A critical review about objective fields in Continuum Mechanics

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*Abstract: The definition of objectivity plays an important role in the enunciation of the Invariance of Material Response (IMR) in the development of the Continuum Mechanics. Unfortunately, the most used framework to formulate the IMR – based on monitoring of one motion by two observers – shows somewhat **inconsistency** with the objectivity definition. Besides, most works present the transformation laws of fields adopting a particular case without need, and this has caused misconception over decades. This article proposes a new approach that allows to formulate the IMR in a coherent way with the definition of objectivity. In addition, the transformation laws of some fields are presented considering a more general case and the classification of a second order tensor regarding its objectivity is done taking into account its mapping type. Further, a straightforward review of objectivity is presented and some conceptual aspects are remarked.*

Keywords: Continuum Mechanics, Objective Fields

INTRODUCTION

Invariance of the Material Response (IMR), whether related to a change in observer or to a Rigid Body Motion (RBM), is an assumption used to impose restrictions upon constitutive equations in Continuum Mechanics. The IMR uses the concept of objective fields and, by doing so, the correct understanding of this concept it is fundamental to apply IMR properly.

Despite its importance, the transformation laws related to a change in observer (or due a RBM) of some quantities are presented differently in the literature. Regarding to the deformation gradient **F**, for example, Gurtin (1981), Bonet and Wood (1997), Holzapfel (2000), Reddy (2008), Gonzalez and Stuart (2008), Gurtin, Fried and Anand (2010), Lai, Krempl and Ruben (2010), Belytschko et al. (2014) and Hackett (2018) differ from Ogden (1984) and Murdoch (2012). In fact, the former authors show a particular case of the equation given by Ogden (1984) and Murdoch (2012), and this has caused a misinterpretation on the subject over years, as commented by Murdoch (2012) and Liu and Sampaio (2013). In addition, the most used framework to formulate IMR (based on monitoring of one motion by two observers) is somewhat inconsistent with the requirement of the objectivity definition.

Although some references like Ogden (1984), Holzapfel (2000), Liu (2003), Murdoch (2003) and Hackett (2018) mention the peculiar characteristic of objectivity related to the deformation gradient, there is not a general treatise covering all mapping types of tensors. That said, the present paper constructs a comprehensive approach that allows to classify if a given second order tensor, together with its mapping type, is objective or not. The same general case for the transformation law of the deformation gradient, given by Ogden (1984) and Murdoch (2012), is obtained, but through the new approach. Besides, the results obtained here can be extended to classify the objectivity of any order tensors taking into account its mapping type. The proposed approach is coherent with the objectivity definition and shows itself to be an alternative framework to elaborate the IMR.

It is worth to mention that the discussion here does not deal with IMR itself, which has controversy details in scientific community as one can see in Murdoch (2005), Frewer (2009) and Liu (2014). The focus in the present paper is on the objectivity classification of fields.

DESCRIPTION OF A RBM

As pointed out by Tenenbaum (2006), a rigid body can be assumed to be an observer. Also, Muschik and Restuccia (2002) highlighted that a referential must incorporate a coordinate system and devices to measure physical quantities. That said, Fig. 1a shows the adopted symbology in this work for an observer $\mathscr{R}$: a cubic rigid body, in dashed line, together with three directions, indicating an orthonormal basis. In this paper, one uses 'observer', 'referential' and 'frame' as synonymous.
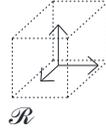
Figure 1 – Symbology of an observer / referential / frame.

Figure 2a shows a region Ω_0 (reference configuration) occupied by a deformable body in a stress-free state at an initial instant t_0. Due a dynamic loading, the body occupies the region Ω_t (current configuration) at some instant t. At t_0, the observer $\mathscr{R}$ identifies a material point $A \in \Omega_0$ by $\mathbf{p}$. This point, at instant t, becomes $A' \in \Omega_t$ and $\mathscr{R}$ monitored it by $\mathbf{x}$. The variables $\mathbf{p}$, $\mathbf{x}$ and t in the Classical Continuum Mechanics are linked by the function $\boldsymbol{\chi}$, that is, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{p}, t)$. In the Fig. 2, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of the three-dimensional Euclidean space.
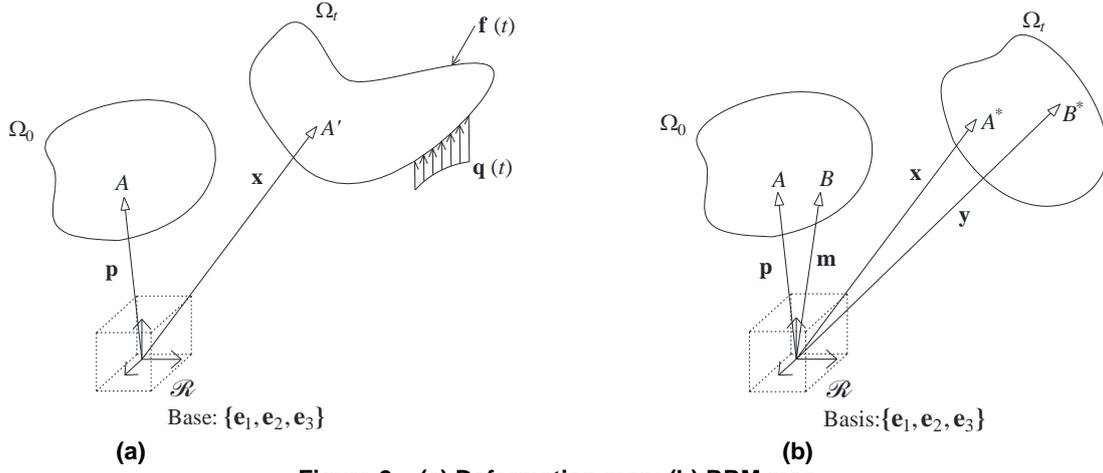

Figure 2 – (a) Deformation map; (b) RBM map.

Figure 2b shows an arbitrary RBM of the body. One notes the distance between any two materials points is preserved, that is, $\|\mathbf{p} - \mathbf{m}\| = \|\mathbf{x} - \mathbf{y}\|$. Developing this expression, according Gurtin (1981), one has:

$$\|\mathbf{p}-\mathbf{m}\|=\|\mathbf{x}-\mathbf{y}\| \Rightarrow \|\mathbf{p}-\mathbf{m}\|=\|\boldsymbol{\chi}(\mathbf{p},t)-\boldsymbol{\chi}(\mathbf{m},t)\| \Rightarrow$$

$$\Rightarrow \nabla_{\mathbf{p}}[(\mathbf{p}-\mathbf{m})\cdot(\mathbf{p}-\mathbf{m})]=\nabla_{\mathbf{p}}\{[\boldsymbol{\chi}(\mathbf{p},t)-\boldsymbol{\chi}(\mathbf{m},t)]\cdot[\boldsymbol{\chi}(\mathbf{p},t)-\boldsymbol{\chi}(\mathbf{m},t)]\} \Rightarrow$$

