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STUDY OF NON-NEWTONIAN FLUID FLOW STABILITY MODELED BY LPTT

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Abstract. *Several flows of practical interest are from viscoelastic fluids, and it is often desirable to know whether these flows propagate in a laminar or turbulent state. The present work investigates the convection of Tollmien-Schlichting waves in a two-dimensional Poiseuille incompressible flow for a viscoelastic fluid. The non-Newtonian fluid is modelled by the Phan-Thien Tanner linear constitutive equation (PTT). Linear Stability Theory (LST) is used to verify the stability of the fluid flows. In LST analysis, the equations are linearized for a viscoelastic fluid flow. The results presented are the neutral stability curves by varying the dimensionless parameters for the viscoelastic fluid model and comparing them with the Newtonian fluid.*

Keywords: *Laminar-Turbulent Transition, Viscoelastic Fluids, LPTT Model, Flow Stability, Linear Stability Theory.*

1. INTRODUCTION

Fluids can be classified as Newtonian and non-Newtonian. In Newtonian fluids, each component of applied shear stress is linearly proportional to the strain rate, the proportionality constant being equal to the dynamic viscosity. In non-Newtonian fluids, this proportionality does not happen. One class of non-Newtonian fluids is viscoelastic. When they are subjected to shear stress, they undergo deformation, and when this ceases, there is a specific recovery of the deformation suffered. The main objective of this work is to study the stability of the two-dimensional incompressible Poiseuille flow of a viscoelastic fluid (modelled by the LPTT). The investigation of this phenomenon is carried out through the analysis of the convection of Tollmien-Schlichting waves for the considered flow, using the techniques of Theory of Linear Stability, to analyze the stability of the flow of viscoelastic fluid of the LPTT type and compare it with that of Newtonian fluids.

The most popular linear viscoelastic model is Maxwell's Beris *et al.* (1987), Mompean and Deville (1997) model, which associates the idea of a fluid that presents characteristics s of both an elastic solid and a viscous Newtonian fluid. Among the main nonlinear viscoelastic models found in the literature the differential models can be mentioned: Oldroyd-B Brasseur *et al.* (1998), Alves *et al.* (2003), White-Metzner White and Metzner (1963), Giesekus Giesekus (1982), Leonov Leonov (1976), FENE Bird *et al.* (1980), PTT Thien and Tanner (1977), Alves *et al.* (2003) and derivatives, PomPom Luo and Tanner (1986), Luo and Tanner (1988). In general, viscoelastic models, can qualitatively describe the nonlinear viscoelastic behavior of polymer melts Bretas (2005). Many elastic instabilities have been reported in recent years, corresponding to experimental or theoretical work using linear stability analysis (Larson *et al.* (1990), Shaqfeh *et al.* (1992), Larson (1992)).

2. METHODOLOGY

The model used in this work is the LPTT (Linear Phan-Thien-Tanner), whose constitutive equation can be written as follows:

$$f(tr(\mathbf{T}))\mathbf{T} + \lambda \overset{\nabla}{\mathbf{T}} = 2\eta_p \mathbf{D}, \tag{1}$$

where $D = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the strain tensor rate, λ is the fluid relaxation time, η is the viscosity contributed by the polymer, and $f(tr(\mathbf{T}))$ is the stress coefficient function.

The PTT model is considered in linear form; that is, the function f is given by:

$$f(tr(\mathbf{T})) = 1 + \frac{\lambda\epsilon}{\eta_p} tr(\mathbf{T}). \tag{2}$$

The notation $f(tr(\mathbf{T}))\mathbf{T}$ is used to represent the trace of the extra tension tensor $\mathbf{T}$ and the symbol $\overset{\nabla}{\mathbf{T}}$ represents the convected derivative given by:

$$\overset{\nabla}{\mathbf{T}} = \frac{D\mathbf{T}}{Dt} - \mathbf{T}\cdot\mathbf{L} - \mathbf{L}^T \cdot \mathbf{T}, \tag{3}$$

where $\mathbf{L} = \nabla\mathbf{u} - \xi\mathbf{D}$ is called the effective velocity gradient, ϵ and ξ are positive model parameters, the constant being ϵ a parameter related to the elongational behavior of the model and ξ affects the shear behavior.

After algebraic manipulations, the equations of the non-Newtonian tensors for the LPTT model are obtained:

$$f(tr(T))T^{xx} + \lambda\left(\frac{\partial T^{xx}}{\partial t} + \frac{\partial(uT^{xx})}{\partial x} + \frac{\partial(vT^{xx})}{\partial y} - 2T^{xx}(1-\xi)\frac{\partial u}{\partial x} - 2T^{xy}\left(1-\frac{1}{2}\xi\right)\frac{\partial u}{\partial y} + T^{xy}\xi\,\frac{\partial v}{\partial x}\right) = 2\eta_p\frac{\partial u}{\partial x},$$
$$f(tr(T))T^{xy} + \lambda\left(\frac{\partial T^{xy}}{\partial t} + \frac{\partial(uT^{xy})}{\partial x} + \frac{\partial(vT^{xy})}{\partial y} - T^{xx}\left(1-\frac{1}{2}\xi\right)\frac{\partial v}{\partial x} + +T^{xx}\,\frac{\xi}{2}\frac{\partial u}{\partial y} - T^{yy}\left(1-\frac{1}{2}\xi\right)\frac{\partial u}{\partial y} + T^{yy}\frac{\xi}{2}\,\frac{\partial v}{\partial x}\right)$$
$$= \eta_p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$
$$f(tr(T))T^{yy} + \lambda\left(\frac{\partial T^{yy}}{\partial t} + \frac{\partial(uT^{yy})}{\partial x} + \frac{\partial(vT^{yy})}{\partial y} - 2T^{yy}(1-\xi)\frac{\partial v}{\partial y} + -2T^{xy}\left(1-\frac{1}{2}\xi\right)\frac{\partial v}{\partial x}\right) + T^{xy}\xi\,\frac{\partial u}{\partial y}\right) = 2\eta_p\frac{\partial v}{\partial y}.$$

2.1 Dimensionless Equations

The parameters adopted are length L, velocity U and density ρ. The dimensionless variables (added by the superscript *) are given below:

$$\mathbf{x}^* = \frac{\mathbf{x}}{L},\;\; \mathbf{u}^* = \frac{\mathbf{u}}{U},\;\; \mathbf{t}^* = \frac{tU}{L},\;\; p^* = \frac{p}{\rho U^2},\;\; \mathbf{T}^* = \frac{\mathbf{T}}{\rho U^2}, \tag{4}$$

When applying changes in variables (4) the dimensionless numbers appear: Reynolds number (Re), Weissenberg number (Wi) and the constant β.

