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Lie Symmetries for the Spinning Top

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Abstract: In this work it is showed the application of Lie symmetries for reducing the order of a system that models a classical problem found in mechanical engineering: a spinning top. The motion equations are given by a three-dimensional system of second-order ordinary differential equations. The system admits three Lie point symmetries which are applied to reduce the order.

Keywords: Lie symmetries, spinning top, reduction of order, analytical solution.

INTRODUCTION

The dynamic analysis of a mechanical system consists, initially, in building a mathematical model from the data collection of a real problem (Chapra, 2010). It is not uncommon for the final results to be far from what would be expected. One way to get around this inconvenience is to solve the differential equation that describes the motion of such a mechanical system in an analytical method (Olver, 1986).

However, it is known that some classes of differential equations, especially the nonlinear ones, are difficult to solve. The Lie symmetries are a powerful tool that can help to reduce and obtain analytical solutions to such equations and systems of equations (Olver, 1986). In recent years, many works have been developed in this area (Basquerotto; Righetto; da Silva, 2017; Basquerotto and Ruiz, 2020; Ruiz; Muriel; Ramírez, 2019).

In order to illustrate the application of Lie symmetries, a spinning top is considered in this work. A problem similar to this has already been considered in previous work but considering stationary precession (Basquerotto; Righetto; da Silva, 2017; Basquerotto and Ruiz, 2020). The problem refers to a spinning top with no constant nutation and precession. From the motion equations, Lie symmetries are computed and applied to reduce the order of the governinig equations.

A SPINNING TOP

Figure (1) shows an axisymmetric spinning top with mass m. In order to parameterize the motion of the top in space, we adopt in the 3-1-3 sequence of three consecutive rotations around three different axes by using the Euler angles. Firstly, it is taken as an inertial reference, $\mathscr{I}$, the axis (X, Y, Z) and $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ centred in O. The first rotation is around the axis $Z \equiv z_1$ with angular velocity $\dot{\psi}$ with an angle of precession ψ on the reference system $\mathscr{B}_1$ with (x_1, y_1, z_1) and $\{\hat{\imath}_1, \hat{\jmath}_1, \hat{k}_1\}$. The second rotation is made around $x_1 \equiv x_2$ in the positive way, with angular velocity $\dot{\theta}$ and with a nutation angle θ on the reference system $\mathscr{B}_2$ with (x_2, y_2, z_2) and $\{\hat{\imath}_2, \hat{\jmath}_2, \hat{k}_2\}$. Finally, the last rotation is around $z_2 \equiv z$ with an angular velocity $\dot{\phi}$ and an angle ϕ on the reference system $\mathscr{B}_3$ with (x, y, z) and $\{\hat{\imath}_3, \hat{\jmath}_3, \hat{k}_3\}$. The center of mass of the top is represented by the point A and the distance from the point O up to this point along the axis z is ℓ.

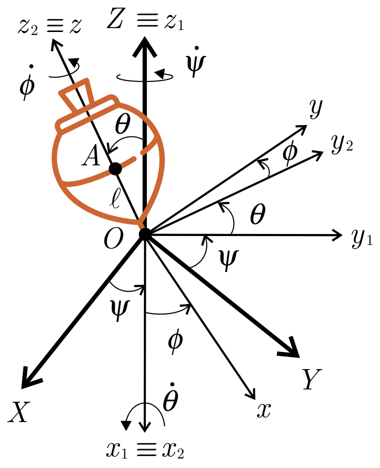

Figure 1 – Spinning top.

The motion equations of the spinning top responsible for describing the precession, nutation and spin, respectively, are given by (Perry, 1957; Santos, 2001; Goldstein, 2002; Ariska and Akhsan, 2018; Weber, 2019):

$$\begin{aligned}
\ddot{\psi} &= \frac{\dot{\theta}}{\sin\theta\,(I+m\ell^2)}\left[\dot{\phi}I_{z_2} + \dot{\psi}\cos\theta\left(I_{z_2} + -2m\ell^2\right)\right] \\
\ddot{\theta} &= \frac{\sin\theta}{I+m\ell^2}\left\{mg\ell + \dot{\psi}\left[-I_{z_2}\dot{\phi} + \dot{\psi}\cos\theta\left(I - I_{z_2} + m\ell^2\right)\right]\right\} \\
\ddot{\phi} &= \frac{\dot{\theta}}{\tan\theta\,(I+m\ell^2)}\left[-I_{z_2}\dot{\phi} + \dot{\psi}\cos\theta\left(2I - I_{z_2} + 2m\ell^2\right)\right] + \frac{1}{I_{z_2}}\left[M + \dot{\psi}\dot{\theta}\sin\theta\,(I_{z_2})\right]
\end{aligned} \tag{1}$$

where $I_{x_2} = I_{y_2} = I$ (if axisymmetric) and I_{z_2} are components of the inertia tensor concerning the axes x_2, y_2, and z_2 respectively, and M is the moment that directly influences in the loss of the motion of the spin of the top.

