DINAME 2023 - Proceedings of the XIX International Symposium on Dynamic Problems of Mechanics,
A.S. De Paula, A.T. Fabro and M.R. Machado (Editors), ABCM.
Pirenópolis, Go, Brazil, February 26th to March 3rd, 2023

Oscillatory response of the simplest electromechanical system

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Abstract: This paper discusses the dynamics of the simplest electromechanical system. The system is composed by two interacting subsystems, a mechanical and an electromagnetic. The system was chosen as simple as possible to highlight the mutual interaction between the two subsystems and to show that this interaction provokes an oscillatory response. Different from purely mechanical systems, here the oscillatory response does not occur due to an interplay of kinetic and potential energies. The system analyzed in this paper does not have elements that can storage potential energies, neither mechanical nor electrical. Natural frequency and normal modes are computed for the electromechanical system. The computed parameters involve mechanical and electromagnetic variables, i.e., they are hybrid, a novelty in the literature. Hybrid model coordinates, resonance and frequency responses graphs are discussed. An energetic analysis is developed. It is shown that the hybrid natural frequency determines the frequency at which occurs the interplay of energy between the mechanical and the electromagnetic subsystems.

Keywords: electromechanical systems, hybrid natural frequency, normal modes, resonance, energetic analysis

INTRODUCTION

Electromechanical systems are an interesting type of dynamical systems. They are composed by two interacting subsystems, a mechanical and an electromagnetic, see Lima and Sampaio (2016), Dantas, Sampaio and Lima (2018) and Rocard (1943). The state of an electromechanical system must involve mechanical and electromagnetic variables, as for example, positions, velocities, angles, currents, and charges, see Lima, Sampaio and Hagedorn 2018 and Lima et al. (2019). The fact that mechanical and electromagnetic variables must appear in the parametrization is reflected in the initial value problem (IVP) that gives the system dynamics. The initial value problem is composed by a set of differential equations and initial conditions with these two types of variables. In the set, the mutual interaction between the mechanical and an electromagnetic subsystems does not appear as a functional relation. The mutual interaction varies with the state of the subsystems and, consequently, depends on initial conditions, see Dantas, Sampaio and Lima (2014).

The dynamic behavior of an electromechanical system depends on this mutual interaction, i.e., the phenomena present in the system response reflect this interplay between the mechanical and electromagnetic subsystems. In this paper, we focus in a special phenomenon: oscillations. We analyze the oscillatory response of the simplest electromechanical system. The system is composed by a DC motor connected to a rigid disc, a motor-disc system. This system has the minimum number of elements necessary to be classified as an electromechanical system. It is a bare minimum to study oscillatory response of electromechanical systems and to make modal analysis. One of the reasons to address the problem in this bare minimum system is to highlight that the mutual interaction between the mechanical and electromagnetic subsystems provokes an oscillatory response. Besides, the system was chosen as simple as possible so that the analyses could be done analytically. Natural frequency and normal modes are computed. Differently from purely mechanical systems, given for example in Meirovitch (1997), here these parameters involve mechanical and electromagnetic variables, i.e., the computed natural frequency and normal modes are hybrid, a novelty in the literature.

The computed hybrid natural frequency is the frequency at which the electromechanical system responds when there is no external excitation acting over it, that is, when the system is free. In our case, this means no external torque acting over the disc and no external source voltage applied over the electric circuit of the DC motor. In this situation, the hybrid natural frequency also represents the frequency at which occurs the interplay of energies between the mechanical and the electromagnetic subsystems.

The dynamics of an electromechanical system can involve the interplay of four types of energies. The mechanical subsystem could have kinetic and potential mechanical energies, and the electromagnetic subsystem could have magnetic and electric energies. For the electromechanical system analyzed in this paper, occurs an interesting phenomena. The interplay of energies between the subsystems involves only two types of energies: a mechanical kinetic energy (due to the movement of the disc) and a magnetic kinetic energy (due to the electric current in the DC motor). The system does not storage potential energies. In the mechanical subsystem, there is no elastic element to storage mechanical potential energy and, in the electromagnetic subsystem, there is no capacitive element to storage electric energy. Thus, the free response of our motor-disc system is characterized by the interplay of kinetic and magnetic energies. This behavior is completely different from what is found in purely mechanical systems, another novelty of the paper. For purely mechanical systems, the free response is usually characterized by an interplay of kinetic and potential energies.