• Reynolds number (Re): Represents the ratio between the inertial forces and the viscous forces of the flow and is defined by:

$$Re = \frac{\rho UL}{\eta_0}, \tag{5}$$

where η_0 is the total dynamic viscosity of the fluid, given by $\eta_0 = \eta_s + \eta_p$, where η_s is the viscosity of the solvent and η_p the viscosity of the polymer.

• Weissenberg number(Wi): For a viscoelastic fluid, is the ratio of a characteristic fluid time scale to a flow time scale:

$$Wi = \frac{\lambda U}{L}, \tag{6}$$

• Constant β: The constant $\beta \in (0,1)$ is a quantity that controls the contribution of the Newtonian solvent, defined by:

$$\beta = \frac{\eta_s}{\eta_0}. \tag{7}$$

Dimensionalized equations of the non-Newtonian tensors for the LPTT model:

$$f(tr(T))T^{xx} + Wi\left(\frac{\partial T^{xx}}{\partial t} + \frac{\partial(uT^{xx})}{\partial x} + \frac{\partial(vT^{xx})}{\partial y} - 2T^{xx}(1-\xi)\frac{\partial u}{\partial x} + -2T^{xy}\left(1-\frac{1}{2}\xi\right)\frac{\partial u}{\partial y} + T^{xy}\xi\,\frac{\partial v}{\partial x}\right) = 2\frac{(1-\beta)}{Re}\frac{\partial u}{\partial x},$$
$$f(tr(T))T^{xy} + Wi\left(\frac{\partial T^{xy}}{\partial t} + \frac{\partial(uT^{xy})}{\partial x} + \frac{\partial(vT^{xy})}{\partial y} - T^{xx}\left(1-\frac{1}{2}\xi\right)\frac{\partial v}{\partial x} + +T^{xx}\,\frac{\xi}{2}\frac{\partial u}{\partial y} - T^{yy}\left(1-\frac{1}{2}\xi\right)\frac{\partial u}{\partial y} + T^{yy}\frac{\xi}{2}\,\frac{\partial v}{\partial x}\right)$$
$$= \frac{(1-\beta)}{Re}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$
$$f(tr(T))T^{yy} + Wi\left(\frac{\partial T^{yy}}{\partial t} + \frac{\partial(uT^{yy})}{\partial x} + \frac{\partial(vT^{yy})}{\partial y} - 2T^{yy}(1-\xi)\frac{\partial v}{\partial y} + -2T^{xy}\left(1-\frac{1}{2}\xi\right)\frac{\partial v}{\partial x}\right) + T^{xy}\xi\,\frac{\partial u}{\partial y}\right) = 2\frac{(1-\beta)}{Re}\frac{\partial v}{\partial y}.$$

2.2 Linear Stability Theory

For analysis using the Theory of Linear Stability in this work, the base flow invariant in the x direction was considered, that is:

$$u = U(y), \quad v = 0, \quad p = P(x,y), \quad T = \mathbf{T}(y). \tag{8}$$

It is considered that the instantaneous flow can be decomposed by a base flow and a perturbed flow; the dependent variables are decomposed as follows:

$$\begin{aligned}
u(x,y,t) &= U(y) + \tilde{u}(x,y,t),\\
v(x,y,t) &= \tilde{v}(x,y,t),\\
p(x,y,t) &= P(x,y) + \tilde{p}(x,y,t),\\
T(x,y,t) &= \mathbf{T}(y) + \tilde{T}(x,y,t).
\end{aligned}$$

SCHLICHTING (1979) states that the perturbations must be small in that nonlinear terms can be neglected compared to linear terms. Thus, the simplified system is given by:

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y} &= 0,\\
\frac{\partial \tilde{u}}{\partial t}+U\frac{\partial \tilde{u}}{\partial x}+\tilde{v}\frac{\partial U}{\partial y} &= -\frac{\partial \tilde{p}}{\partial x}+\frac{\beta}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial x^2}+\frac{\partial^2 \tilde{u}}{\partial y^2}\right)+\frac{\partial \tilde{T}^{xx}}{\partial x}+\frac{\partial \tilde{T}^{xy}}{\partial y},\\
\frac{\partial \tilde{v}}{\partial t}+U\frac{\partial \tilde{v}}{\partial x} &= -\frac{\partial \tilde{p}}{\partial y}+\frac{\beta}{Re}\left(\frac{\partial^2 \tilde{v}}{\partial x^2}+\frac{\partial^2 \tilde{v}}{\partial y^2}\right)+\frac{\partial \tilde{T}^{xy}}{\partial x}+\frac{\partial \tilde{T}^{yy}}{\partial y}.
\end{aligned}$$

