupled with an aluminium elastic base, which allows their synchronization.**

## BIFURCATION ANALYSIS

Numerical continuation studies are performed for one parameter,  $p$ , to identify the ranges of system parameters and the limits where rotations exist for both pendula. The computation is performed using ABESPOL (Chong, 2016) which connects to COCO (Dankowicz & Schilder, 2013) through an interface. The tool performed direct numerical integrations using the ranges of values obtained in the previous section. Using this, the one-parameter continuation is computed. The results obtained are performed for  $\omega = 2.00$  and  $\delta = 0.00$ ; range of  $p \in [0.01274 - 1.40000]$ , results are shown in Fig. 2(a) for pendulum 1 and Fig. 2(b) for pendulum 2. Solutions show the complexity of the system due to the coexistence of stable solutions of pendulum in antiphase and phase rotatory motions and oscillations for both pendulums; green dotted lines represent stable solutions while red dotted lines represent the unstable solutions. Some exemplary phase planes are presented on the graph for pendula antiphase rotations and oscillations. From the continuation study, a saddle node bifurcation is identified at  $p = 0.012739$  and a period doubling at  $p = 0.42600$  for pendulum 1, and a saddle node bifurcation at  $p = 0.012739$  for pendulum 2 when  $\omega = 2.00$  and  $\delta = 0.00$ . From the results, only the first saddle node differentiates between stable and unstable orbits, therefore further work will focus on the limits between the rotations and oscillations. Moreover, the coexistence between oscillatory responses and antiphase motions is presented for values of  $p < 0.32000$  and the coexistence between phase rotations and oscillations occurs in the range  $p \in [0.790439 - 0.848629]$ , when  $\omega = 2.00$  and  $\delta = 0.00$  for both pendula. To target the anti-phase rotatory motions some basins of attraction are computed with ABESPOL and some examples are presented in Fig. 2(c) for pendulum 1 and in Fig. 2(d) for pendulum 2.

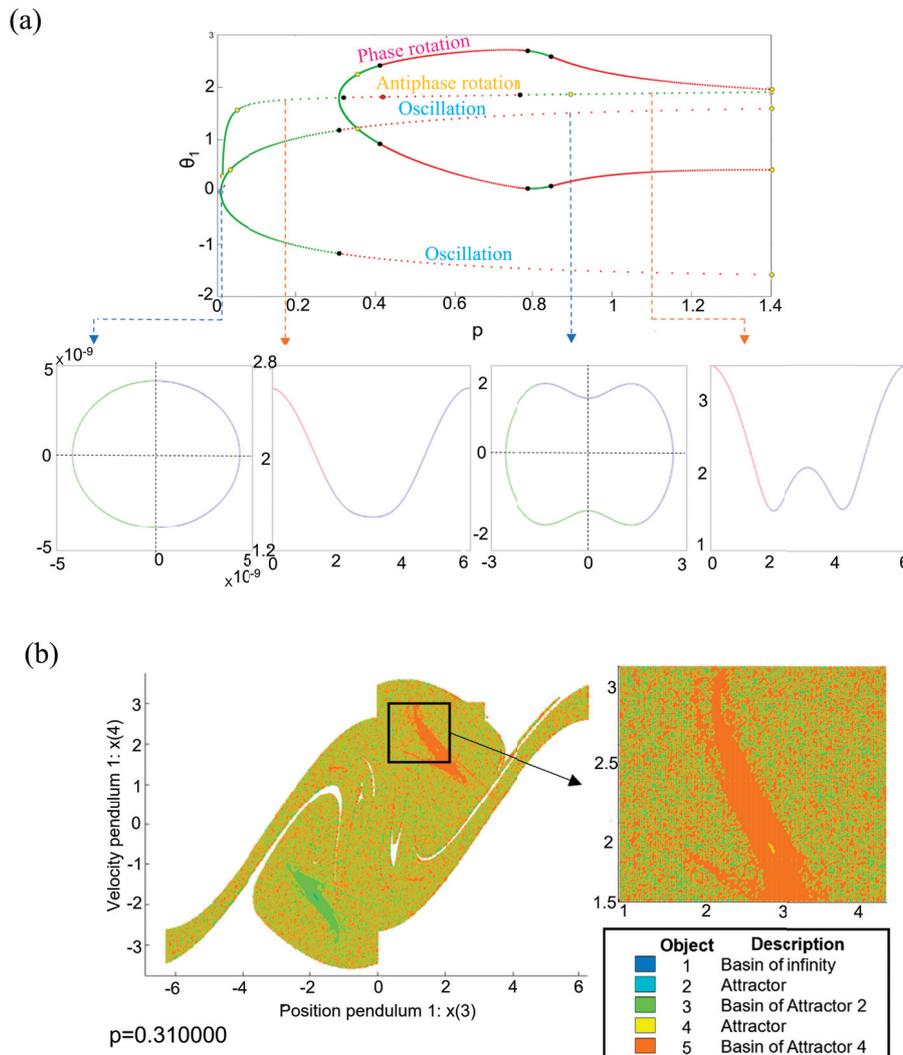


Figure 2 – (a) Bifurcation diagram with phase portraits with initial conditions  $x(0) = x'(0) = 0$ ,  $\theta_1(0) = 1.5708$ ,  $\theta_1'(0) = 2.5$ ,  $\theta_2(0) = 0.5708$  and  $\theta_2'(0) = -2.5$ . Case for  $\omega=2.00$  and  $\delta=0.00$  from  $p=[0 \ 1.40000]$ . Stable and unstable solutions are presented in green and red respectively, saddle node bifurcations are represented as red dots. (b) Pendulum 1 basin of attraction for  $p=0.31000$ .

## CONCLUDING REMARKS

This work investigates the nonlinear dynamics of a two-pendula system coupled with an elastic base when harmonically excited from the base in the vertical direction. It defined ranges of system parameters where stable orbits for pendula rotations exist. Moreover, direct numerical integration with Runge Kutta are computed to narrower the range of system parameters for one-parameter continuation studies giving ranges of  $\omega \in [2.00 - 2.20]$ ,  $p \in [0.05 - 0.5]$  and  $\delta \in [0.00 - 0.06]$ , where  $\omega$  and  $p$  are the non-dimensionalised variables of the frequency and amplitude of excitation respectively and  $\delta$  the parameter mismatch. One-parameter continuation studies are computed with ABESPOL for  $\omega = 2.00$ ,  $\delta = 0.00$  and  $p \in [0.01279 - 1.40000]$  from which a saddle node is identified at  $p = 0.01279$  for anti-phase rotatory motions for both pendula and a period doubling bifurcation is identified at  $p = 0.42600$  for pendulum 1. From the bifurcation analysis it was identified the coexistence between anti-phase and phase rotations for values of  $p \in [0.790439 - 0.8486629]$  and coexistence between oscillations and anti-phase rotations for values of  $p < 0.32000$ , is identified when  $\omega = 2.00$  and  $\delta = 0.00$ . Several basins of attraction are computed to target the stable anti-phase rotatory motions for the pendula system and its applicability to a novel pendula WEC able to convert the rotatory motions, induced by the pseudo-harmonic motions of the waves, into electricity.

An experimental rig has been manufactured for model adjustment, it comprises two pendula coupled with an aluminium elastic base, which allows pendula synchronization; a unique pendula supporting shaft that, due to its design, allows independent motion for each pendulum. The pendulum's bobs are made of two identical pieces allowing the bobs to move along the rod for setting different pendulums lengths and hence, test the model with the parameter mismatch  $\delta \neq 0$  for model adjustments.

## ACKNOWLEDGMENTS

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