The dynamics of an electromechanical system is usually parametrized with mechanical and electromagnetic variables, as positions, velocities, angles, currents, and charges. Since these variables are native and intrinsic to the problem, they are the most natural variables to parametrize the system dynamics. With such kind of variables, the set of differential equations present in the IVP that characterizes the dynamics of an electromechanical system is a coupled set of equations.

The choice to parametrize the dynamics with native and intrinsic variables, easier to understand and visualize, generates a coupled set of equations in the IVP. However, if the dynamics were parametrized with a special set of variables, obtained from the hybrid normal modes and called modal coordinates, the set of equations would become uncoupled. The hybrid normal modes forms a basis of a vector space that can be used to represent the system dynamics.

Writing the system dynamics in terms of the modal coordinates turns possible to compute the system response for external excitations in a simple way. For the system analyzed in this paper, we focus on the computation of the response for harmonic external excitations and graphs of frequency response. Since the motor-disc system is linear and conservative, when it is excited harmonically with frequency equal to the natural frequency of the system, resonance appears, i.e., an electromechanical resonance, another novelty of the paper.

The literature dealing with dynamics of electromechanical systems is vast. There are several papers addressing the problem in different applications. Some examples are Alan and Bediz (2021), Bai et al. (2018), Larbi, Deü and Ohayon (2012), Thomas, Ducarne and Deü (2011), Tan et al. (2021). However, the majority of the literature usually does not focus on the behavior of the electromechanical coupling itself. They focus on electromechanical coupling combined with other phenomena, which hinders the understanding of the role of the electromechanical coupling in a system response.

In this paper, we address the problem purposefully in the simplest electromechanical system. There are no other phenomena present than electromechanical coupling. We believe that the analyses performed here can help in the understanding of the dynamics of electromechanical systems in different applications.

This paper is organized as follows. First is it presented the dynamics of the motor-disc, i.e., the initial value problem (IVP) that describes the dynamics of the analyzed electromechanical system. A discussion about the terms that appear in the IVP is done. After, the homogeneous solution of the IVP, that is, the system response when there is no external excitation acting over it is calculated. The hybrid natural frequency and normal modes are also computed and an energetic analysis is developed. It is show that the hybrid natural frequency determines the frequency at which occurs the interplay of energies between the mechanical and the electromagnetic subsystems. The decoupling of the equations of the IVP that gives the system dynamics using modal coordinates is made and finally, the system responses for harmonic external excitations and resonance are presented.

DYNAMICS OF THE ELECTROMECHANICAL SYSTEM

The electromechanical system analyzed in this paper is a DC motor connected to a disc as shown in Fig. 1.

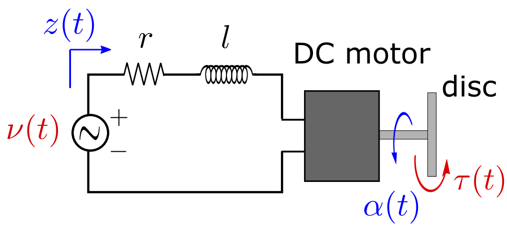


Figure 1 – Electromechanical system.

The initial value problem that characterizes the system dynamics is defined as follows. Find $(\alpha,\, z)$ such that, for all $t > 0$,

$$\begin{aligned} l\ddot{z}(t) + r\,\dot{z}(t) + k_e\dot{\alpha}(t) &= \nu(t)\,, \\ j_m\ddot{\alpha}(t) + b_m\dot{\alpha}(t) - k_e\dot{z}(t) &= \tau(t)\,, \end{aligned} \tag{1}$$

with the initial conditions $\dot{\alpha}(0) = \theta_0$, $\alpha(0) = \alpha_0$, $\dot{z}(0) = c_0$ and $z(0) = z_0$. In these equations, t is the time, ν is the source voltage, z is the electric charge, $\dot{\alpha}$ is the angular speed of the disc, l is the electric inductance, j_m is the disc moment of inertia, b_m is the damping ratio in the transmission of the torque generated by the motor, k_e is the motor electromagnetic force constant, r is the electrical resistance, and τ is an external torque made over the disc.