For the equation of the non-Newtonian tensor T^{xx}, one has that:

$$f(tr(T)) = \left[1+\underbrace{\epsilon\frac{ReWi}{(1-\beta)}}_{constante\ (C)}(T^{xx}+T^{yy})\right]T^{xx} = \left[1+C(T^{xx}+\tilde{T}^{xx})+(T^{yy}+\tilde{T}^{yy})\right](T^{xx}+\tilde{T}^{xx}) = \tilde{T}^{xx}+ C\left(2T^{xx}\tilde{T}^{xx}+T^{xx}\tilde{T}^{yy}+T^{yy}\tilde{T}^{xx}\right).$$

$$C\left(2T^{xx}\tilde{T}^{xx}+T^{xx}\tilde{T}^{yy}+T^{yy}\tilde{T}^{xx}\right)\tilde{T}^{xx}+Wi\left(\frac{\partial \tilde{T}^{xx}}{\partial t}+U\frac{\partial \tilde{T}^{xx}}{\partial x}+\tilde{v}\frac{\partial \mathbf{T}^{xx}}{\partial y}-2\tilde{T}^{xy}\frac{\partial U}{\partial y}+\xi\tilde{T}^{xy}\frac{\partial U}{\partial y}-2\mathbf{T}^{xy}\frac{\partial \tilde{u}}{\partial y}\right.$$
$$\left.+\xi\mathbf{T}^{xy}\frac{\partial \tilde{u}}{\partial y}+T^{xx}\frac{\partial \tilde{v}}{\partial y}-T^{xx}\frac{\partial \tilde{u}}{\partial x}+2\xi T^{xx}\frac{\partial \tilde{u}}{\partial x}+\xi T^{xy}\frac{\partial \tilde{v}}{\partial x}\right)=2\frac{(1-\beta)}{Re}\frac{\partial \tilde{u}}{\partial x},$$

For the equation of the non-Newtonian tensor T^{xy}, one has that:

$$C\left(T^{xy}\tilde{T}^{xx}+T^{xx}\tilde{T}^{xy}+T^{yy}\tilde{T}^{xy}+T^{xy}\tilde{T}^{yy}\right)\tilde{T}^{xy}+Wi\left(\frac{\partial \tilde{T}^{xy}}{\partial t}+U\frac{\partial \tilde{T}^{xy}}{\partial x}+\tilde{u}\frac{\partial \mathbf{T}^{xy}}{\partial x}+\tilde{v}\frac{\partial \mathbf{T}^{xy}}{\partial y}-\frac{\xi}{2}\tilde{T}^{xx}\frac{\partial U}{\partial y}+\frac{\xi}{2}\tilde{T}^{yy}\frac{\partial U}{\partial y}\right.$$
$$\left.+\frac{\xi}{2}T^{xx}\frac{\partial \tilde{u}}{\partial y}-T^{yy}\frac{\partial \tilde{u}}{\partial y}+\frac{\xi}{2}T^{yy}\frac{\partial \tilde{u}}{\partial y}+\mathbf{T}^{yy}\frac{\partial \tilde{u}}{\partial y}+T^{xy}\frac{\partial \tilde{v}}{\partial y}+\tilde{T}^{xy}\frac{\partial U}{\partial x}-T^{xx}\frac{\partial \tilde{v}}{\partial x}+\frac{\xi}{2}T^{xx}\frac{\partial \tilde{v}}{\partial x}+\frac{\xi}{2}\,T^{yy}\frac{\partial \tilde{v}}{\partial x}\right)=\frac{(1-\beta)}{Re}\left[\frac{\partial \tilde{v}}{\partial x}+\frac{\partial \tilde{u}}{\partial y}\right],$$

For the equation of the non-Newtonian tensor T^{yy}, one has that:

$$C\left(2T^{yy}\tilde{T}^{yy}+T^{yy}\tilde{T}^{xx}+T^{xx}\tilde{T}^{yy}\right)\tilde{T}^{yy}+Wi\left(\frac{\partial \tilde{T}^{yy}}{\partial t}+U\frac{\partial \tilde{T}^{yy}}{\partial x}+\tilde{v}\frac{\partial \mathbf{T}^{yy}}{\partial y}+\xi\tilde{T}^{xy}\frac{\partial U}{\partial y}+\xi T^{xy}\frac{\partial \tilde{u}}{\partial y}-\mathbf{T}^{yy}\frac{\partial \tilde{v}}{\partial y}+\right.$$
$$\left.2\xi\mathbf{T}^{yy}\frac{\partial \tilde{v}}{\partial y}+T^{yy}\frac{\partial \tilde{u}}{\partial x}-2T^{xy}\frac{\partial \tilde{v}}{\partial x}+\xi T^{xy}\frac{\partial \tilde{v}}{\partial x}\right)=2\frac{(1-\beta)}{Re}\frac{\partial \tilde{u}}{\partial x}.$$

The resulting equations are linear, and the coefficients of the equations do not depend on t and x; the solutions can be searched using the separation of variables method as follows:

$$\begin{aligned}
\tilde{u}(x,y,t) &= \overline{u}(y)e^{i(\alpha x-\omega t)},\\
\tilde{v}(x,y,t) &= \overline{v}(y)e^{i(\alpha x-\omega t)},\\
\tilde{p}(x,y,t) &= \overline{p}(y)e^{i(\alpha x-\omega t)},\\
\tilde{T}(x,y,t) &= \overline{T}(y)e^{i(\alpha x-\omega t)},
\end{aligned}$$

$i=\sqrt{-1}$, α being the wave number in the x direction and u, v, p and T being the amplitudes of the perturbations. Where ω is the frequency with which the perturbations, of wavelength $\lambda = 2\pi/\alpha$ and wave speed $c=\frac{\omega}{\alpha}$, propagate. We also consider $\overline{u}, \overline{v}$ and $\overline{p}$ as amplitudes of the perturbations. Assuming that these equations make up a solution to the simplified system, the conjugate complexes also make up a possible solution to the system in question. A linear combination of the solutions is taken as the solution belonging to the set of real numbers.