$$\Rightarrow (\nabla_{\mathbf{p}}\mathbf{p}-\nabla_{\mathbf{p}}\mathbf{m})^{\mathrm{T}}(\mathbf{p}-\mathbf{m})=[\nabla_{\mathbf{p}}\boldsymbol{\chi}(\mathbf{p},t)-\nabla_{\mathbf{p}}\boldsymbol{\chi}(\mathbf{m},t)]^{\mathrm{T}}[\boldsymbol{\chi}(\mathbf{p},t)-\boldsymbol{\chi}(\mathbf{m},t)] \overset{(I)}{\Rightarrow}$$

$$\overset{(I)}{\Rightarrow} \mathbf{p}-\mathbf{m}=[\nabla_{\mathbf{p}}\boldsymbol{\chi}(\mathbf{p},t)]^{\mathrm{T}}[\boldsymbol{\chi}(\mathbf{p},t)-\boldsymbol{\chi}(\mathbf{m},t)] \Rightarrow \nabla_{\mathbf{m}}(\mathbf{p}-\mathbf{m})=\nabla_{\mathbf{m}}\{[\nabla_{\mathbf{p}}\boldsymbol{\chi}(\mathbf{p},t)]^{\mathrm{T}}[\boldsymbol{\chi}(\mathbf{p},t)-\boldsymbol{\chi}(\mathbf{m},t)]\} \Rightarrow$$

$$\Rightarrow [\nabla_{\mathbf{m}}\boldsymbol{\chi}(\mathbf{m},t)]^{-1}=[\nabla_{\mathbf{p}}\boldsymbol{\chi}(\mathbf{p},t)]^{\mathrm{T}} \tag{1}$$

The term $\nabla_{\mathbf{p}}\boldsymbol{\chi}(\mathbf{m}, t)$ in step (I) means that the gradient is performed *after* $\boldsymbol{\chi}$ is written as a function of $\mathbf{m}$ and t, and that is why it is null. Now, Eq. (1) only makes sense if $\nabla_{\mathbf{m}}\boldsymbol{\chi}$ [$\nabla_{\mathbf{p}}\boldsymbol{\chi}$] does not depend on $\mathbf{m}$ [$\mathbf{p}$], that is, it must depend only of time. Therefore, $\nabla_{\mathbf{m}}\boldsymbol{\chi} = \nabla_{\mathbf{p}}\boldsymbol{\chi}$ and, from Eq. (1), $\nabla_{\mathbf{p}}\boldsymbol{\chi}$ is orthogonal. That said, the Taylor series of $\boldsymbol{\chi}$ has the following form:

$$\boldsymbol{\chi}(\mathbf{p},t)=\boldsymbol{\chi}(\mathbf{m},t)+\mathbf{Q}(t)(\mathbf{p}-\mathbf{m}) \Rightarrow \boldsymbol{\chi}(\mathbf{p},t)=\mathbf{Q}(t)\mathbf{p}+[\mathbf{Q}(t)\mathbf{m}+\boldsymbol{\chi}(\mathbf{m},t)] \tag{2}$$

with $\mathbf{Q}(t) = \nabla_{\mathbf{p}}\boldsymbol{\chi}$. If $\mathbf{p}$ is fixed and $\mathbf{m}$ vary in Eq. (2), one realizes that $\boldsymbol{\chi}(\mathbf{p}, t)$ does not change. Thus, the term $[\mathbf{Q}(t)\mathbf{m} + \boldsymbol{\chi}(\mathbf{m}, t)]$ is a function only of time, and Eq. (2) becomes:

$$\boldsymbol{\chi}(\mathbf{p},t)=\mathbf{Q}(t)\mathbf{p}+\mathbf{c}(t) \tag{3}$$

Due to physical restrictions, $\mathbf{Q}(t)$ in Eq. (3) is a *proper* orthogonal tensor. Lastly, it will be useful to know the transformation rule of a vector that connects both origins of the coordinate systems of two observers, as shown by Fig. 3. The vector $\overrightarrow{OO'}$ seen by $\mathscr{R}'$ is named as $\mathbf{a}'$ and it seen by $\mathscr{R}$ is named as $\mathbf{a}$. According to Boldrini et al. (1986), one has:
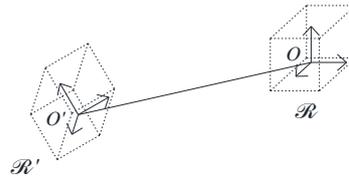
$$\mathbf{a}'=\mathbf{Q}(t)\mathbf{a} \tag{4}$$

Figure 3 – Scheme for Eq. (4).

PROPOSED APPROACH

Figure 4a shows a new referential $\mathscr{R}^*$ placed arbitrary in space at t_0. As an observer can be seen as a rigid body, $\mathscr{R}$ and $\mathscr{R}^*$ are separated by a RBM. That said, one defines the configuration monitored by $\mathscr{R}^*$ as the body moved by the same RBM applied from $\mathscr{R}$ to $\mathscr{R}^*$, as shown in Figure 4b.
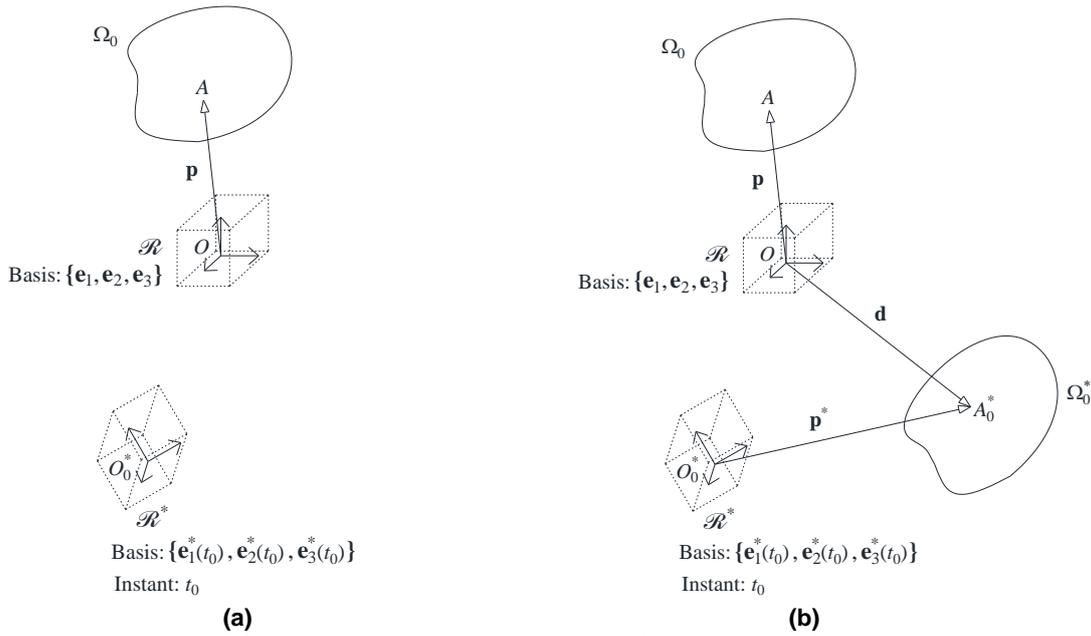

Figure 4 – (a) New observer $\mathscr{R}^*$ at t_0; (b) Same RBM of $\mathscr{R}$ to $\mathscr{R}^*$ and of Ω_0 to Ω_0^*.