The system state is given by four variables, two of them mechanical (angular velocity and position of the disc) and two of them electromagnetic (charge and current in the motor). These four variables are native and intrinsic to the problem, natural variables to parametrize the system state. The system dynamics, parametrized with these four variables, is given by an initial value problem comprising a set of two coupled differential equations. The coupling between the mechanical and electromagnetic subsystems is not given by a functional relation. It depends on the system state and, consequently, depends on initial conditions. Writing Eq. (1) in matrix form, and assuming $b_m = 0$ and $r = 0$ to get a conservative system, we obtain:

$$\begin{bmatrix} l & 0 \\ 0 & j_m \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\alpha}(t) \end{bmatrix} + \begin{bmatrix} 0 & k_e \\ -k_e & 0 \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ \dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} \nu(t) \\ \tau(t) \end{bmatrix}, \tag{2}$$

$$M\ddot{\mathbf{Y}}(t) + G\dot{\mathbf{Y}}(t) = \mathbf{F}(t), \tag{3}$$

where M and G will be called inertia and gyroscopic matrices respectively and $\mathbf{Y} = \begin{bmatrix} z \\ \alpha \end{bmatrix}$. The initial conditions become $\dot{\mathbf{Y}}(0) = \begin{bmatrix} c_0 \\ \theta_0 \end{bmatrix}$ and $\mathbf{Y}(0) = \begin{bmatrix} z_0 \\ \alpha_0 \end{bmatrix}$. Matrix G is skew symmetric, i.e., $G^T = -G$, where $\square^T$ indicates the transpose. It is interesting to notice that despite calling M and G inertia and gyroscopic matrices, an usual terminology used in mechanical systems, see Udwadia (2019) and Lancaster (2013), here these two matrices have a different physical interpretation from the traditional one. M is not an inertia matrix like those that appear in purely mechanical systems. M is composed by elements that represent inertia of two different natures, a mechanical and an electromagnetic. G is also not a traditional gyroscopic matrix. In purely mechanical systems, a gyroscopic matrix usually couples motions in different directions. Here G couples the mechanical and electromagnetic subsystems. It is responsible for the interplay of energies between these two subsystems.

Besides having inertia and gyroscopic matrices with different physical interpretation, Eq. (3) has another big difference from what is found in dynamics of purely mechanical systems. It does not have a matrix composed by elements that can storage potential energies to be called stiffness matrix. Our system system has neither elements that can store mechanical potential energy, as springs, nor elements that can storage electric energy, as capacitors. Its dynamics equation is composed only by inertia and gyroscopic matrices.

The absence of a stiffness matrix brings some novelties. The first one is that even without a stiffness matrix, the system exhibits an oscillatory response, as it will shown in the next Section. This is a not usual behavior found in purely mechanical systems for which oscillatory responses usually involve inertia and stiffness matrices, in other words, interplay between kinetic and potential energies. The second novelty brought by the absence of a stiffness matrix is that it makes our system different from those analyzed in references that makes modal analysis including gyroscopic matrices. To the best of our knowledge, modal analysis of systems with just inertia and gyroscopic matrices is a novelty in the literature. No references dealing with it were found after an extensive literature review. Just references that makes modal analysis of systems with inertia, gyroscopic and stiffness matrices were found, as for example, Meirovitch (1975) and Zheng, Ren and Williams (1997). In these references, the presence of a stiffness matrix is mandatory.

Before starting the computation of the hybrid natural frequency and normal modes of our electromechanical system, let us rewrite Eq. (3) as a first order differential equation. Making $\dot{\alpha} = \theta$ and $\dot{z} = c$, where θ represents the angular velocity of the disc and c represents the current in the electric circuit of the DC motor, it is possible to rewrite Eq. (3) as

$$M\dot{\mathbf{X}}(t) + G\mathbf{X}(t) = \mathbf{F}(t), \tag{4}$$

where $\mathbf{X} = \dot{\mathbf{Y}} = \begin{bmatrix} c \\ \theta \end{bmatrix}$. The initial condition turns into $\mathbf{X}(0) = \begin{bmatrix} c_0 \\ \theta_0 \end{bmatrix}$. Once the solution of the IVP involving Eq. (4) is obtained, it can be integrated to become the solution of the IVP involving Eq. (3). The constants of integration that will appear in this integration should be computed so that the initial condition $\mathbf{Y}(0)$ is satisfied.