Substituting these linear combinations into the equations obtained for the perturbations, one has:

Continuity:

$$i\alpha\overline{u}+\overline{v}'=0. \tag{9}$$

For the momentum equation in the x direction, we have:

$$Re(i\alpha U-i\omega)\overline{u}-\beta(\overline{u}''-\alpha^2\overline{u})+Re\overline{v}U' = i\alpha(\overline{T}^{xx}-\overline{p})+\overline{T}'^{xy}. \tag{10}$$

For the momentum equation in the y direction, we have:

$$Re(i\alpha U-i\omega)\overline{v}-\beta(\overline{v}''-\alpha^2\overline{v}) = i\alpha\overline{T}^{xy}+\overline{T}'^{yy}-\overline{p}. \tag{11}$$

For the tensors T^{xx}, T^{xy} and T^{yy}, we have:

$$C\left(2T^{xx}\overline{T}^{xx}+T^{xx}\overline{T}^{yy}+T^{yy}\overline{T}^{xx}\right)\overline{T}^{xx}+Wi\left(-i\omega\overline{T}'^{xx}+i\alpha U\overline{T}'^{xx}+\overline{v}\mathbf{T}^{xx}-2\overline{T}^{xy}\frac{\partial U}{\partial y}+\xi\overline{T}^{xy}\frac{\partial U}{\partial y}-2\mathbf{T}^{xy}\frac{\partial\overline{u}}{\partial y}\right.$$
$$\left.+\xi\mathbf{T}^{xy}\frac{\partial\overline{u}}{\partial y}+T^{xx}\frac{\partial\overline{v}}{\partial y}-i\alpha\overline{u}T^{xx}+2i\alpha\xi\overline{u}T^{xx}+i\alpha\xi\overline{v}T^{xy}\right)=\frac{2i\alpha(1-\beta)}{Re\;\overline{u}},$$

$$C\left(T^{xy}\overline{T}^{xx}+T^{xx}\overline{T}^{xy}+T^{yy}\overline{T}^{xy}+T^{xy}\overline{T}^{yy}\right)\overline{T}^{xy}+Wi\left(-i\omega\overline{T}^{xy}+i\alpha U\overline{T}^{xy}+\overline{v}\mathbf{T}^{xy}+\frac{\xi}{2}\overline{T}^{xx}\frac{\partial U}{\partial y}-\frac{\xi}{2}\overline{T}^{yy}\frac{\partial U}{\partial y}+\right.$$
$$\left.\frac{\xi}{2}\overline{T}^{yy}\frac{\partial U}{\partial y}+\frac{\xi}{2}T^{xx}\frac{\partial\overline{u}}{\partial y}-\mathbf{T}^{yy}\frac{\partial\overline{u}}{\partial y}+\frac{\xi}{2}T^{yy}\frac{\partial\overline{u}}{\partial y}+\mathbf{T}^{xy}\frac{\partial\overline{v}}{\partial y}+i\alpha\overline{u}T^{xy}-i\alpha\overline{v}T^{xx}+i\alpha\frac{\xi}{2}\overline{v}T^{xx}+i\alpha\frac{\xi}{2}\overline{v}T^{yy}\right)=\frac{(1-\beta)}{Re(\frac{\partial u}{\partial y+i\alpha\overline{v}})},$$

$$C\left(2T^{yy}\overline{T}^{yy}+T^{yy}\overline{T}^{xx}+T^{xx}\overline{T}^{yy}\right)\overline{T}^{yy}+Wi\left(-i\omega\overline{T}^{yy}+i\alpha U\overline{T}^{yy}+\overline{v}\mathbf{T}^{yy}+\xi\overline{T}^{xy}\frac{\partial U}{\partial y}+\xi T^{xy}\frac{\partial\overline{u}}{\partial y}-\mathbf{T}^{yy}\frac{\partial\overline{v}}{\partial y}+\right.$$

$$\left.2\,\xi\mathbf{T}^{yy}\frac{\partial\overline{v}}{\partial y}+i\alpha T^{yy}\overline{u}-2i\overline{v}\alpha T^{xy}+i\alpha\xi\overline{v}T^{xy}\right)=\frac{2i\alpha(1-\beta)}{Re\;\overline{v}}. \tag{12}$$

2.3 Baseflow

An explicit solution was obtained using the numerical solution seen in the equations (9)-(12) for the baseflow for the LPTT fluid. This solution was obtained using the Mathematica and Matlab/Octave tools, and a paper with the results is submitted to the Journal of Non-Newtonian Fluid Mechanics ARAUJO *et al.* (2022).

2.4 Numerical Method

A 2D spatial LST code was used to analyze the spatial stability of flow in a channel for the LPTT viscoelastic model. α is searched for stability analysis of a perturbation given ω (temporal frequency). A spectrum of eigenvalues is obtained where a search is done for the most unstable eigenvalue. The system for the LPTT viscoelastic model starting from the equations (9)-(12) is as follows:

Id is the m x n identity matrix of Chebyshev modes in the matrix below. Zero represents a matrix m x n size of Chebyshev modes. $1i$ is the imaginary unit of a complex number that can be written that way in Matlab or octave. It is a matrix of blocks, and each block has the Chebyshev number squared.

$varTxx_u = -2\cdot Wi\cdot Txyb\cdot Dy + Wi\cdot xi\cdot Txyb\cdot Dy;$
$varTxx_v = +Wi\cdot dTxxbdy\cdot Id;$
$varTxx_{Txy} = -2\cdot Wi\cdot dUdy\cdot Id + Wi\cdot xi\cdot dUdy\cdot Id;$
$varTxx_{Tyy} = +cte\cdot Txxb\cdot Id;$
$varTxx_{\alpha u} = +21i\cdot Wi\cdot Txxb\cdot Id - 2\cdot 1i\cdot Wi\cdot xi\cdot Txxb\cdot Id + 2\cdot 1i\cdot (1-\beta)/Re\cdot Id;$
$varTxx_{\alpha v} = -1i\cdot Wi\cdot xi\cdot Txyb\cdot Id;$
$varTxx_{\alpha Txx} = -1i\cdot Wi\cdot U\cdot Id;$