In Figure 4b, the point A_0^* indicates the point A after the RBM. The vector $\overrightarrow{OA_0^*}$ is labeled by $\mathscr{R}$ perspective and the vector $\overrightarrow{O_0^*A_0^*}$ by $\mathscr{R}^*$ perspective. For $t > t_0$, when the body starts moving/deforming, $\mathscr{R}^*$ also moves in an arbitrary way. Thus, for a given time t, $\mathscr{R}^*$ identifies the body moved by the same RBM applied from $\mathscr{R}$ to $\mathscr{R}^*$ at this instant, as shown in Figure 5.
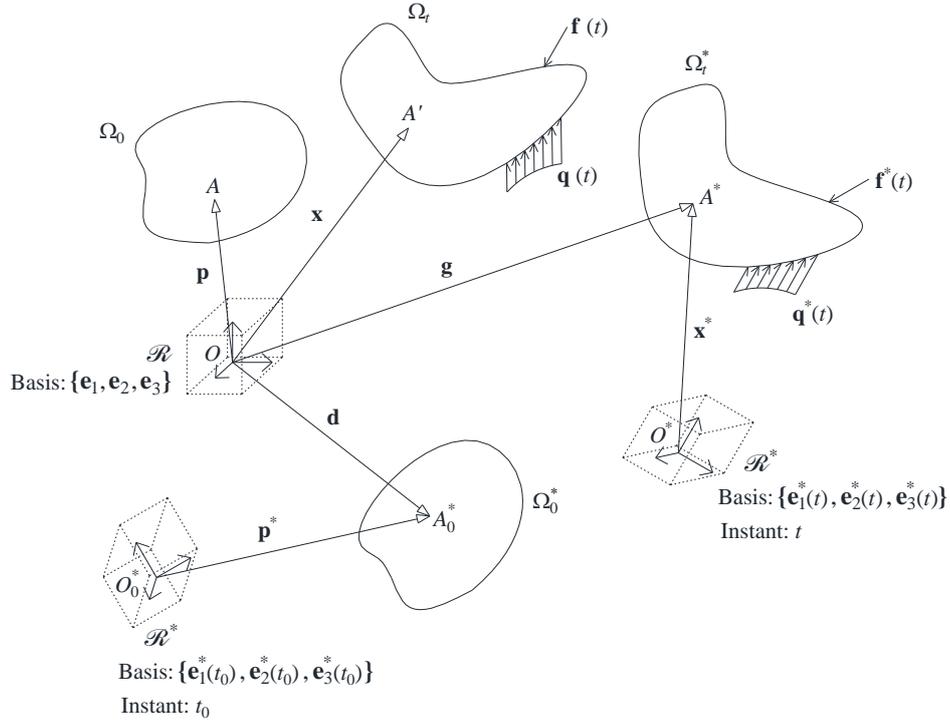
Figure 5 – Proposed framework.

In Figure 5, the point A^* represents the point A' after the RBM. The vector $\overrightarrow{OA^*}$ is labeled by $\mathscr{R}$ perspective and the vector $\overrightarrow{O^*A^*}$ by $\mathscr{R}^*$ perspective. Now, by Eq. (3), the RBM description of Ω_0 to Ω_0^* seen by $\mathscr{R}$ is:

$$\mathbf{d} = \mathbf{Q}_{\mathrm{I}}(t_0)\mathbf{p} + \mathbf{c}_{\mathrm{I}}(t_0) \tag{5}$$

Again, as a referential is understood as a rigid body, one can interpret Ω_0^* as a frame *at instant t_0* that sees the RBM of $\mathscr{R}$ to $\mathscr{R}^*$. That said, by Eq. (3) one has:

$$[\mathbf{Q}_{\mathrm{I}}(t_0)]^{\mathrm{T}}(-\mathbf{p}^*) = \mathbf{Q}_{\mathrm{I}}(t_0)(-\mathbf{d}) + \mathbf{c}_{\mathrm{I}}(t_0) \Rightarrow \mathbf{d} = [\mathbf{Q}_{\mathrm{I}}(t_0)]^{\mathrm{T}} [\mathbf{Q}_{\mathrm{I}}(t_0)]^{\mathrm{T}} \mathbf{p}^* + [\mathbf{Q}_{\mathrm{I}}(t_0)]^{\mathrm{T}}\mathbf{c}_{\mathrm{I}}(t_0) \tag{6}$$

In Eq. (6), the negative sign in $\mathbf{p}^*$ and $\mathbf{d}$ are essentials for indicate the same point during the RBM. Besides, Eq. (6) is written taking the canonical basis for Ω_0^* and, in view of Eq. (4), the vector $\mathbf{p}^*$ must be transformed by $[\mathbf{Q}_{\mathrm{I}}(t_0)]^{\mathrm{T}}$. Substituting Eq. (6) in Eq. (5) one has:

$$[\mathbf{Q}_{\mathrm{I}}(t_0)]^{\mathrm{T}} [\mathbf{Q}_{\mathrm{I}}(t_0)]^{\mathrm{T}} \mathbf{p}^* + [\mathbf{Q}_{\mathrm{I}}(t_0)]^{\mathrm{T}}\mathbf{c}_{\mathrm{I}}(t_0) = \mathbf{Q}_{\mathrm{I}}(t_0)\mathbf{p} + \mathbf{c}_{\mathrm{I}}(t_0) \Rightarrow \mathbf{p}^* = \mathbf{Q}_{\mathrm{II}}(t_0)\mathbf{p} + \mathbf{c}_{\mathrm{II}}(t_0) \tag{7}$$

where in Eq. (7) one has defined $Q_{\mathrm{II}}(t_0) = Q_{\mathrm{I}}^3(t_0)$ and $c_{\mathrm{II}}(t_0) = Q_{\mathrm{I}}(t_0) [Q_{\mathrm{I}}(t_0) - \mathbf{I}] c_{\mathrm{I}}(t_0)$. Now, by Eq. (3), the RBM description of Ω_t to Ω_t^* witnessed by $\mathscr{R}$ is given in Eq. (8).

$$\mathbf{g} = \mathbf{Q}_{\mathrm{III}}(t)\mathbf{x} + \mathbf{c}_{\mathrm{III}}(t) \tag{8}$$