Since l and j_m are considered to be non-zero, M is an invertible matrix. Thus, Eq. (4) can be rewritten as

$$\dot{\mathbf{X}}(t) = -M^{-1}\,G\mathbf{X}(t) + M^{-1}\,\mathbf{F}(t) = A\mathbf{X}(t) + \mathbf{B}(t), \tag{5}$$

where $A = -M^{-1}G$ and $\mathbf{B}(t) = M^{-1}\mathbf{F}(t) = \begin{bmatrix} \nu(t)/l \\ \tau(t)/j_m \end{bmatrix}$. The solution of Eq. (5) can be written as $\mathbf{X}(t) = \mathbf{X}_h(t) + \mathbf{X}_p(t)$, where X_h is the general solution of the associated homogeneous equation ($\dot{\mathbf{X}}_h = A\,\mathbf{X}_h$) and X_p is a particular solution of the non-homogeneous equation.

HOMOGENEOUS SOLUTION

We propose as solution to the associated homogeneous equation $\mathbf{X}_h = \mathbf{U}\,e^{\lambda\,t}$, where $\mathbf{U}$ is a non-zero constant vector and λ a scalar. Substituting the proposed general solution into the the associated homogeneous equation, we get $(A - \lambda\,I)\mathbf{U} = \mathbf{0}$, which forms an eigenvalue problem.

Natural frequency and normal modes of the electromechanical system

Since $\mathbf{U} \neq \mathbf{0}$, the matrix $(A - \lambda\,I)$ is singular. Thus:

$$\det\left(A-\lambda I\right)=0 \quad\Rightarrow\quad \lambda^2+\frac{k_e^2}{l\,j_m}=0 \quad\Rightarrow\quad \lambda_{1,2}=\pm\frac{k_e}{\sqrt{l\,j_m}}\,i, \tag{6}$$

where $i=\sqrt{-1}$. Substituting the two eigenvalues $\lambda_{1,2}$ into the eigenvalue problem, it is possible to write $(A-\lambda_1 I)\mathbf{U}_1=\mathbf{0}$ and $(A-\lambda_1 I)\mathbf{U}_2=\mathbf{0}$. For $\lambda_1=\frac{k_e}{\sqrt{l\,j_m}}i$, the associated eigenvector is $\mathbf{U}_1=\begin{bmatrix} i\,j_m/\sqrt{l\,j_m} \\ 1\end{bmatrix}$. For $\lambda_2=-\frac{k_e}{\sqrt{l\,j_m}}i$, the associated eigenvector is $\mathbf{U}_2=\begin{bmatrix} -i\,j_m/\sqrt{l\,j_m} \\ 1\end{bmatrix}$. The eigenvalues $\lambda_{1,2}$ give a natural frequency of the system $\omega_n=\frac{k_e}{\sqrt{l\,j_m}}$. The eigenvectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are normal modes. Observe that the natural frequency, ω_n, and the normal modes are hybrid. They involve mechanical and electromagnetic parameters. Since two pairs of eigenvalues and eigenvectors were found, the general solution of the associated homogeneous equation will be a linear combination of the two found solutions $e^{\lambda_1 t}\mathbf{U}_1$ and $e^{\lambda_2 t}\mathbf{U}_2$. It can be written as:

$$\mathbf{X}_h(t)=a\,e^{\lambda_1 t}\mathbf{U}_1+b\,e^{\lambda_2 t}\mathbf{U}_2=\begin{bmatrix}\cos\left(\frac{k_e}{\sqrt{l\,j_m}}t\right)\frac{j_m}{\sqrt{l\,j_m}}h-\sin\left(\frac{k_e}{\sqrt{l\,j_m}}t\right)\frac{j_m}{\sqrt{l\,j_m}}d \\ \cos\left(\frac{k_e}{\sqrt{l\,j_m}}t\right)d+\sin\left(\frac{k_e}{\sqrt{l\,j_m}}t\right)h\end{bmatrix}, \tag{7}$$

where a and b are constants, $d=a+b$ and $h=i(a-b)$. In the case of $\mathbf{B}=\mathbf{0}$, Eq. (5) becomes homogeneous and the constants a and b are computed so that Eq. (7) satisfies the initial condition $\mathbf{X}(0)$. Thus,

$$h=\frac{\sqrt{l\,j_m}}{j_m}c_0, \qquad d=\theta_0\,. \tag{8}$$

Figures 2(a) and 2(b) show the solution of Eq. (5) for the homogeneous case, i.e. $\mathbf{B}=\mathbf{0}$. The parameter values used in the construction of the graphs are listed in Table 1. The motor parameters were obtained from the specifications of the motor Maxon DC brushless number 411678. The initial conditions are $c_0=1.000$ Amp and $\theta_0=1.000$ rad/s.