AN INTRODUCTION TO THE LIE SYMMETRIES

This section briefly introduces Lie symmetries. The reader is referred to (Olver, 1986; Bluman, 1989; Ruiz; Muriel; Ramírez, 2019; Basquerotto and Ruiz, 2020; Basquerotto et al., 2022) for the foundations concerning Lie symmetries. Due to the particular form of the problem considered in this work, we deal with a three-dimensional second-order system of the form:

$$\Delta_q\left(t,\psi,\theta,\phi,\dot{\psi},\dot{\theta},\dot{\phi},\ddot{\psi},\ddot{\theta},\ddot{\phi}\right) = 0, \qquad q = 1,2,3, \tag{2}$$

where t is the independent variable, ψ, θ and ϕ are the dependent variables and the derivatives with respect to t are denoted by the dot symbol above. Let us consider a vector field χ defined on an open set of the configuration space with coordinates (t,ψ,θ,ϕ):

$$\chi = \xi\frac{\partial}{\partial t} + \eta_\psi\frac{\partial}{\partial\psi} + \eta_\theta\frac{\partial}{\partial\theta} + \eta_\phi\frac{\partial}{\partial\phi}, \tag{3}$$

where ξ, η_ψ, η_θ and η_ϕ are infinitesimal functions. For the case of a second-order system of the form (2), one can consider an operator $\mathscr{U}''$ of the form:

$$\mathscr{U}'' = \beta_\psi^{(1)}\frac{\partial}{\partial\dot{\psi}} + \beta_\theta^{(1)}\frac{\partial}{\partial\dot{\theta}} + \beta_\phi^{(1)}\frac{\partial}{\partial\dot{\phi}} + \beta_{\ddot{\psi}}^{(2)}\frac{\partial}{\partial\ddot{\psi}} + \beta_\theta^{(2)}\frac{\partial}{\partial\ddot{\theta}} + \beta_\phi^{(2)}\frac{\partial}{\partial\ddot{\phi}}, \tag{4}$$

where

$$\beta_\psi^{(1)} = \mathscr{D}_t(\eta_\psi) - \dot{\psi}\mathscr{D}_t(\xi), \quad \beta_\theta^{(1)} = \mathscr{D}_t(\eta_\theta) - \dot{\theta}\mathscr{D}_t(\xi) \quad \beta_\phi^{(1)} = \mathscr{D}_t(\eta_\phi) - \dot{\phi}\mathscr{D}_t(\xi), \tag{5}$$

and

$$\begin{aligned}
&\beta_\psi^{(2)} = \mathscr{D}_t\left(\beta_\psi^{(1)}\right) - \ddot{\psi}\mathscr{D}_t(\xi), \quad \beta_\theta^{(2)} = \mathscr{D}_t\left(\beta_\theta^{(1)}\right) - \ddot{\theta}\mathscr{D}_t(\xi), \\
&\beta_n^{(2)} = \mathscr{D}_t\left(\beta_\phi^{(1)}\right) - \ddot{\phi}\mathscr{D}_t(\xi),
\end{aligned} \tag{6}$$

being $\mathscr{D}_t$ the total derivative operator with respect to t:

$$\mathscr{D}_t = \frac{\partial}{\partial t} + \dot{\psi}\frac{\partial}{\partial\psi} + \dot{\theta}\frac{\partial}{\partial\theta} + \dot{\phi}\frac{\partial}{\partial\phi} + \ddot{\psi}\frac{\partial}{\partial\dot{\psi}} + \ddot{\theta}\frac{\partial}{\partial\dot{\theta}} + \ddot{\phi}\frac{\partial}{\partial\dot{\phi}} + \cdots. \tag{7}$$

A vector field χ of the form (3) is a Lie point symmetry of the system (2) if the following condition is satisfied:

$$\left(\chi+\mathscr{U}''\right)(\Delta_q)=0,\ \ q=1,2,3,\quad \text{whenever}\ \begin{cases}\Delta_1=0,\\ \Delta_2=0,\\ \Delta_3=0.\end{cases} \tag{8}$$

Condition (8) involves t, ϕ, θ and ψ and their derivatives with respect t, as well as ξ and η_ψ, η_θ, η_ϕ and their partial derivatives with respect to t, ψ, θ and ϕ. After eliminating all dependencies through the derivatives involving ψ, θ and ϕ, the remaining partial derivatives coefficients from ψ, θ and ϕ to zero can be equated. This provides a system of partial differential equations to determine the functions ξ, η_θ, η_Ω and η_n. These equations are known as determining equations for the given system.