Taking Ω_t^* as an observer of the RBM of $\mathscr{R}$ to $\mathscr{R}^*$ *at instant t*, one has:

$$[\mathbf{Q}_{\mathrm{III}}(t)]^{\mathrm{T}}(-\mathbf{x}^*) = \mathbf{Q}_{\mathrm{III}}(t)(-\mathbf{g}) + \mathbf{c}_{\mathrm{III}}(t) \Rightarrow \mathbf{g} = [\mathbf{Q}_{\mathrm{III}}(t)]^{\mathrm{T}} [\mathbf{Q}_{\mathrm{III}}(t)]^{\mathrm{T}} \mathbf{x}^* + [\mathbf{Q}_{\mathrm{III}}(t)]^{\mathrm{T}}\mathbf{c}_{\mathrm{III}}(t) \tag{9}$$

where the conventions in Eq. (9) are similar to those applied in Eq. (6). Substituting Eq. (9) in Eq. (8) one has:

$$[\mathbf{Q}_{\mathrm{III}}(t)]^{\mathrm{T}} [\mathbf{Q}_{\mathrm{III}}(t)]^{\mathrm{T}} \mathbf{x}^* + [\mathbf{Q}_{\mathrm{III}}(t)]^{\mathrm{T}}\mathbf{c}_{\mathrm{III}}(t) = \mathbf{Q}_{\mathrm{III}}(t)\mathbf{x} + \mathbf{c}_{\mathrm{III}}(t) \Rightarrow \mathbf{x}^* = \mathbf{Q}_{\mathrm{IV}}(t)\mathbf{x} + \mathbf{c}_{\mathrm{IV}}(t) \tag{10}$$

where in Eq. (10) one has defined $Q_{\mathrm{IV}}(t) = Q_{\mathrm{III}}^3(t)$ and $c_{\mathrm{IV}}(t) = Q_{\mathrm{III}}(t) [Q_{\mathrm{III}}(t) - \mathbf{I}] c_{\mathrm{III}}(t)$. Next, one defines the functions $\mathbf{Q}(t)$ and $\mathbf{c}^*(t)$ by parts as:

$$\mathbf{Q}(t) = \begin{cases} \mathbf{Q}_{II}(t_0) & \text{for } t = t_0 \\ \mathbf{Q}_{IV}(t) & \text{for } t > t_0 \end{cases} \quad \text{and} \quad \mathbf{c}^*(t) = \begin{cases} \mathbf{c}_{II}(t_0) & \text{for } t = t_0 \\ \mathbf{c}_{IV}(t) & \text{for } t > t_0 \end{cases} \tag{11}$$

Using the idea of Eq. (11) in Eq. (7) and in Eq. (10), one has:

$$\mathbf{p}^* = \mathbf{Q}(t_0)\mathbf{p} + \mathbf{c}^*(t_0) \tag{12}$$

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}^*(t) \quad \text{or} \quad \chi^*(\mathbf{p}^*,t) = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}^*(t) \tag{13}$$

where one has defined $\mathbf{x}^* = \chi^*(\mathbf{p}^*, t)$. The functions $\mathbf{Q}(t)$ and $\mathbf{c}^*(t)$ in Eqs. (12) and (13) are assumed to be continuous and differentiable. One must note that Eq. (13) is found in the literature but its interpretation may change according to the way the IMR is formulated.

Objective fields

According to Ogden (1984), any field can be written as a function of $\mathbf{p}$ and t or as a function of $\mathbf{x}$ and t. The first case is named as the material description and the second as the spatial description of the field. For a tensor field $\mathbf{A}$, for which $\mathbf{h} = \mathbf{A}\mathbf{w}$ holds, it is said material if both $\mathbf{w}$ and $\mathbf{h}$ are materials. When $\mathbf{w}$ is material [spatial] and $\mathbf{h}$ is spatial [material], $\mathbf{A}$ is said material-spatial [spatial-material]. At last, if both $\mathbf{w}$ and $\mathbf{h}$ are spatial, so $\mathbf{A}$ is said spatial. Therefore, as mentioned by Ogden (1984), there is a difference between 'material tensor field' and 'material description of a tensor field'. The former refers to the mapping done by the tensor while the latter to the function chosen to express the tensor. Thus, 'material tensor field' indicates a tensor that assigns to each material vector another material vector, as stated before, and 'material description of a tensor field' means a tensor written as a function of $\mathbf{p}$ and t. Analogously, 'spatial tensor field' indicates a tensor that assigns to each spatial vector another spatial vector and 'spatial description of a tensor field' means a tensor written as a function of $\mathbf{x}$ and t.

Before presenting the definition of objectivity, one notes that the differential relation of Eq. (12) leads to:

$$\mathbf{dp}^* = \mathbf{Q}(t_0)\mathbf{dp} \Rightarrow \|\mathbf{dp}^*\|^2 = \|\mathbf{Q}(t_0)\mathbf{dp}\|^2 = [\mathbf{Q}(t_0)\mathbf{dp}] \cdot [\mathbf{Q}(t_0)\mathbf{dp}] = \mathbf{dp} \cdot \mathbf{dp} = \|\mathbf{dp}\|^2 \Rightarrow \|\mathbf{dp}^*\| = \|\mathbf{dp}\| \tag{14}$$

Following Holzapfel (2000), from Eq. (14) one affirms that:

$$\frac{\mathbf{dp}^*}{\|\mathbf{dp}^*\|} = \mathbf{Q}(t_0)\frac{\mathbf{dp}}{\|\mathbf{dp}\|} \tag{15}$$

Since $\mathbf{dp}^*/\|\mathbf{dp}^*\|$ and $\mathbf{dp}/\|\mathbf{dp}\|$ are unit vectors, one may interpret them as vectors of basis, so Eq. (15) becomes:

$$\mathbf{e}_i^*(t_0) = \mathbf{Q}(t_0)\mathbf{e}_i \tag{16}$$

A similar procedure applied to Eq. (13) leads to:

$$\mathbf{e}_i^*(t) = \mathbf{Q}(t)\mathbf{e}_i \tag{17}$$

Equations (16) and (17) are represent in Figure 5. Now, the definition of objective fields can be presented properly. As reported by Ogden (1984), a material scalar field α is said objective if:

$$\alpha^*(\mathbf{p}^*,t) = \alpha(\mathbf{p},t) \tag{18}$$

The objectivity requires a comparison between the components of a given field, seen by an observer $\mathscr{R}$, with the components of this same field after an arbitrary RBM seen by an observer $\mathscr{R}^*$ that suffered the same RBM of the body. Thus, the proposed framework of the Fig. 5 is coherent with this definition. Standard texts on this subject present the objectivity but them variables with star (∗) do not represent the field seen by a referential that moves together with the body. Now, for a spatial scalar field, one has:

$$\alpha^*(\mathbf{x}^*,t) = \alpha(\mathbf{x},t) \tag{19}$$

Next, a material vector field $\mathbf{w}$ is said objective if each of its component follows Eq. (18), that is, if $w_i^*(\mathbf{p}^*, t) = w_i(\mathbf{p}, t)$. Then, one can proceed in the following manner:

$$w_i^*(\mathbf{p}^*,t) = w_i(\mathbf{p},t) \Rightarrow \mathbf{w}^*(\mathbf{p}^*,t) \cdot \mathbf{e}_i^*(t_0) = \mathbf{w}(\mathbf{p},t) \cdot \mathbf{e}_i \overset{(I)}{\Rightarrow} \mathbf{w}^*(\mathbf{p}^*,t) \cdot \mathbf{Q}(t_0)\mathbf{e}_i = \mathbf{w}(\mathbf{p},t) \cdot \mathbf{e}_i \overset{(II)}{\Rightarrow}$$