Table 1 – Parameter values

$l=1.880\times10^{-4}$ H	$k_e=5.330\times10^{-2}$ V/(rad/s)
$j_m=1.210\times10^{-4}$ kg m^2	

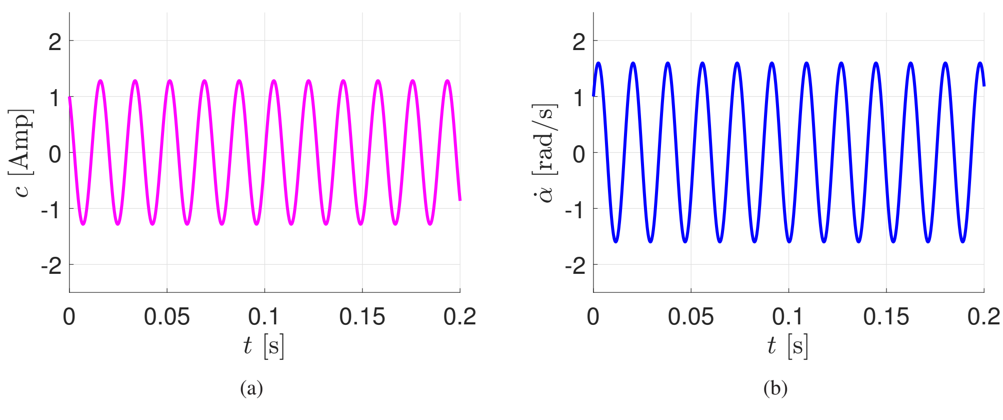


Figure 2 – (a) Current and (b) angular velocity the disc for the homogeneous case, i.e. $\mathbf{B}=\mathbf{0}$.

ENERGETIC ANALYSIS

There are many references dealing with dynamics of systems with gyroscopic matrices, as Giannini (2017), Udwadia (2017), Udwadia (2020a) and Udwadia (2020b).The study of such kind of systems is not new and has been attracting the attention of researchers for a long time, Lancaster (1966).One of the topics of interest regarding systems with gyroscopic matrices is stability analysis. The paper Lancaster (2013) addresses the problem with a mathematical approach and presents algebraic results for stability of time invariant systems with gyroscopic matrices.

An useful and simple criteria to determine whether a system with gyroscopic and conservative forces is stable was defined in Huseyin, Hagedorn and Teschner (1983). Following this criteria, the homogeneous electromechanical system analyzed here is stable. It is possible affirm also that it is conservative, as shown in the following.

To make an energetic analysis of our electromechanical system, we start multiplying Eq. (3) on the left by $\dot{\mathbf{Y}}^T$. Considering a homogeneous system with $\mathbf{F} = \mathbf{0}$, we get

$$\begin{aligned} \dot{\mathbf{Y}}^T(t)M\ddot{\mathbf{Y}}(t) + \dot{\mathbf{Y}}^T(t)G\dot{\mathbf{Y}}(t) &= 0 \\ l\ddot{z}\dot{z} + j_m\ddot{\alpha}\dot{\alpha} &= 0 \\ \frac{d}{dt}\left[\frac{1}{2}l\dot{z}^2(t) + \frac{1}{2}j_m\dot{\alpha}^2(t)\right] &= 0\,. \end{aligned} \tag{9}$$

The term $\frac{1}{2}l\dot{z}^2$ represents the magnetic energy of the system and the term $\frac{1}{2}j_m\dot{\alpha}^2$ the kinetic energy. The system do not have potential energies, neither mechanical nor electrical. Observing Eq. (10) it is possible verify that the sum of the magnetic and kinetic energies is constant, that is, the analyzed homogeneous electromechanical system is conservative. The free response of our motor-disc system is characterized by the interplay of kinetic and magnetic energies. This energy interplay provokes an oscillatory response. Observe that what usually provokes oscillatory response in purely mechanical systems is the interplay of kinetic and potential energies.