$varTxy_u = +1/2\cdot Wi\cdot xi\cdot Txxb\cdot Dy - Wi\cdot Tyyb\cdot Dy + 0.5\cdot Wi\cdot xi\cdot Tyyb\cdot Dy - (1-\beta)/Re\cdot Dy;$
$varTxy_v = +Wi\cdot dTxybdy\cdot Id + Wi\cdot xi\cdot Txyb\cdot Dy;$
$varTxy_{Txx} = +cte\cdot Txyb\cdot Id + 0.5\cdot Wi\cdot xi\cdot dUdy\cdot Id;$
$varTxy_{Tyy} = +cte\cdot Txyb\cdot Id - Wi\cdot dUdy\cdot Id + 0.5\cdot Wi\cdot xi\cdot dUdy\cdot Id;$
$varTxy_{\alpha u} = -1i\cdot Wi\cdot xi\cdot Txyb\cdot Id;$
$varTxy_{\alpha v} = +1i\cdot Wi\cdot Txxb\cdot Id - 0.5\cdot Wi\cdot xi\cdot Txxb\cdot Id - 0.5\cdot 1i\cdot Wi\cdot xi\cdot Tyyb\cdot Id + 1i\cdot (1-\beta)/Re\cdot Id;$
$varTxy_{\alpha Txy} = -1i\cdot Wi\cdot U\cdot Id;$

$varTyy_u = +Wi\cdot xi\cdot Txyb\cdot Dy;$
$varTyy_v = +Wi\cdot dTyybdy\cdot Id - 2Wi\cdot Tyyb\cdot Dy - 2\cdot Wi\cdot xi\cdot Tyyb\cdot Dy + 2\cdot (1-\beta)/Re\cdot Dy;$
$varTyy_{Txx} = +cte\cdot Tyyb\cdot Id;$
$varTyy_{Txy} = +Wi\cdot xi\cdot dUdy\cdot Id;$
$varTyy_{\alpha v} = +2\cdot 1i\cdot Wi\cdot Txyb\cdot Id - 1i\cdot Wi\cdot xi\cdot Txyb\cdot Id;$
$varTyy_{\alpha Tyy} = -1i\cdot Wi\cdot U\cdot Id;$

$var = -1i\cdot omega\cdot Id - (\beta/Re)\cdot Dy2;$
$varTxx_{Txx} = +Id + 2\cdot cte\cdot Txxb\cdot Id + cte\cdot Tyyb\cdot Id - 1i\cdot omega\cdot Wi\cdot Id;$
$varTxy_{Txy} = +Id + cte\cdot Txxb\cdot Id + cte\cdot Tyyb\cdot Id - 1i\cdot omega\cdot Wi\cdot Id;$
$varTyy_{Tyy} = +Id + cte\cdot Txxb\cdot Id + 2\cdot cte\cdot Tyyb\cdot Id - 1i\cdot omega\cdot Wi\cdot Id;$

Where the matrix L is given by:

$$\begin{bmatrix} \text{zero} & \text{zero} & \text{Dy} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} \\ \text{var} & \text{zero} & \text{dUdy} & \text{zero} & \text{zero} & \text{zero} & \text{-Dy} & \text{zero} \\ \text{zero} & \text{Id} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} \\ \text{zero} & \text{zero} & \text{var} & \text{zero} & \text{Dy} & \text{zero} & \text{zero} & \text{-Dy} \\ \text{zero} & \text{zero} & \text{zero} & \text{Id} & \text{zero} & \text{zero} & \text{zero} & \text{zero} \\ \text{varTxx}_u & \text{zero} & \text{varTxx}_v & \text{zero} & \text{zero} & \text{varTxx}_{Txx} & \text{varTxx}_{Txy} & \text{varTxx}_{Tyy} \\ \text{varTxy}_u & \text{zero} & \text{varTxy}_v & \text{zero} & \text{zero} & \text{varTxy}_{Txx} & \text{varTxy}_{Txy} & \text{varTxy}_{Tyy} \\ \text{varTyy}_u & \text{zero} & \text{varTyy}_v & \text{zero} & \text{zero} & \text{varTyy}_{Txx} & \text{varTyy}_{Txy} & \text{varTyy}_{Tyy} \end{bmatrix},$$

and the matrix F is given by:

$$\begin{bmatrix} \text{-1i*Id} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} \\ \text{-1i*U} & \text{-}(\beta/Re)*Id & \text{zero} & \text{zero} & \text{-1i*Id} & \text{1i*Id} & \text{zero} & \text{zero} \\ \text{Id} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} \\ \text{zero} & \text{zero} & \text{-1i*U} & \text{-}(\beta/Re)*Id & \text{zero} & \text{zero} & \text{1i*Id} & \text{zero} \\ \text{zero} & \text{zero} & \text{Id} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{zero} \\ \text{varTxx}_{\alpha u} & \text{zero} & \text{varTxx}_{\alpha v} & \text{zero} & \text{zero} & \text{varTxx}_{\alpha Txx} & \text{zero} & \text{zero} \\ \text{varTxy}_{\alpha u} & \text{zero} & \text{varTxy}_{\alpha v} & \text{zero} & \text{zero} & \text{zero} & \text{varTxy}_{\alpha Txy} & \text{zero} \\ \text{zero} & \text{zero} & \text{varTyy}_{\alpha v} & \text{zero} & \text{zero} & \text{zero} & \text{zero} & \text{varTyy}_{\alpha Tyy} \end{bmatrix}$$

The system is constructed as follows $L*[u;\alpha*u;v;\alpha*v;p;Txx;Txy;Tyy]^T = \alpha*F*[u;\alpha*u;v;\alpha*v;p;Txx;Txy;Tyy]^T$.

Figure 1 shows the eigenvalue spectra obtained by the eigenvalue problem. In the second figure, a search for the most unstable eigenvalue is performed.

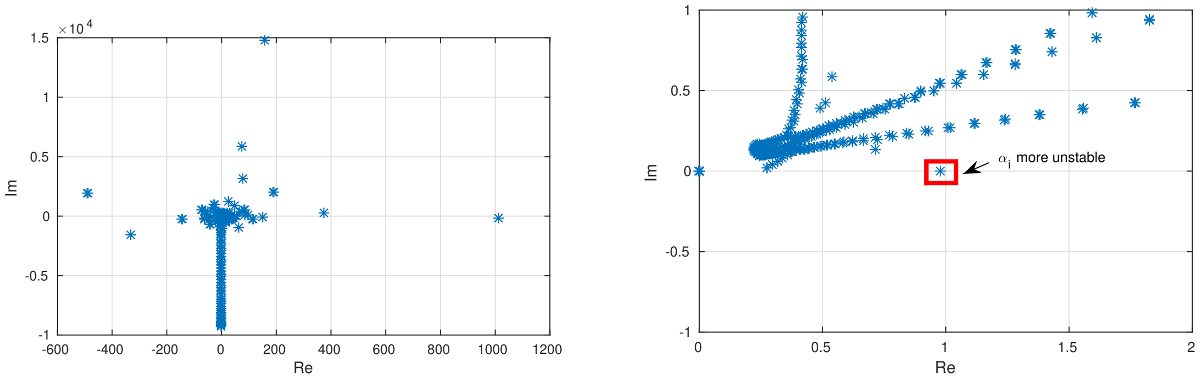

Figure 1. Spectrum of eigenvalues.