$$\overset{(II)}{\Rightarrow} [\mathbf{Q}(t_0)]^{\mathrm{T}} \mathbf{w}^*(\mathbf{p}^*,t) \cdot \mathbf{e}_i = \mathbf{w}(\mathbf{p},t) \cdot \mathbf{e}_i \overset{(III)}{\Rightarrow} [\mathbf{Q}(t_0)]^{\mathrm{T}} \mathbf{w}^*(\mathbf{p}^*,t) = \mathbf{w}(\mathbf{p},t) \Rightarrow \mathbf{w}^*(\mathbf{p}^*,t) = \mathbf{Q}(t_0)\mathbf{w}(\mathbf{p},t) \tag{20}$$

where in (I) it was used Eq. (16), in (II) the transpose definition $\mathbf{S}\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{S}^{\mathrm{T}}\mathbf{v}$ ($\mathbf{S}$ is a second order tensor) and in (III) the arbitrariness of $\mathbf{e}_i$. Analogously, a spatial vector field $\mathbf{w}$ is said objective if each of its component follows Eq. (19), that is, if $w_i^*(\mathbf{x}^*, t) = w_i(\mathbf{x}, t)$. Thus:

$$w_i^*(\mathbf{x}^*,t) = w_i(\mathbf{x},t) \Rightarrow \mathbf{w}^*(\mathbf{x}^*,t) \cdot \mathbf{e}_i^*(t) = \mathbf{w}(\mathbf{x},t) \cdot \mathbf{e}_i \overset{(I)}{\Rightarrow} \mathbf{w}^*(\mathbf{x}^*,t) \cdot \mathbf{Q}(t)\mathbf{e}_i = \mathbf{w}(\mathbf{x},t) \cdot \mathbf{e}_i \Rightarrow \mathbf{w}^*(\mathbf{x}^*,t) = \mathbf{Q}(t)\mathbf{w}(\mathbf{x},t) \tag{21}$$

where in (I) it was used Eq. (17). Now, following Lai (2010), a second order tensor $\mathbf{A}$ is said objective if:

$$A_{ij}^* = A_{ij} \Rightarrow \mathbf{e}_i^*(t) \cdot \mathbf{A}^*\mathbf{e}_j^*(t) = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j \tag{22}$$

For a material $\mathbf{A}^*$, both $\mathbf{e}_j^*$ and $\mathbf{A}^*\mathbf{e}_j^*$ in Eq. (22) are materials. As already known, an inner product between vectors is feasible if, and only if, both vectors are materials or both are spatial. That said, since $\mathbf{A}^*\mathbf{e}_j^*$ is material, so it does $\mathbf{e}_i^*$. Thus, one can use Eq. (16) in Eq. (22):

$$\mathbf{Q}(t_0)\mathbf{e}_i \cdot \mathbf{A}^*\mathbf{Q}(t_0)\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j \Rightarrow \mathbf{A}^* = \mathbf{Q}(t_0)\mathbf{A}[\mathbf{Q}(t_0)]^{\mathrm{T}} \tag{23}$$

where it was used the arbitrariness of $\mathbf{e}_i$ and $\mathbf{e}_j$. If $\mathbf{A}^*$ in Eq. (22) is material-spatial, so $\mathbf{A}^*\mathbf{e}_j^*$ is spatial. This implies that $\mathbf{e}_i^*$ is also spatial. Therefore, one can apply Eq. (16) for $\mathbf{e}_j^*$ and Eq. (17) for $\mathbf{e}_i^*$ in Eq. (22):

$$\mathbf{Q}(t)\mathbf{e}_i \cdot \mathbf{A}^*\mathbf{Q}(t_0)\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j \Rightarrow \mathbf{A}^* = \mathbf{Q}(t)\mathbf{A}[\mathbf{Q}(t_0)]^{\mathrm{T}} \tag{24}$$

If $\mathbf{A}^*$ is spatial-material, then $\mathbf{A}^*\mathbf{e}_j^*$ in Eq. (22) is material. Thus, $\mathbf{e}_i^*$ must be material. Hence, using Eq. (16) for $\mathbf{e}_i^*$ and Eq. (17) for $\mathbf{e}_j^*$ in Eq. (22), one has:

$$\mathbf{Q}(t_0)\mathbf{e}_i \cdot \mathbf{A}^*\mathbf{Q}(t)\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j \Rightarrow \mathbf{A}^* = \mathbf{Q}(t_0)\mathbf{A}[\mathbf{Q}(t)]^{\mathrm{T}} \tag{25}$$

Lastly, if $\mathbf{A}^*$ is spatial, both $\mathbf{e}_j^*$ and $\mathbf{A}^*\mathbf{e}_j^*$ in Eq. (22) are spatial. In this case, $\mathbf{e}_i^*$ is also spatial. So, substituting Eq. (17) in Eq. (22), one gives:

$$\mathbf{Q}(t)\mathbf{e}_i \cdot \mathbf{A}^*\mathbf{Q}(t)\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j \Rightarrow \mathbf{A}^* = \mathbf{Q}(t)\mathbf{A}[\mathbf{Q}(t)]^{\mathrm{T}} \tag{26}$$

Equations (18) to (21) and (23) to (26) are summarized in Tab. 1.

Table 1 – Objective fields definitions.

		Objectivity Definition
Scalar	Material	$\alpha^*(\mathbf{p}^*, t) = \alpha(\mathbf{p}, t)$
	Spatial	$\alpha^*(\mathbf{x}^*, t) = \alpha(\mathbf{x}, t)$
Vector	Material	$\mathbf{w}^*(\mathbf{p}^*, t) = \mathbf{Q}(t_0)\,\mathbf{w}(\mathbf{p}, t)$
	Spatial	$\mathbf{w}^*(\mathbf{x}^*, t) = \mathbf{Q}(t)\,\mathbf{w}(\mathbf{x}, t)$
Second Order Tensor	Material	$\mathbf{A}^*(\mathbf{p}^*, t) = \mathbf{Q}(t_0)\,\mathbf{A}(\mathbf{p}, t)\,[\mathbf{Q}(t_0)]^{\mathrm{T}}$
	Material-Spatial	$\mathbf{A}^*(\mathbf{p}^*, t) = \mathbf{Q}(t)\,\mathbf{A}(\mathbf{p}, t)\,[\mathbf{Q}(t_0)]^{\mathrm{T}}$
	Spatial -Material	$\mathbf{A}^*(\mathbf{x}^*, t) = \mathbf{Q}(t_0)\,\mathbf{A}(\mathbf{x}, t)\,[\mathbf{Q}(t)]^{\mathrm{T}}$
	Spatial	$\mathbf{A}^*(\mathbf{x}^*, t) = \mathbf{Q}(t)\,\mathbf{A}(\mathbf{x}, t)\,[\mathbf{Q}(t)]^{\mathrm{T}}$