The total energy present in our homogeneous electromechanical system is defined by the initial conditions of current in the electric circuit of the motor, $c(0) = c_0$, and speed of the disc, $\dot{\alpha}(0) = \theta_0$. Thus:

$$\frac{1}{2}l\dot{z}^2(t) + \frac{1}{2}j_m\dot{\alpha}^2(t) \;=\; \frac{1}{2}lc_0^2 + \frac{1}{2}j_m\theta_0^2\,. \tag{10}$$

The free response of the system is characterized by an interplay of kinetic and magnetic energies. The phase portrait of $\dot{\alpha}$ and c is a center around the point $(0,0)$, Jordan and Smith (2007). It is show in Fig. 3 for different values of initial conditions. The parameter values used to construct the graph are given in Table 1.

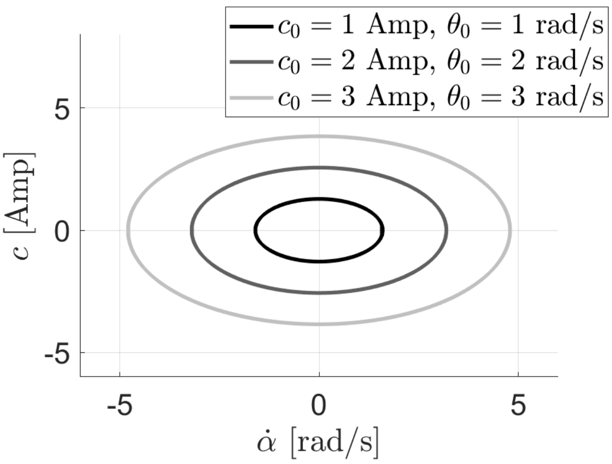


Figure 3 – Phase portraits of $\dot{\alpha}$ and c for different values of initial conditions.

DECOUPLING THE IVP THAT GIVES THE SYSTEM DYNAMICS USING THE NORMAL MODES

With the eigenvalues and eigenvectors, spectral and modal matrices can be writing as:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{k_e}{\sqrt{l\,j_m}}\,i & 0 \\ 0 & -\frac{k_e}{\sqrt{l\,j_m}}\,i \end{bmatrix}, \tag{11}$$

$$P = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} i j_m/\sqrt{l\,j_m} & -i j_m/\sqrt{l\,j_m} \\ 1 & 1 \end{bmatrix}. \tag{12}$$

It is possible to compute: $P^{-1} = \frac{1}{2\,i\,j_m/\sqrt{l\,j_m}}\begin{bmatrix} 1 & i j_m/\sqrt{l\,j_m} \\ -1 & i j_m/\sqrt{l\,j_m} \end{bmatrix}$. The matrix A can be written as $A = P\Lambda P^{-1}$. Since one of the objectives of the paper is the computation of the response for harmonic external excitations, we are interested on external excitations of the source voltage v of the form $v_0 \cos(\omega t)$ or $v_0 \sin(\omega t)$, and external torque τ of the form $\tau_0 \cos(\omega t)$ or $\tau_0 \sin(\omega t)$. Here, v_0 and τ_0 represents the amplitudes of the external excitations and ω their frequencies. Calling by $\mathbf{X}_1$ and $\mathbf{X}_2$ the system states when the system is forced with the functions sine and cosine, respectively, it possible to write:

$$\dot{\mathbf{X}}_1(t) = A\mathbf{X}_1(t) + \mathbf{B}_1(t) = A\mathbf{X}_1(t) + \begin{bmatrix} v_0/l \\ \tau_0/j_m \end{bmatrix} \cos(\omega t), \tag{13}$$

$$\dot{\mathbf{X}}_2(t) = A\mathbf{X}_2(t) + \mathbf{B}_2(t) = A\mathbf{X}_2(t) + \begin{bmatrix} v_0/l \\ \tau_0/j_m \end{bmatrix} \sin(\omega t). \tag{14}$$