THE LIE SYMMETRIES OF THE SPINNING TOP

By applying the Lie conditions, given by Eqs. (3) and (4), in the system of eqs. (1), it is possible to find the following determining equations: $\dfrac{\partial\eta_\phi}{\partial\phi}=0$, $\dfrac{\partial\eta_\phi}{\partial\psi}=0$, $\dfrac{\partial\eta_\phi}{\partial t}=0$, $\dfrac{\partial\eta_\phi}{\partial\theta}=0$, $\dfrac{\partial\eta_\psi}{\partial\phi}=0$, $\dfrac{\partial\eta_\psi}{\partial\psi}=0$, $\dfrac{\partial\eta_\psi}{\partial t}=0$, $\dfrac{\partial\eta_\psi}{\partial\theta}=0$, $\dfrac{\partial\xi}{\partial\phi}=0$, $\dfrac{\partial\xi}{\partial\psi}=0$, $\dfrac{\partial\xi}{\partial\theta}=0$, $\dfrac{\partial\xi}{\partial t}=0$ e $\eta_\theta=0$. By solving the determining equations, it is possible to obtain the following infinitesimals generators to Eq. (1) listed in the table (1).

Table 1 – Infinitesimals generators of the spinning top.

	ξ	η_ϕ	η_ψ	η_θ
$\mathscr{X}_1=\dfrac{\partial}{\partial t}$	1	0	0	0
$\mathscr{X}_2=\dfrac{\partial}{\partial\psi}$	0	0	1	0
$\mathscr{X}_3=\dfrac{\partial}{\partial\phi}$	0	1	0	0

The infinitesimal generator $\mathscr{X}_1$ is related to time translation, whereas $\mathscr{X}_2$ and $\mathscr{X}_3$ correspond to the rotation on the axes where the angles are measured by ψ and ϕ, respectively.

Reducing the Order of the System of Equations

This subsection aims to show how a Lie symmetry can be used to reduce by one the order of second-order system equations of the form (1). By computing the local action associated with the vector fields:

$$\bar{t}=t+\varepsilon_1,\quad \bar{\theta}=\theta,\quad \bar{\psi}=\psi+\varepsilon_2,\quad \bar{\phi}=\phi+\varepsilon_3, \tag{9}$$

where $(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in\mathbb{R}^3$. The prolonged action becomes:

$$\dot{\bar{\theta}}=\frac{D_t(\bar{\theta})}{D_t(\bar{t})}=\dot{\theta},\quad \dot{\bar{\psi}}=\frac{D_t(\bar{\psi})}{D_t(\bar{t})}=\dot{\psi},\quad \dot{\bar{\phi}}=\frac{D_t(\bar{\phi})}{D_t(\bar{t})}=\dot{\phi}. \tag{10}$$

Thus, we obtain the following complete set of first-order differential invariants for the group action:

$$y=\theta,\quad w=\dot{\theta},\quad z=\dot{\psi},\quad \mu=\dot{\phi}. \tag{11}$$

Second-order invariants can be obtained by derivation:

$$\dot{w}=\frac{D_t(w)}{D_t(y)}=\frac{\ddot{\theta}}{\dot{\theta}},\quad \dot{z}=\frac{D_t(z)}{D_t(y)}=\frac{\ddot{\psi}}{\dot{\theta}},\quad \dot{\mu}=\frac{D_t(\mu)}{D_t(y)}=\frac{\ddot{\phi}}{\dot{\theta}}. \tag{12}$$

It can be checked that system (1) can be expressed in terms of the invariants $\{y,w,z,\mu,\dot{w},\dot{z},\dot{\mu}\}$ as the following reduced system:

$$\begin{aligned}
&\dot{\mu}w-\frac{w\left(zI_{z_2}+\mu\cos(y)\left(-2m\ell^2+I_{z_2}\right)\right)}{\sin(y)\left(m\ell^2+I\right)}=0\\
&w\dot{w}-\frac{\sin(y)\left(mg\ell+\mu\left(-zI_{z_2}+\mu\cos(y)\left(m\ell^2+I+I_{z_2}\right)\right)\right)}{m\ell^2+I}=0\\
&w\dot{z}-\frac{w\left(-zI_{z_2}+\mu\cos(y)\left(2m\ell^2+2I-I_{z_2}\right)\right)}{\tan(y)\left(m\ell^2+I\right)}+\frac{M+w\mu\sin(y)\left(I_{z_2}\right)}{I_{z_2}}=0
\end{aligned} \tag{13}$$

Observe that (13) is a first-order system where y acts as independent variable and w, z and μ correspond to the dependent variables.

FINAL REMARKS

Sophus Lie developed a method that unified several integration procedures for solving differential equations. Lie established the concept of the symmetry group of transformations, and many fundamental advances in physics and mathematics achieved today started with his theorem. This work has analyzed the Lie Symmetries of a spinning top considering nutation and precession as time-varying. We could note that it is possible to find the infinitesimals generators related to these physical quantities and use them to reduce the order of the system of equations.

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