TRANSFORMATION LAWS FOR SOME KINECMATIC FIELDS

The gradient of Eq. (13) with respect to **p** leads to:

$$\nabla_{\mathbf{p}}\left[\chi^*\left(\mathbf{p}^*,t\right)\right]=\nabla_{\mathbf{p}}[\mathbf{Q}(t)\chi(\mathbf{p},t)+\mathbf{c}(t)]\overset{(I)}{\Rightarrow}\left(\nabla_{\mathbf{p}^*}\chi^*\right)\left(\nabla_{\mathbf{p}}\mathbf{p}^*\right)=\mathbf{Q}(t)\left(\nabla_{\mathbf{p}}\chi\right)\overset{(II)}{\Rightarrow}\mathbf{F}^*\,\mathbf{Q}(t_0)=\mathbf{Q}(t)\mathbf{F}\Rightarrow$$

$$\Rightarrow\mathbf{F}^*\left(\mathbf{p}^*,t\right)=\mathbf{Q}(t)\mathbf{F}(\mathbf{p},t)[\mathbf{Q}(t_0)]^{\mathrm{T}} \tag{27}$$

where in (I) the chain rule was used and in (II) the term $\nabla_{\mathbf{p}}\mathbf{p}^*$ was obtained by taking the gradient with respect to **p** of Eq. (12). The tensors $\mathbf{F}(\mathbf{p},t)=\nabla_{\mathbf{p}}\chi(\mathbf{p},t)$ and $\mathbf{F}^*(\mathbf{p}^*,t)=\nabla_{\mathbf{p}}\chi^*(\mathbf{p}^*,t)$ are the deformation gradients seen by $\mathscr{R}$ and $\mathscr{R}^*$, respectively, with the assumptions $\det\mathbf{F}>0$ and $\det\mathbf{F}^*>0$. That said, the polar decomposition of **F** is given by $\mathbf{F}=\mathbf{RU}=\mathbf{VR}$, where both **U** and **V** are positive definite tensors and **R** is a proper orthogonal tensor. Further, one defines the right Cauchy-Green tensor as $\mathbf{C}=\mathbf{U}^2=\mathbf{F}^{\mathrm{T}}\mathbf{F}$ and the left Cauchy-Green tensor as $\mathbf{B}=\mathbf{V}^2=\mathbf{FF}^{\mathrm{T}}$. The polar decomposition in relation to $\mathscr{R}^*$ follows by just putting a star ($*$) in all variables.

Now, as presented in the previous item, the way a tensor transforms from $\mathscr{R}$ to $\mathscr{R}^*$ does not determine if it is objective. Its mapping type must be known. So, one determines the mapping type of some tensors in what follows. First, Eq. (12) itself implies that **p** is not objective since it is material and does not obey Eq. (20). Analogously, Eq. (13) itself implies that **x** is not objective since it is spatial and does not follow Eq. (21). Now, a differential relation of $\mathbf{x}=\chi(\mathbf{p},t)$ gives (with time fixed):

$$\mathbf{dx}=\mathbf{F}\,\mathbf{dp} \tag{28}$$

As **dp** is material and **dx** is spatial, then **F** is a material-spatial tensor. Thus, since Eq. (27) obeys Eq. (24), **F** is objective. This contrasts with $\mathbf{F}^*=\mathbf{Q}(t)\mathbf{F}$, a commonly found formula that, actually, follows from Eq. (27) with $\mathbf{Q}(t_0)=\mathbf{I}$. Given a spatial vector **s**, one has from Eq. (28):

$$\mathbf{dx}\cdot\mathbf{s}=\mathbf{F}\,\mathbf{dp}\cdot\mathbf{s}\Rightarrow\mathbf{dx}\cdot\mathbf{s}=\mathbf{dp}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{s} \tag{29}$$

Tensor $\mathbf{F}^{\mathrm{T}}$ must map **s** into a material vector in Eq. (29), thus $\mathbf{F}^{\mathrm{T}}$ is spatial-material. Pre-multiplying Eq. (28) by $\mathbf{F}^{\mathrm{T}}$, one follows:

$$\mathbf{F}^{\mathrm{T}}\mathbf{dx}=\mathbf{F}^{\mathrm{T}}\mathbf{F}\,\mathbf{dp}\Rightarrow\mathbf{F}^{\mathrm{T}}\mathbf{dx}=\mathbf{C}\,\mathbf{dp} \tag{30}$$

The left side of Eq. (30) is a material vector, so **C** is material. Still in Eq. (30), one has:

$$\mathbf{F}^{\mathrm{T}}\mathbf{dx}=\mathbf{U}^2\,\mathbf{dp}\Rightarrow\mathbf{F}^{\mathrm{T}}\mathbf{dx}=\mathbf{U}\left(\mathbf{U}\,\mathbf{dp}\right) \tag{31}$$

Again, as $\mathbf{F}^{\mathrm{T}}\mathbf{dx}$ is a material vector, **U** is material since, within parentheses in Eq. (31), **U** is applied upon a material vector **dp**. Pre-multiplying Eq. (30) by **F** one provides:

$$\mathbf{FF}^{\mathrm{T}}\mathbf{dx}=\mathbf{F}\left(\mathbf{C}\,\mathbf{dp}\right)\Rightarrow\mathbf{B}\mathbf{dx}=\mathbf{F}\left(\mathbf{C}\,\mathbf{dp}\right) \tag{32}$$

As **F** is material-spatial, Eq. (32) shows that **B** is a spatial tensor. From Eq. (32) one can proceed:

$$\mathbf{V}^2\mathbf{dx}=\mathbf{F}\left(\mathbf{C}\,\mathbf{dp}\right)\Rightarrow\mathbf{V}\left(\mathbf{V}\mathbf{dx}\right)=\mathbf{F}\left(\mathbf{C}\,\mathbf{dp}\right) \tag{33}$$

As the right side of Eq. (33) is a spatial vector, and **V** is applied upon a spatial vector **dx** in the left side, then **V** is spatial. Lastly, according to Ogden (1984) on page 91, **H** is a two-point tensor, which means that, in this case, **H** is material-spatial.

Moreover, the polar decomposition applied in Eq. (28) gives $\mathbf{dx}=\mathbf{R}(\mathbf{U}\mathbf{dp})$ and, as **U** is material, so **R** is material-spatial. Now, with the mapping types known, further transformation laws are presented followed with the result concerning objectivity of some fields. Using Eq. (27), one has for **C**:

$$\mathbf{C}^*=\left(\mathbf{F}^*\right)^{\mathrm{T}}\mathbf{F}^*=\mathbf{Q}(t_0)\mathbf{F}^{\mathrm{T}}[\mathbf{Q}(t)]^{\mathrm{T}}\mathbf{Q}(t)\mathbf{F}[\mathbf{Q}(t_0)]^{\mathrm{T}}\Rightarrow\mathbf{C}^*\left(\mathbf{p}^*,t\right)=\mathbf{Q}(t_0)\mathbf{C}(\mathbf{p},t)[\mathbf{Q}(t_0)]^{\mathrm{T}} \tag{34}$$