Multiplying Eq. (14) by the complex i and adding it to Eq. (13) we get:

$$\dot{\mathbf{S}}(t) = A\mathbf{S}(t) + \mathbf{E}(t) = A\mathbf{S}(t) + \begin{bmatrix} v_0/l \\ \tau_0/j_m \end{bmatrix} e^{i\omega t}, \tag{15}$$

where $\mathbf{S} = \mathbf{X}_1 + i\mathbf{X}_2$. Creating a new variable $\mathbf{S} = P\mathbf{Q}$, called modal variable, Eq. (15) can be rewritten as

$$\begin{array}{rcccl} P\dot{\mathbf{Q}}(t) & = & AP\mathbf{Q}(t) & + & \mathbf{E}(t), \\ P^{-1}P\dot{\mathbf{Q}}(t) & = & P^{-1}AP\mathbf{Q}(t) & + & P^{-1}\mathbf{E}(t), \\ \dot{\mathbf{Q}}(t) & = & \Lambda\mathbf{Q}(t) & + & P^{-1}\mathbf{E}(t), \end{array} \tag{16}$$

Thus, parametrizing the state of the electromechanical system with $\mathbf{Q}$, a modal variable with components q_1 and q_2, the equations that give the system dynamics become uncoupled and can be solved independently. They are:

$$\begin{cases} \dot{q}_1(t) - \dfrac{ik_e}{\sqrt{l\,j_m}} q_1(t) = \dfrac{1}{\dfrac{2\,i\,j_m}{\sqrt{l\,j_m}}} \left(\dfrac{v_0}{l} + \dfrac{i\,j_m}{\sqrt{l\,j_m}} \dfrac{\tau_0}{j_m} \right) e^{i\omega t}, \\ \dot{q}_2(t) + \dfrac{ik_e}{\sqrt{l\,j_m}} q_2(t) = \dfrac{1}{\dfrac{2\,i\,j_m}{\sqrt{l\,j_m}}} \left(-\dfrac{v_0}{l} + \dfrac{i\,j_m}{\sqrt{l\,j_m}} \dfrac{\tau_0}{j_m} \right) e^{i\omega t}. \end{cases} \tag{17}$$

PARTICULAR SOLUTION IN TERMS OF THE MODAL COORDINATES

In this section of the paper, the particular solutions of each equation of Eq. (17) is computed. These solutions are computed for two different cases. The first the case is when the system is harmonically excited at a frequency different from the natural frequency, ω_n. The second case is when it is excited at a frequency equal to ω_n.

External excitation at frequency different from the natural frequency of the system

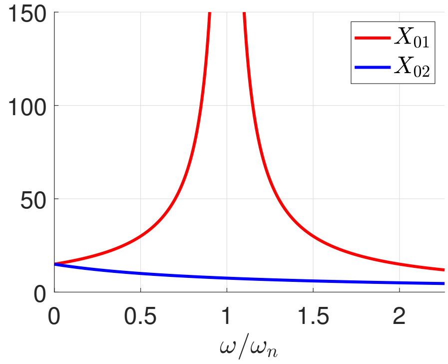
Considering that $\omega \neq \frac{k_e}{\sqrt{l\,j_m}}$, we propose as particular solution to the first differential of Eq. (17) the expression $q_{1p}(t) = X_{01}\, e^{i(\omega t + \beta_1)}$, where X_{01} and β_1 are constants to be determined. They represent, respectively, the amplitude of the proposed particular solution and the angular phase between the excitation and the particular solution. Substituting the proposed particular solution into the first equation of Eq. (17) and, analyzing the modulus and phase of the complex terms of the obtained expression, it is possible to compute X_{01} and β_1 as:

$$X_{01} = \frac{\left| \dfrac{1}{2} \sqrt{\left(\dfrac{v_0^2}{l\,j_m} + \dfrac{\tau_0^2}{j_m^2} \right)} \right|}{\left| \omega - \dfrac{k_e}{\sqrt{l\,j_m}} \right|}, \qquad \beta_1 = \begin{cases} \arctan\left(\dfrac{-v_0}{\sqrt{l\,j_m}} \dfrac{j_m}{\tau_0} \right) - \dfrac{3\pi}{2}, \text{ if } \omega < \dfrac{k_e}{\sqrt{l\,j_m}}, \\ \arctan\left(\dfrac{-v_0}{\sqrt{l\,j_m}} \dfrac{j_m}{\tau_0} \right) - \dfrac{\pi}{2}, \text{ if } \omega > \dfrac{k_e}{\sqrt{l\,j_m}}. \end{cases} \tag{18}$$