2.5 Verification

The verification of the LST code implemented for the plane, two-dimensional, incompressible and isothermal Poiseuille problem, using the LPTT viscoelastic fluid model, was performed. The verification was performed by comparing the developed code considering 1 and $\epsilon = 1 \times 10^{-5}$ (zero becomes unstable, and the code does not converge to the solution) in the equation 1 representing the LPTT model with the one that had already been developed for the Oldroyd-B model Gervazoni (2016). It is worth noting that when $\xi = 0$ and $\epsilon = 0$ in the 1 equation, the constitutive equation obtained is from the Oldroyd-B model. From the neutral curves of Figures 2 and 3, a close agreement between the Oldroyd-B and LPTT viscoelastic models is observed. Thus, the verification performed indicates that the implemented code can simulate and analyze the stability of two-dimensional flows with the LPTT model.

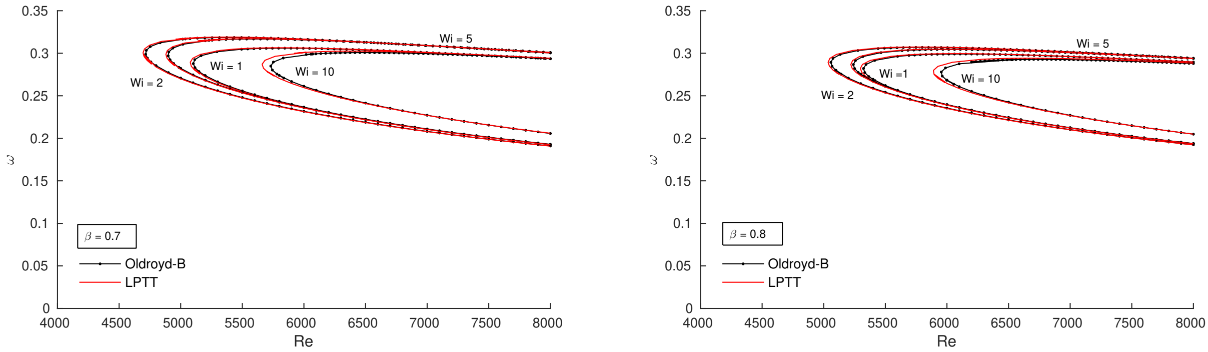

Figure 2. Comparison of neutral curves for the Oldroyd-B (○) black curve and LPTT (-) red curve models.

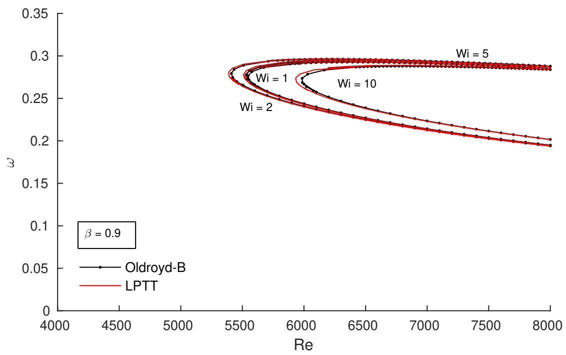

Figure 3. Comparison of neutral curves for the Oldroyd-B (○) black curve and LPTT (-) red curve models.

3. RESULTS

The results are presented where the stability of a two-dimensional flow for a viscoelastic fluid of the LPTT type, using the Linear Stability Theory. Spatial analysis was performed by elaborating stability curves for each set of parameters. Numerical simulations were performed to find the values of α_i for various Reynolds values and frequency ω.

Checking the influence of the Weissenberg number (Wi):

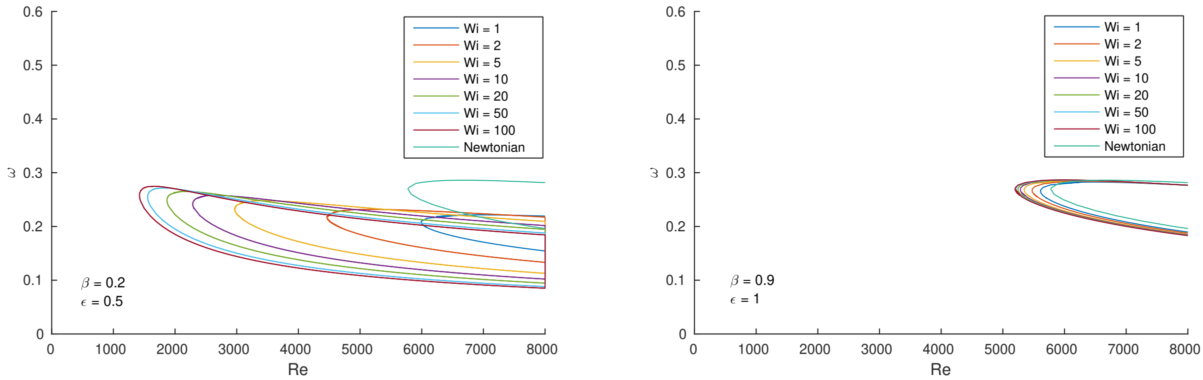

Figure 4. Spatial neutral stability diagram for $\beta = 0.2$ and 0.9, $\xi = 0$, $\epsilon = 0.5$ and 0.75 e $Wi = 1,\ 2,\ 5,\ 10,\ 20,\ 50$ and 100.