Since **C** is material and Eq. (34) follows the format of Eq. (23), then **C** is objective. Next, it is known that a positive definite tensor **A** can expressed by $\mathbf{A}={}_{\mathbf{A}}\mathbf{Q}\,({}_{\mathbf{A}}\boldsymbol{\Lambda})\,{}_{\mathbf{A}}\mathbf{Q}^{\mathrm{T}}$, where ${}_{\mathbf{A}}\mathbf{Q}$ is the orthogonal tensor (modal tensor of **A**) filled by the normalized eigenvectors of **A** (arranged in columns) and ${}_{\mathbf{A}}\boldsymbol{\Lambda}$ is the diagonal tensor filled by the eigenvalues of **A** (spectral tensor). That said, one can uses $\mathbf{C}=\mathbf{U}^2={}_{\mathbf{U}}\mathbf{Q}\,({}_{\mathbf{U}}\boldsymbol{\Lambda})^2\,{}_{\mathbf{U}}\mathbf{Q}^{\mathrm{T}}$ in Eq. (34) to get:

$$\left(\mathbf{U}^*\right)^2 = \mathbf{Q}(t_0)\,_{\mathrm{U}}\mathbf{Q}\left(_{\mathrm{U}}\boldsymbol{\Lambda}\right)^2 \,_{\mathrm{U}}\mathbf{Q}^{\mathrm{T}}\left[\mathbf{Q}(t_0)\right]^{\mathrm{T}} \Rightarrow \left(\mathbf{U}^*\right)^2 = \mathbf{Q}(t_0)\,_{\mathrm{U}}\mathbf{Q}\left(_{\mathrm{U}}\boldsymbol{\Lambda}\right)^2 \left[\mathbf{Q}(t_0)\,_{\mathrm{U}}\mathbf{Q}\right]^{\mathrm{T}} \Rightarrow \mathbf{U}^* = \mathbf{Q}(t_0)\,_{\mathrm{U}}\mathbf{Q}\left(_{\mathrm{U}}\boldsymbol{\Lambda}\right)\left[\mathbf{Q}(t_0)\,_{\mathrm{U}}\mathbf{Q}\right]^{\mathrm{T}} \Rightarrow$$

$$\Rightarrow \mathbf{U}^* = \mathbf{Q}(t_0)\,_{\mathrm{U}}\mathbf{Q}\left(_{\mathrm{U}}\boldsymbol{\Lambda}\right)\,_{\mathrm{U}}\mathbf{Q}^{\mathrm{T}}\left[\mathbf{Q}(t_0)\right]^{\mathrm{T}} \Rightarrow \mathbf{U}^*\left(\mathbf{p}^*,t\right) = \mathbf{Q}(t_0)\mathbf{U}(\mathbf{p},t)\left[\mathbf{Q}(t_0)\right]^{\mathrm{T}} \tag{35}$$

Since $\mathbf{U}$ is material and Eq. (35) obeys Eq. (23), so $\mathbf{U}$ is objective. Using Eqs. (27) and (35) in $\mathbf{F}^* = \mathbf{R}^*\mathbf{U}^*$, one concludes that $\mathbf{R}^* = \mathbf{Q}(t)\,\mathbf{R}\,[\mathbf{Q}(t_0)]^{\mathrm{T}}$. Therefore, as this expression obeys Eq. (24), $\mathbf{R}$ is objective. Further, the transformation law for $\mathbf{B}$ is:

$$\mathbf{B}^* = \mathbf{F}^*\left(\mathbf{F}^*\right)^{\mathrm{T}} = \mathbf{Q}(t)\mathbf{F}\left[\mathbf{Q}(t_0)\right]^{\mathrm{T}}\mathbf{Q}(t_0)\mathbf{F}^{\mathrm{T}}\left[\mathbf{Q}(t)\right]^{\mathrm{T}} \Rightarrow \mathbf{B}^*\left(\mathbf{x}^*,t\right) = \mathbf{Q}(t)\mathbf{B}(\mathbf{x},t)\left[\mathbf{Q}(t)\right]^{\mathrm{T}} \tag{36}$$

As $\mathbf{B}$ is spatial and Eq. (36) follows Eq. (26), $\mathbf{B}$ is objective. Performing the same procedure done with $\mathbf{U}$ in Eq. (35), one applies $\mathbf{B} = \mathbf{V}^2 = \,_{\mathrm{v}}\mathbf{Q}\,(_{\mathrm{v}}\boldsymbol{\Lambda})^2\,_{\mathrm{v}}\mathbf{Q}^{\mathrm{T}}$ in Eq. (36) to get:

$$\left(\mathbf{V}^*\right)^2 = \mathbf{Q}(t)\,_{\mathrm{v}}\mathbf{Q}\left(_{\mathrm{v}}\boldsymbol{\Lambda}\right)^2 \,_{\mathrm{v}}\mathbf{Q}^{\mathrm{T}}\left[\mathbf{Q}(t)\right]^{\mathrm{T}} \Rightarrow \left(\mathbf{V}^*\right)^2 = \mathbf{Q}(t)\,_{\mathrm{v}}\mathbf{Q}\left(_{\mathrm{v}}\boldsymbol{\Lambda}\right)^2 \left[\mathbf{Q}(t)\,_{\mathrm{v}}\mathbf{Q}\right]^{\mathrm{T}} \Rightarrow \mathbf{V}^* = \mathbf{Q}(t)\,_{\mathrm{v}}\mathbf{Q}\left(_{\mathrm{v}}\boldsymbol{\Lambda}\right)\left[\mathbf{Q}(t)\,_{\mathrm{v}}\mathbf{Q}\right]^{\mathrm{T}} \Rightarrow$$

$$\Rightarrow \mathbf{V}^* = \mathbf{Q}(t)\,_{\mathrm{v}}\mathbf{Q}\left(_{\mathrm{v}}\boldsymbol{\Lambda}\right)\,_{\mathrm{v}}\mathbf{Q}^{\mathrm{T}}\left[\mathbf{Q}(t)\right]^{\mathrm{T}} \Rightarrow \mathbf{V}^*\left(\mathbf{x}^*,t\right) = \mathbf{Q}(t)\mathbf{V}(\mathbf{x},t)\left[\mathbf{Q}(t)\right]^{\mathrm{T}} \tag{37}$$

Since $\mathbf{V}$ is spatial and Eq. (37) is in the same format of Eq. (26), then $\mathbf{V}$ is objective. Now, the differential relation of Eq. (12) provides:

$$\mathbf{dp}^* = \mathbf{Q}(t_0)\mathbf{dp} \tag{38}$$

and the differential relation of Eq. (13) gives:

$$\mathbf{dx}^* = \mathbf{Q}(t)\mathbf{dx} \tag{39}$$

and one concludes, since $\mathbf{dp}$ is material and $\mathbf{dx}$ is spatial, that $\mathbf{dp}$ and $\mathbf{dx}$ are objective for they follow Eq. (20) and Eq. (21), respectively. Continuing with the material time derivative of Eq. (13), one yields:

$$\dot{\boldsymbol{\chi}}^*\left(\mathbf{p}^*,t\right) = \dot{\mathbf{Q}}(t)\boldsymbol{\chi}(\mathbf{p},t) + \mathbf{Q}(t)\dot{\boldsymbol{\chi}}(\mathbf{p},t) + \dot{\mathbf{c}}^*(t) \tag{40}$$