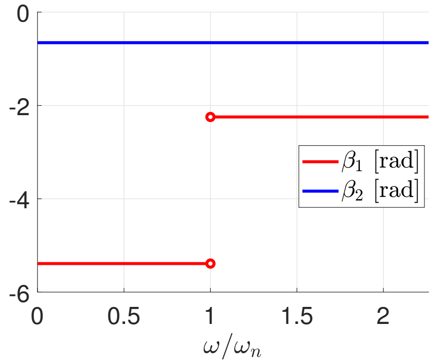
To second equation of Eq. (17), we propose as particular solution the expression $q_{2p}(t) = X_{02}\, e^{i(\omega t + \beta_2)}$, where X_{02} and β_2 are constants to be determined. They represent, respectively, the amplitude of the proposed particular solution and the angular phase between the excitation and the particular solution. Substituting the proposed particular solution into the second equation of Eq. (17), and analyzing the modulus and phase of the complex terms of the obtained expression, X_{02} and β_2 are computed:

$$X_{02} = \frac{\left| \dfrac{1}{2} \sqrt{\left(\dfrac{v_0^2}{l\,j_m} + \dfrac{\tau_0^2}{j_m^2} \right)} \right|}{\left| \omega + \dfrac{k_e}{\sqrt{l\,j_m}} \right|}, \qquad \beta_2 = \arctan\left(\frac{v_0}{\sqrt{l\,j_m}} \frac{j_m}{\tau_0} \right) - \frac{\pi}{2}. \tag{19}$$

Figures 4(a) and 4(b) show the frequency responses and phase graphs for the particular solutions of equations given in Eq. (17). To plot the graphs, it was considered the following values: $v_0 = 1.000$ V and $\tau_0 = 1.000$ Nm. The values used to l, j_m and k_e are given in Table 1.



(a)



(b)

Figure 4 – (a) Frequency response graphs and (b) phase graphs for the particular solutions of Eq. (17)**.**

External excitation at frequency equal to the natural frequency of the system

Considering that $\omega = \omega_n = \frac{k_e}{\sqrt{l\,j_m}}$, resonance occurs for the first equation of Eq. (17). Thus, the particular solution of this equation is not periodic and its amplitude grows linearly over time. This particular solution can be written as $q_{1pr}(t) = X_{01r}\,t\,e^{i(\omega t+\beta_{1r})}$, where X_{01r} and β_{1r} are constants to be determined. Substituting it into the first equation of Eq. (17) and analyzing the modulus and phase of the complex terms of the obtained expression, it is possible to compute X_{01r} and β_{1r}:

$$X_{01r} = \frac{1}{2}\sqrt{\left(\frac{v_0^2}{l\,j_m} + \frac{\tau_0^2}{j_m^2}\right)}, \qquad \beta_{1r} = \arctan\left(\frac{-v_0}{\sqrt{l\,j_m}}\frac{j_m}{\tau_0}\right). \tag{20}$$

For the second equation of Eq. (17), we propose as particular solution $q_{2pr}(t) = X_{02r}\,e^{i(\omega t+\beta_{2r})}$, where X_{02r} and β_{2r} are constants to be determined. Substituting it into the second equation of Eq. (17) and analyzing the modulus and phase of the complex terms obtained, it is possible to write:

$$X_{02r} = \frac{\left|\frac{1}{2}\sqrt{\left(\frac{v_0^2}{l\,j_m} + \frac{\tau_0^2}{j_m^2}\right)}\right|}{\left|\omega + \frac{k_e}{\sqrt{l\,j_m}}\right|}, \qquad \beta_{2r} = \arctan\left(\frac{v_0}{\sqrt{l\,j_m}}\frac{j_m}{\tau_0}\right) - \frac{\pi}{2}. \tag{21}$$

CONCLUSIONS

In this paper, the oscillatory response of the simplest electromechanical system was analyzed. The free response of the system is characterized by the interplay of kinetic and magnetic energies. This behavior is completely different from what is found in purely mechanical systems. For purely mechanical systems, the free response is usually characterized by an interplay of kinetic and potential energies.

Natural frequency and normal modes, were computed for the electromechanical system. Since the computed parameters involve mechanical and electromagnetic variables, they are hybrid. The hybrid natural frequency is the frequency at which occurs the interplay of energies between the mechanical and the electromagnetic subsystems. The hybrid normal modes forms a basis of a vector space that can be used to represent and decouple the system dynamics.

ACKNOWLEDGMENTS

The authors acknowledge the support given by FAPERJ, CNPq and CAPES.

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