In figure 4, the neutral stability curves of the LPTT model as the value of Wi increases, there is a shift of the neutral curve to the left, indicating that the flow becomes unstable for smaller Reynolds number values. This shift is much less pronounced when the Weissenberg number (Wi) has values of 50 and 100. Increasing the value of the constant (Wi) to 0.9 (closer to the Newtonian fluid), it is noticed that the neutral stability curves of the LPTT model as the value of (Wi) increases. The neutral curve shifts to the left, indicating that the flow becomes unstable for smaller Reynolds number values. This shift is much less pronounced when the Weissenberg number (Wi) has values of 50 and 100.

Checking the influence of the constant β:

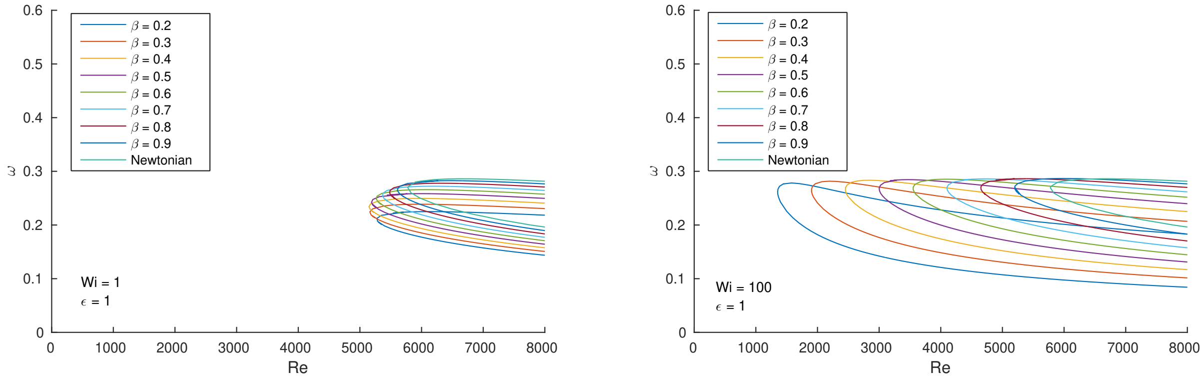

Figure 5. Spatial neutral stability diagram for $\beta = 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8$ and 0.9, $\xi = 0$, $\epsilon = 1$ and $Wi = 1$ and 100.

In Figure 5, it is observed that in the neutral stability curves of the LPTT model, as soon as the value of β increases, the value of ω increases. In the Figure on the left, one notices that for values near $\beta = 0.2$ and for values near $\beta = 0.9$, the neutral curve shifts to the right. For values of β between $\beta = 0.8$ and $\beta = 0.4$, the departure from the neutral curve tends to the left, indicating that the flow becomes unstable for smaller Reynolds number values. The critical Reynolds number is close to 5000. In the Figure to the right, it is observed that as the value of β decreases, the neutral stability curve shifts to the left, where it is notable that the flow becomes unstable for smaller Reynolds number values. It is noteworthy that as the value of β increases, the neutral stability curves of the LPTT model get closer to the neutral curve of the Newtonian model.

Checking the influence of the constant ϵ:

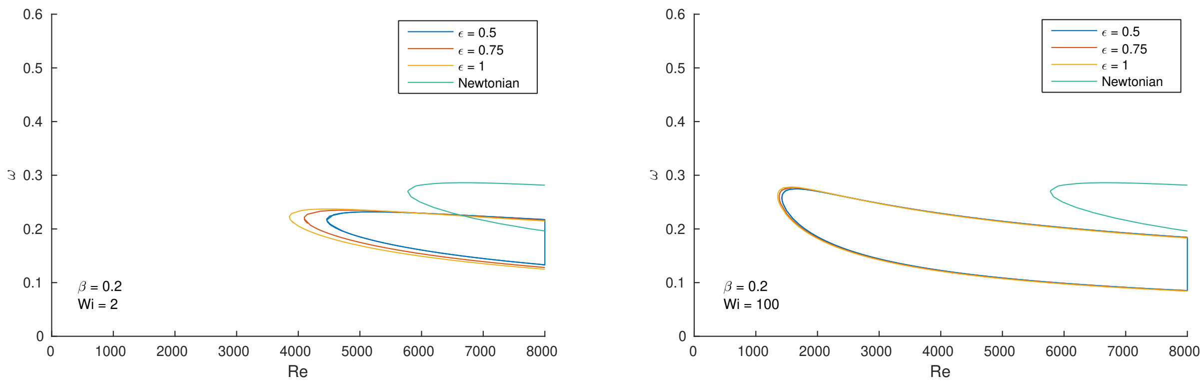

Figure 6. Spatial neutral stability diagram for $\beta = 0.2$, $\xi = 0$, $\epsilon = 0.5,\ 0.75$, and 1 and $Wi = 2$ and 100.

Analyzing Figure 6, the neutral stability curves of the LPTT model as the value of ϵ increases, the neutral curve shifts to the left. This departure is most noticeable in the Figure on the left, showing that the flow becomes unstable for smaller Reynolds number values. In the Figure on the right, it is observed that when the Weissenberg number (Wi) increases, the neutral curve shifts to the left, but the spacing between the neutral curves is less pronounced. In these figures, the Newtonian fluid flow is more stable than the LPTT fluid flows.

Checking the influence of the constant ξ:

It can be observed in the Figure 7 on the left, that the neutral stability curves of the LPTT model as the value of Wi increases and ξ decreases for $\xi = 0$, there is a shift of the neutral curve to the left, indicating that the flow becomes unstable for smaller Reynolds number values. In the Figure on the right, as the value of ϵ increases for $\epsilon = 1$ and the value of ξ decreases for $xi = 0$, the displacement of the neutral stability curve tends to the left, making the flow more unstable for smaller Reynolds number values. Analyzing the figures, it can be seen that the neutral stability curves of the LPTT model get closer to the neutral curve of the Newtonian model as the value of $beta$ increases and when the value of Wi decreases.