Since $\mathbf{v}(\mathbf{x}, t) = [\dot{\boldsymbol{\chi}}(\mathbf{p}, t)]_{\mathrm{sd}}$ and $\mathbf{v}^*(\mathbf{x}^*, t) = [\dot{\boldsymbol{\chi}}^*(\mathbf{p}^*, t)]_{\mathrm{sd}}$ are the *spatial description* of the velocity with respect to $\mathscr{R}$ and $\mathscr{R}^*$, respectively, one concludes that:

$$\mathbf{v}^*\left(\mathbf{x}^*,t\right) = \dot{\mathbf{Q}}(t)\mathbf{x} + \mathbf{Q}(t)\mathbf{v}(\mathbf{x},t) + \dot{\mathbf{c}}^*(t) \tag{41}$$

which is the transformation law for the velocity. Once more, taking the material time derivative of Eq. (40), one gives:

$$\ddot{\boldsymbol{\chi}}^*\left(\mathbf{p}^*,t\right) = \ddot{\mathbf{Q}}(t)\boldsymbol{\chi}(\mathbf{p},t) + 2\dot{\mathbf{Q}}(t)\dot{\boldsymbol{\chi}}(\mathbf{p},t) + \mathbf{Q}(t)\ddot{\boldsymbol{\chi}}(\mathbf{p},t) + \ddot{\mathbf{c}}^*(t) \tag{42}$$

Since $\mathbf{a}(\mathbf{x}, t) = [\ddot{\boldsymbol{\chi}}(\mathbf{p}, t)]_{\mathrm{sd}}$ and $\mathbf{a}^*(\mathbf{x}^*, t) = [\ddot{\boldsymbol{\chi}}^*(\mathbf{p}^*, t)]_{\mathrm{sd}}$ are the spatial description of the acceleration with respect to $\mathscr{R}$ and $\mathscr{R}^*$, respectively, one has from Eq. (42):

$$\mathbf{a}^*\left(\mathbf{x}^*,t\right) = \ddot{\mathbf{Q}}(t)\mathbf{x} + 2\dot{\mathbf{Q}}(t)\mathbf{v}(\mathbf{x},t) + \mathbf{Q}(t)\mathbf{a}(\mathbf{x},t) + \ddot{\mathbf{c}}^*(t) \tag{43}$$

which is the transformation rule for the acceleration. Thus, since $\mathbf{v}$ and $\mathbf{a}$ are spatial vectors by construction, they are not objective because Eqs. (41) and (43) do not follow Eq. (21). After, the gradient of Eq. (41) with respect to $\mathbf{x}$ yields to:

$$\nabla_{\mathbf{x}}\left[\mathbf{v}^*\left(\mathbf{x}^*,t\right)\right] = \nabla_{\mathbf{x}}\left[\dot{\mathbf{Q}}(t)\mathbf{x} + \mathbf{Q}(t)\mathbf{v}(\mathbf{x},t) + \dot{\mathbf{c}}^*(t)\right] \Rightarrow \left[\nabla_{\mathbf{x}^*}\mathbf{v}^*\left(\mathbf{x}^*,t\right)\right]\left(\nabla_{\mathbf{x}}\mathbf{x}^*\right) = \dot{\mathbf{Q}}(t) + \mathbf{Q}(t)\nabla_{\mathbf{x}}\left[\mathbf{v}(\mathbf{x},t)\right] \Rightarrow$$

$$\Rightarrow \mathbf{L}^*\left(\mathbf{x}^*,t\right)\mathbf{Q}(t) = \dot{\mathbf{Q}}(t) + \mathbf{Q}(t)\mathbf{L}(\mathbf{x},t) \Rightarrow \mathbf{L}^*\left(\mathbf{x}^*,t\right) = \mathbf{Q}(t)\mathbf{L}(\mathbf{x},t)\left[\mathbf{Q}(t)\right]^{\mathrm{T}} + \boldsymbol{\Omega}(t) \tag{44}$$

where $\mathbf{L}(\mathbf{x}, t) = \nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x}, t)$ and $\mathbf{L}^*(\mathbf{x}^*, t) = \nabla_{\mathbf{x}}\mathbf{v}^*(\mathbf{x}^*, t)$ are the velocity gradients according to $\mathscr{R}$ and $\mathscr{R}^*$, respectively. Tensor $\boldsymbol{\Omega}(t)$ is usually referred as frame-spin. Since $\boldsymbol{\Omega}(t)$ is skew and the addition decomposition of $\mathbf{L}$ is $\mathbf{L} = \mathbf{D} + \mathbf{W}$, with $\mathbf{D} = \mathrm{sym}\,\mathbf{L}$ and $\mathbf{W} = \mathrm{skw}\,\mathbf{L}$, one affirms that:

$$\mathbf{L}^* = \mathbf{Q}(t)\mathbf{L}\left[\mathbf{Q}(t)\right]^{\mathrm{T}} + \boldsymbol{\Omega}(t) \Rightarrow \mathbf{D}^* + \mathbf{W}^* = \mathbf{Q}(t)(\mathbf{D} + \mathbf{W})\left[\mathbf{Q}(t)\right]^{\mathrm{T}} + \boldsymbol{\Omega}(t) \Rightarrow$$

$$\Rightarrow \mathbf{D}^* + \mathbf{W}^* = \mathbf{Q}(t)\mathbf{D}\left[\mathbf{Q}(t)\right]^{\mathrm{T}} + \mathbf{Q}(t)\mathbf{W}\left[\mathbf{Q}(t)\right]^{\mathrm{T}} + \boldsymbol{\Omega}(t) \Rightarrow \begin{cases} \mathbf{D}^*\left(\mathbf{x}^*,t\right) = \mathbf{Q}(t)\mathbf{D}(\mathbf{x},t)\left[\mathbf{Q}(t)\right]^{\mathrm{T}} \\ \mathbf{W}^*\left(\mathbf{x}^*,t\right) = \mathbf{Q}(t)\mathbf{W}(\mathbf{x},t)\left[\mathbf{Q}(t)\right]^{\mathrm{T}} + \boldsymbol{\Omega}(t) \end{cases} \tag{45}$$

which is the transformation rule for **D** (stretching tensor) and for **W** (spin tensor), respectively. Moreover, since **L** is spatial by definition, then so does **D** e **W**. That said, one compares Eq. (44) and Eq. (45) with Eq. (26) and concludes that both **L** e **W** are not objective, and **D** is objective. Results concerning objectivity of vectors and tensors discussed in this section are listed in Tab. 2.

Table 2 – Summary of the study.

			Objective?
Vector	**p**	Material	No
	dp	Material	Yes
	x, v, a	Spatial	No
	dx	Spatial	Yes
Second Order Tensor	**F**, **R**	Material-Spatial	Yes
	C, U	Material	Yes
	B, V, D	Spatial	Yes
	L, W	Spatial	No

CONCLUSIONS

This paper has presented a new approach that allows to formulate the IMR in a coherent way with the definition of objectivity. The transformation laws of some important fields were written (Tab. 2) considering a more general case, that is, without the widely used assumption $\mathbf{Q}(t_0) = \mathbf{I}$. It was highlighted that the classification of a second order tensor regarding its objectivity must consider its mapping type (Tab. 1). At last, the proposed framework (Fig. 5) showed itself both mathematically and physically consistent.

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