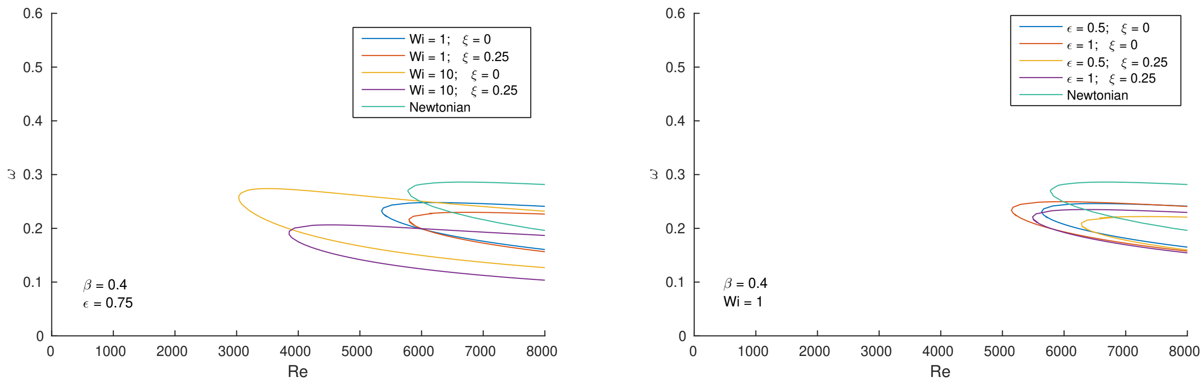

Figure 7. Spatial neutral stability diagram for $\beta = 0.4$, $\xi = 0$ and 0.25, $\epsilon = 0.5, 0.75$ and 1 and $Wi = 1$ and 10.

4. CONCLUSIONS

The incompressible, isothermal, two-dimensional flow equations for a non-Newtonian viscoelastic fluid were presented. The viscoelastic model adopted was the Linear Phan Thien Tanner model (LPTT). Spatial analysis was used to investigate the stability of viscoelastic fluid flows using the Linear Stability Theory through neutral stability curves. The neutral stability curves were evaluated only through the two-dimensional perturbations for different values of dimensionless parameters of the model. The numerical results obtained using the LST technique were satisfactory in analyzing the stability of viscoelastic flows.

5. REFERENCES

Alves, M.A., Oliveira, P.J. and Pinho, F.T., 2003. “Benchmark solutions for the flow of oldroyd-b and ptt fluids in planar contractions”. *Journal of Non-Newtonian Fluid Mechanics*, Vol. 110, No. 1, pp. 45–75.

ARAUJO, M.T.d., Furlan, L.J.d.S., Souza, L.F.d. and Brandi, A.C., 2022. “Semi-analytical method for channel and pipe flows for linear phan-thien-tanner fluid model with a solvent contribution”. *Available at SSRN 4078503*.

Beris, A., Armstrong, R. and Brown, R., 1987. “Spectral/finite-element calculations of the flow of a maxwell fluid between eccentric rotating cylinders”. *Journal of non-newtonian fluid mechanics*, Vol. 22, No. 2, pp. 129–167.

Bird, R., Dotson, P. and Johnson, N., 1980. “Polymer solution rheology based on a finitely extensible bead—spring chain model”. *Journal of Non-Newtonian Fluid Mechanics*, Vol. 7, No. 2-3, pp. 213–235.

Brasseur, E., Fyrillas, M.M., Georgiou, G.C. and Crochet, M.J., 1998. “The time-dependent extrudate-swell problem of an oldroyd-b fluid with slip along the wall”. *Journal of Rheology*, Vol. 42, No. 3, pp. 549–566.

Bretas, R.E., 2005. *Reologia de polímeros fundidos*. Editora da UFSCar.

Gervazoni, E.S., 2016. “Análise de estabilidade linear de escoamentos bidimensionais do fluido oldroyd-b”.

Giesekus, H., 1982. “A simple constitutive equation for polymer fluids based on the concept of deformation-dependent tensorial mobility”. *Journal of Non-Newtonian Fluid Mechanics*, Vol. 11, No. 1-2, pp. 69–109.

Larson, R.G., 1992. “Instabilities in viscoelastic flows”. *Rheologica Acta*, Vol. 31, No. 3, pp. 213–263.

Larson, R.G., Shaqfeh, E.S. and Muller, S.J., 1990. “A purely elastic instability in taylor–couette flow”. *Journal of Fluid Mechanics*, Vol. 218, pp. 573–600.

Leonov, A., 1976. “Nonequilibrium thermodynamics and rheology of viscoelastic polymer media”. *Rheologica acta*, Vol. 15, No. 2, pp. 85–98.

Luo, X.L. and Tanner, R., 1986. “A streamline element scheme for solving viscoelastic flowproblems part ii: integral constitutive models”. *Journal of Non-Newtonian Fluid Mechanics*, Vol. 22, No. 1, pp. 61–89.

Luo, X.L. and Tanner, R., 1988. “Finite element simulation of long and short circular die extrusion experiments using integral models”. *International journal for numerical methods in engineering*, Vol. 25, No. 1, pp. 9–22.

Mompean, G. and Deville, M., 1997. “Unsteady finite volume simulation of oldroyd-b fluid through a three-dimensional planar contraction”. *Journal of Non-Newtonian Fluid Mechanics*, Vol. 72, No. 2-3, pp. 253–279.

SCHLICHTING, H., 1979. *Boundary-layer theory*. [S.l.]: McGraw-Hill, IncH.,.

Shaqfeh, E.S., Muller, S.J. and Larson, R.G., 1992. “The effects of gap width and dilute solution properties on the viscoelastic taylor-couette instability”. *Journal of Fluid Mechanics*, Vol. 235, pp. 285–317.

Thien, N.P. and Tanner, R.I., 1977. “A new constitutive equation derived from network theory”. *Journal of Non-Newtonian Fluid Mechanics*, Vol. 2, No. 4, pp. 353–365.

White, J. and Metzner, A., 1963. “Development of constitutive equations for polymeric melts and solutions”. *Journal of Applied Polymer Science*, Vol. 7, No. 5, pp. 1867–1889.